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A NOTE OF STABILITY OF WEAKLY EFFICIENT SOLUTION SET FOR OPTIMIZATION WITH SET-VALUED MAPS [1]

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In this paper, we study the stability of weakly efficient solution sets for optimization problems with set-valued maps. We introduce the concept of essential weakly efficient solutions and essential components of weakly efficient solution sets. We first show that most optimization problems with set-valued maps (in the sense of Baire category) are stable. Secondly, we obtain some sufficient conditions for the existence of one essential weakly efficient solution or one essential component of the weakly efficient solution set .
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1 Introduction and Preliminaries

In this paper, our principle aim is to study the stability of the set of weakly efficient solutions for optimization problems with set-valued maps. We give an example that shows that the weakly efficient solution set for the optimization problem with set-valued maps is not stable. We then introduce the concept of an essential weakly efficient solution and essential component for optimization problems with set-valued maps. We first show that most optimization problems with set-valued maps (in the sense of Baire category) are stable. Secondly, we obtain some sufficient conditions for the existence of an essential weakly efficient solution or an essential component of the weakly efficient solution set for optimization with set-valued maps.

Let X be a nonempty subset in a topological vector space E, and let Y and Z be two ordered linear topological spaces with positive cones C and D, with $\text{int}C \neq \emptyset$ and $\text{int}D \neq \emptyset$, where $\text{int}C$ denotes the interior of the set C.

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In [6], the authors consider the following optimization problem:
(MP) $\min_{\text{int}C} F(x)$
s.t. $(-G(x)) \cap D \neq \emptyset$
where $F : X \to 2^Y$ and $G : X \to 2^Z$ are set-valued mappings.

We denote by V the feasible set of (MP), i.e.

$$V := \{x \in X : \quad (-G(x)) \cap D \neq \emptyset\}$$

Definition 1.1: A point $x_0 \in V$ is said to be a weakly efficient solution of (MP) F iff there exists $y_0 \in F(x_0)$, and for each $x \in V$, there exists no $y \in F(x)$ satisfying $y_0 \in y + \text{int}C$, i.e.

$$(y_0 - F(V)) \bigcap (\text{int}C) = \emptyset$$

where $F(V) = \cup_{x \in V} F(x)$.

Definition 1.2: Let T be a set-valued map from Hausdorff topological spaces E to H.

(1) T is upper semicontinuous (usc) (respectively, lower semicontinuous, lsc) at $x_0 \in E$ if for each open set O in H with $O \supset T(x_0)$ (respectively, $O \cap T(x_0) \neq \emptyset$), there exists an open neighborhood $N(x_0)$ of x_0 such that $O \supset T(x)$ (respectively, $O \cap T(x) \neq \emptyset$) for each $x \in N(x_0)$.

(2) T is usc (respectively, lsc) on E if T is usc (respectively, lsc) at every point $x \in E$.

(3) T is continuous at $x_0 \in E$ if it is both usc and lsc at x_0; T is continuous on E if it is continuous at every point $x \in E$.

Lemma 1.1:([1]) *Let X and Y be two Hausdorff topological spaces and X is compact, $T : X \to 2^Y$ be upper semicontinuous with compact values, then $T(X) = \cup_{x \in X} T(x)$ is compact.*

Lemma 1.2:([5]) *Let X and Y be two topological spaces with Y regular. If T is an usc set-valued map from X to Y with $T(x)$ closed for each $x \in X$, then T is closed, i.e., for x_α, $x_0 \in X$ with $x_\alpha \to x_0$, for any $y_\alpha \in T(x_\alpha)$ and $y_\alpha \to y_0$, then $y_0 \in T(x_0)$.*

Lemma 1.3: *Let X be a nonempty compact subset of a Hausdorff topological space E, and let (Z, D) be an ordered linear topological space with a closed positive cone D. If $G : X \to 2^Z$ is usc with compact values, then $V = \{x \in X : \ (-G(x)) \cap D \neq \emptyset\} \subset X$ is compact.*

Proof: Since $G : X \to 2^Z$ is usc with compact values, by Lemma 1.1, $G(X) = \cup_{x \in X}(G(x))$ is compact. Let a net $\{x_\alpha\} \subset V$ and $x_\alpha \to x^* \in X$. That is, $(-G(x_\alpha)) \cap D \neq \emptyset$, take $y_\alpha \in (-G(x_\alpha)) \cap D$, then $\{y_\alpha\} \subset (-G(X))$, since $(-G(X))$ is compact, $\{y_\alpha\}$ has a cluster point $y^* \in (-G(X))$. We may assume that $y_\alpha \to y^*$, by Lemma 1.2, $y^* \in (-G(x^*))$. Further, D is closed, $y^* \in D$, hence, $y^* \in (-G(x^*)) \cap D$, then $x^* \in V$, V is closed and hence V is compact. □

Lemma 1.4:([4]) *If X is (completely) metrizable, Z is a Baire space, and $T : Z \to K(X)$ is an usc with compact values mapping, then the subset of points where T is lsc is a (dense) residual set in Z.*

Lemma 1.5:([2]) *Let $\{A_\alpha\}_{\alpha \in \Gamma}$ be a net in $K(X)$ that converges to $A \in K(X)$ in the Vietoris topology (see [5]). Then any net $\{x_\alpha\}_{\alpha \in \Gamma}$ with $x_\alpha \in A_\alpha$ for each $\alpha \in \Gamma$ has a cluster point in A where $K(X)$ is the space of all non-empty compact subsets of X.*

2 Weakly Efficient Solution Mapping

We suppose that X is a nonempty compact subset of a Hausdorff topological space E, (Y,C), (Z,D) are two ordered Banach spaces with closed positive cones, and $\text{int}C \neq \emptyset$, $\text{int}D \neq \emptyset$. Then $G : X \to 2^Z$ is usc with nonempty compact values, and $V = \{x \in X : (-G(x)) \cap D \neq \emptyset\}$. Define
$M = \{F : X \to 2^Y : F$ continuous with compact values, $\exists x_0 \in V$, $\exists y_0 \in F(x_0)$ such that $(y_0 - F(V)) \cap \text{int}C = \emptyset\}$.

Then for any $F, F' \in M$, we define

$$\rho(F,F') = \sup_{x\in X} H(F(x), F'(x))$$

where H is the Hausdorff metric defined on Y. Clearly, (M,ρ) is a complete metric space.

For any $F \in M$, we denote by $S(F)$ the weakly efficient solution set of the (MP) F. That is, $S(F) = \{x \in X : \exists y \in F(x)$ such that $(y - F(V)) \cap \text{int}C = \emptyset\}$. Then we have $S(F) \neq \emptyset$, where S is a weakly efficient solution set-valued mapping from M to X. Then we have the following usc of the mapping S.

Theorem 2.1: *The mapping $S : M \to 2^X$ is usc with nonempty compact values.*

Proof: For any $F \in M$, we need to prove $S(F) \subset X$ is compact. Let a net $\{x_\alpha\} \subset S(F)$ and $x_\alpha \to x^* \in X$. By $x_\alpha \in S(F)$, there exists $y_\alpha \in F(x_\alpha) \subset F(V)$ such that

$$(y_\alpha - F(V)) \cap (\text{int}C) = \emptyset$$

By Lemma 1.3, V is compact, by Lemma 1.1, $F(V)$ is compact, then $\{y_\alpha\}$ has a cluster point $y^* \in F(V)$. We may assume that $y_\alpha \to y^*$, where by Lemma 1.2, $y^* \in F(x^*)$.

Suppose that $(y^* - F(V)) \cap \text{int}C \neq \emptyset$, then there exists $z_0 \in F(V)$ such that $y^* - z_0 \in \text{int}C$. But, $\text{int}C$ is an open set and $y_\alpha \to y^*$. Then there exists α_0 such that $y_\alpha - z_0 \in \text{int } C$ for any $\alpha \geq \alpha_0$ which contradicts the fact that $(y_\alpha - F(V)) \cap \text{int}C = \emptyset$. Hence $(y^* - F(V)) \cap \text{int } C = \emptyset$ and $x^* \in S(F)$, $S(F)$ is compact.

Because X is compact, to prove the usc of mapping S, we only need to show that

$$\text{Graph}S = \{(F,x) \in M \times X : x \in S(F)\}$$

$$= \{(F,x) \in M \times X : \exists y \in F(x) \text{such that} (y - F(V)) \cap \text{int}C = \emptyset\}$$

is closed.

Let a net $\{(F_\alpha, x_\alpha)\} \subset \text{Graph}S$ and $(F_\alpha, x_\alpha) \to (F^*, x^*) \in M \times X$. Then $\exists y_\alpha \in F_\alpha(x_\alpha)$ such that $(y_\alpha - F_\alpha(V)) \cap \text{int}C = \emptyset$.

Because $F_\alpha \to F^*$ and $x_\alpha \to x^*$, we have $F_\alpha(x_\alpha) \to F^*(x_\alpha) \to F^*(x^*)$. Further, because $F_\alpha(x_\alpha)$ is compact, by Lemma 1.5, $\{y_\alpha\}$ has a cluster point $y^* \in F^*(x^*)$ so we may assume $y_\alpha \to y^*$.

Suppose that $(y^* - F^*(V)) \cap \text{int}C \neq \emptyset$, then $\exists z_0 \in F^*(V)$ such that $y^* - z_0 \in \text{int}C$. Then $\exists q_0 \in V$ such that $z_0 \in F^*(q_0)$, since $F_\alpha(q_0) \to F^*(q_0)$. There exists $z_\alpha \in F_\alpha(q_0) \subset F_\alpha(V)$ such that $z_\alpha \to z_0$, then $y_\alpha - z_\alpha \to y^* - z_0$ and $y^* - z_0 \in \text{int}C$. There exists α_0 such that $\forall \alpha > \alpha_0$, $y_\alpha - z_\alpha \in \text{int}C$ which contradicts the fact that $(y_\alpha - F_\alpha(V)) \cap \text{int}C = \emptyset$. Hence $(y^* - F^*(V)) \cap \text{int}C = \emptyset$, $(F^*, x^*) \in \text{Graph}S$ and S is usc. Therefore, S is usc with nonempty compact values. □

Definition 2.1: For $F \in M$, the weakly efficient solution set $S(F)$ is called stable if the mapping S is continuous at F.

Remark 2.1: There exists $F \in M$ such that $S(F)$ is not stable.

Example 2.1: Let $X = [0,2]$, $Y = Z = (-\infty, +\infty)$, $C = D = [0, +\infty)$, $V = \{x \in X : G(x) \cap D \neq \emptyset\} = [0,2]$, and $F : X \to 2^Y$ as:

$$F(x) = \begin{cases} [0,4], & \text{if } x \in [0,1] \\ [x-1,\ 4-(x-1)], & \text{if } x \in (1,\ 2]. \end{cases}$$

Then $F(V) = [0,4]$. For any ε with $0 < \varepsilon < 1$, take

$$F^\varepsilon(x) = \begin{cases} [\frac{\varepsilon}{2}(1-x), 4-\frac{\varepsilon}{2}(1-x)], & \text{if } x \in [0,1] \\ [x-1,\ 4-(x-1)], & \text{if } x \in (1,\ 2]. \end{cases}$$

Then $F^\varepsilon(V) = [0,4]$ and $S(F^\varepsilon) = \{1\}$. Hence, $\rho(F, F^\varepsilon) = \frac{\varepsilon}{2} \to 0$ $(\varepsilon \to 0)$. But, $S(F^\varepsilon) \not\to S(F)$ $(\varepsilon \to 0)$. That is, S is not continuous at F, and hence, $S(F)$ is not stable.

3 Essential Weakly Efficient Solution

To study the stability of a weakly efficient solution set, we first introduce the following notions.

Definition 3.1: For $F \in M$,

(1) $x \in S(F)$ is said to be an essential weakly efficient solution of (MP) F relative to M if, for any open neighborhood $N(x)$ of x in X, there exists an open neighborhood $N(F)$ of F in M such that for any $F' \in N(F)$, $S(F') \cap N(x) \neq \emptyset$,

(2) F is essential relative to M if every $x \in S(F)$ is an essential weakly efficient solution relative to M.

Remark 3.1: If X is a complete metric space, Definition 3.1 shows that if the (MP) F is essential, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any other (MP) $F' \in M$ with $\rho(F, F') < \delta$, having at least one weakly efficient solution x' in the ε-neighborhood of the weakly efficient solution set of F. By Theorem 2.1, the set-valued mapping S is continuous at F when F is essential. Therefore, the essential property of F characterizes the continuous property of its weakly efficient solution mapping S at F.

By Theorem 2.1 and Definition 3.1, we have the following theorem:

Theorem 3.1:

(1) *S is lsc at $F \in M$ if and only if F is essential relative to M.*

(2) *S is continuous at $F \in M$ if and only if F is essential relative to M.*

Proof:

(1) From the fact that $F \in M$ is essential relative to M, for any open set O with $O \cap S(F) \neq \emptyset$, take $x_0 \in O \cap S(F)$. Then $x_0 \in O$, since O is an open set. There exists an open neighborhood $N(x_0)$ of x_0 in X such that $N(x_0) \subset O$, and $x_0 \in S(F)$ is an essential weakly efficient solution relative to M. By Definition 3.1, there exists an open neighborhood $N(F)$ of F in M such that $\forall F' \in N(F)$, $S(F') \cap N(x_0) \neq \emptyset$. Then $S(F') \cap O \neq \emptyset$ and S is lsc at F.

Conversely, suppose S is lsc at F. Then $\forall x \in S(F)$, for any open neighborhood $N(x)$ of x in X with $N(x) \cap S(F) \neq \emptyset$. there exists an open neighborhood $N(F)$ of F in M such that $\forall F' \in N(F)$, $N(x) \cap S(F') \neq \emptyset$. Hence x is an essential weakly efficient solution and F is essential relative to M.

(2) The result follows from Theorem 3.1 (1) and Theorem 2.1. □

By Lemma 1.4 , Theorem 2.1 and Theorem 3.1, it is easy to prove the following result (proof omitted).

Theorem 3.2: *Let X be a complete metric space , then there exists a dense residual subset $Q \subset M$ such that each $F \in Q$ is essential relative to M.*

If X is a complete metric space with metric d, then $K(X)$ is a complete metric space when equipped with the Hausdorff metric h induced by d. By Corollary 4.2.3 [5, **?**], the Vietoris topology on $K(X)$ coincides with the topology induced by the Hausdorff metric h. Then the mapping $S : M \to K(X)$ is continuous at $F \in M$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $h(S(F), S(F')) < \varepsilon$ whenever $F' \in M$ and $\rho(F, F') < \delta$. That is, the weakly efficient solution set $S(F)$ of F is stable: $S(F')$ is close to $S(F)$ whenever F' is close to F for all $F' \in M$. Theorem 3.1(2) implies that if $F \in M$, then F is essential relative to M if and only if the weakly efficient solution set $S(F)$ is stable.

The following Example 3.1 shows that there exists $F \in M$, $\forall x \in S(F)$, where x is not an essential weakly efficient solution relative to M, and hence $S(F)$ is not stable.

Example 3.1: Let $X = [0, 1]$, $Y = Z = (-\infty, +\infty)$, $C = D = [0, +\infty)$, $V = \{x \in X : \ G(x) \cap D \neq \emptyset\} = [0, 1]$, $F : X \to 2^Y$, $\forall x \in [0, 1]$, $F(x) = [0, 1]$. Then $F(V) = [0, 1]$, and $\forall x \in [0, 1]$, there exists $y_0 = 0 \in F(x)$ such that

$$(y_0 - F(V)) \cap \text{int}C = (0 - [0, 1]) \cap (0, +\infty) = \emptyset.$$

Then $S(F) = [0, 1]$.

For any $x_0 \in S(F)$, take $\delta > 0$ such that $N(x_0, \delta) \subset [0, 1]$ (if $x_0 = 1$, take $N(1, \delta) = (1 - \delta, 1] \subset [0, 1]$; if $x_0 = 0$, take $N(0, \delta) = [0, \delta) \subset [0, 1]$).

For any ε with $0 < \varepsilon < 1$, we define

$$F^\varepsilon(x) = \begin{cases} [0, 1], & \text{if } x \in [0, x_0 - \delta], \\ [\frac{\varepsilon}{\delta}(x - (x_0 - \delta)),\ 1], & \text{if } x \in (x_0 - \delta, x_0], \\ [-\frac{\varepsilon}{\delta}(x - (x_0 + \delta)),\ 1], & \text{if } x \in (x_0,\ x_0 + \delta), \\ [0, 1], & \text{if } x \in [x_0 + \delta, 1]. \end{cases}$$

Then $F^\varepsilon : X \to 2^Y$ is continuous and $F^\varepsilon(V) = [0, 1]$, $\rho(F, F^\varepsilon) < \varepsilon$.

But, $\forall x \in [0, 1] \setminus N(x_0, \delta)$, take $y_0 = 0 \in F^\varepsilon(x)$ such that $(0 - [0, 1]) \cap \text{int}C = \emptyset$
$\forall x \in N(x_0, \delta)$, $\forall y \in F^\varepsilon(x)$, $(y - [0, 1]) \cap \text{int}C \neq \emptyset$.

Then $S(F^\varepsilon) = [0, 1] \setminus N(x_0, \delta)$, $S(F^\varepsilon) \cap N(x_0, \delta) = \emptyset$, therefore $\forall x_0 \in S(F)$, x_0 is not an essential weakly efficient solution relative to M.

We have a sufficient condition that $F \in M$ is essential relative to M.

Theorem 3.3: *If $F \in M$ and $S(F)$ is a singleton set, then F is essential relative to M.*

Proof: Suppose that $S(F) = \{x\}$, let O be any open set in X such that $S(F) \cap O \neq \emptyset$, then $x \in O$, so that $S(F) \subset O$. Since S is usc at $F \in M$, there exists an open neighborhood $N(F)$ of F in M such that $S(F') \subset O$ for each $F' \in N(F)$ Then $S(F') \cap O \neq \emptyset$ for each $F' \in N(F)$. Thus, S is lsc at F, by Theorem 3.1 and F is essential relative to M. □

4 Essential Component of Weakly Efficient Solution Set

In Example 3.1, the $S(F)$ is not stable, but, $S(F) = [0,1]$ is connected.

For $F \in M$, the component of a point $x \in S(F)$ is the union of all connected subsets of $S(F)$ which contain the point x, (see pp.356 in [3]). Components are connected closed subsets of $S(F)$ and are also compact as $S(F)$ is compact. It is easy to see that the components of two distinct points of $S(F)$ either coincide or are disjoint so that all components constitute a decomposition of $S(F)$ into connected pairwise disjoint compact subset. That is,

$$S(F) = \bigcup_{\alpha\in\Lambda} C_\alpha(F)$$

where Λ is an index set. For any $\alpha \in \Lambda$, $C_\alpha(F)$ is a nonempty connected compact set and for any α, $\beta \in \Lambda(\alpha \neq \beta)$, $C_\alpha(F) \cap C_\beta(F) = \emptyset$.

Definition 4.1: For $F \in M$, $C_\alpha(F)$ is called an essential component of $S(F)$ if for any open set O containing $C_\alpha(F)$, $\exists\delta > 0$ such that $\forall F' \in M$ with $\rho(F, F') < \delta$, $S(F') \cap O \neq \emptyset$.

Remark 4.1: For $F \in M$, if $x \in S(F)$ is an essential weakly efficient solution, then the component which contains the point x is an essential component.

Theorem 4.1: *For $F \in M$, if $S(F)$ is connected, then $S(F)$ is an essential component.*

Proof: For $F \in M$, $S(F)$ is connected, since S is usc at F. For any open set O containing $S(F)$, there exists $\delta > 0$ such that $S(F') \subset O$ for any $F' \in M$ with $\rho(F, F') < \delta$, then $S(F') \cap O \neq \emptyset$, hence $S(F)$ is an essential component. □

Example 4.1: Let $X = [0,3]$, $Y = Z = (-\infty, +\infty)$, $C = D = [0, +\infty)$, $V = \{x \in X : G(x) \cap D \neq \emptyset\} = [0,3]$, and $F : X \to 2^Y$ as

$$F(x) = \begin{cases} [0,3], & \text{if } x\in[0,1] \\ [x-1,\ 3-(x-1)], & \text{if } x\in(1,\ 1.5] \\ [-x+2,\ 3-(-x+2)], & \text{if } x \in (1.5,\ 2) \\ [0,\ 3], & \text{if } x \in [2,\ 3], \end{cases}$$

then $F \in M$, $F(V) = [0,3]$, and

$\forall x \in (1,2)$, $\forall y \in F(x)$, $(y - [0,3]) \cap \text{int}C \neq \emptyset$;

$\forall x \in [0,1] \cup [2,3]$, $\exists y_0 = 0 \in F(x) = [0,3]$, $(0 - [0,3]) \cap \text{int}C = \emptyset$.

Hence $S(F) = [0,1] \cup [2,3]$, $C_1(F) = [0,1]$, and $C_2(F) = [2,3]$.

For $C_1(F) = [0,1]$, take $\delta > 0$ such that $O([0,1],\delta) \cap X = [0,\ 1+\delta) \subset [0,\ 1.5)$, then $(O([0,1],\delta) \cap X) \cap [2,3] = [0,\ 1+\delta) \cap [2,3] = \emptyset$.

For any ε with $0 < \varepsilon < 3 - \delta$, we define

$$F^\varepsilon(x) = \begin{cases} [\varepsilon,\ 3], & \text{if } x\in[0,1] \\ [(1-\frac{\varepsilon}{\delta})x + \frac{\varepsilon}{\delta} - 1 + \varepsilon,\ 3-(x-1)], & \text{if } x\in(1,\ 1+\delta) \\ [x-1,\ 3-(x-1)], & \text{if } x \in [1+\delta,\ 1.5] \\ [-x+2,\ 3-(-x+2)], & \text{if } x \in (1.5,\ 2) \\ [0,\ 3], & \text{if } x \in [2,\ 3], \end{cases}$$

then $F^\varepsilon \in M$, $F^\varepsilon(V) = [0,3]$, $\rho(F, F^\varepsilon) < \varepsilon$, and

$\forall x \in [0,2)$, $\forall y \in F^\varepsilon(x)$, $(y - [0,3]) \cap \text{int}C \neq \emptyset$;

$\forall x \in [2,3],\ \ \exists y_0 = 0 \in F^{\varepsilon}(x) = [0,3],\ \ (0 - [0,3]) \cap \mathrm{int}C = \emptyset.$

Then $S(F^{\varepsilon}) = [2,3]$ and $S(F^{\varepsilon}) \cap [0,\ 1+\delta) = \emptyset$. Hence $C_1(F) = [0,1]$ is not an essential component of $S(F)$. Similarly, one may prove that $C_2(F) = [2,3]$ is not an essential component of $S(F)$. Hence $S(F)$ has no essential component.

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