ON SOLUTIONS OF GENERAL NONLINEAR STOCHASTIC INTEGRAL EQUATIONS

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We study the existence, uniqueness, and stability of random solutions of a general class of nonlinear stochastic integral equations by using the Banach fixed point theorem. The results obtained in this paper generalize the results of Szynal and Wędrychowicz (1993).

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1. Introduction

Stochastic or random integral equations are extremely important in the study of many physical phenomena in life sciences and engineering [3, 14, 16]. There are currently two basic versions of stochastic integral equations being studied by probabilists and mathematical statisticians, namely, those integral equations involving Ito-Doob type of stochastic integrals and those which can be formed as probabilistic analogues of classical deterministic integral equations whose formulation involves the usual Lebesgue integral. Equations of the later category have been studied extensively.

Several papers have appeared on the problem of existence of solutions of nonlinear stochastic integral equations, and the results are established by using various fixed point techniques [1, 6–11]. Further, asymptotic behavior and stability of solutions of stochastic integral equations are discussed in [2, 4, 5, 12, 13]. In this paper we will prove an existence and uniqueness theorem for a general class of nonlinear stochastic integral equations and to investigate the asymptotic behavior of their solutions. The results are based on a construction of the real Banach space of tempered functions, which contains the space $D([0,\infty))$ of real right continuous functions having left-hand limits. The results of this paper generalize the results of Szynal and Wędrychowicz [15].

2. Preliminaries

Let (R,B,ν) be a measurable space with the Lebesgue measure ν on (R,B), where B denotes the Borel σ-field of subsets of R. Let $L_p(R,B,\nu)$, $1 \le p < \infty$, denote the set of all

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ν-measurable functions $x : R \to R$, such that the functions $|x(\cdot)|^p$ are ν-measurable. The norm of $x \in L_p(R,B,\nu)$ is defined by

$$\|x\|_{L_p} = \left(\int_R |x(t)|^p d\nu(t) \right)^{1/p}. \tag{2.1}$$

Let $L([0,\infty)) = L_\infty([0,\infty),B,\nu)$ be the space of ν-essentially bounded functions on $[0,\infty)$. Assume that $p(\cdot) \in L([0,\infty))$ is a fixed positive function. The triplet (Ω,A,P) denotes a complete probability space. By $\mathcal{L}_1^p(R_+,L_2(\Omega,A,P),p)$ (or shortly $\mathcal{L}_1^p$) we mean a space of all functions $x(t,\cdot)$ in R_+ which are integrable with respect to Lebesgue measure ν, with values $X(t)$ being random variables in $L_2(\Omega,A,P)$, and the topology is generated by

$$\|x\|_p = \int_0^\infty p(t)\nu - \operatorname*{ess\,sup}_{s\in[0,t]} \|x(s)\|_{L_2} d\nu(t), \tag{2.2}$$

where $\nu - \operatorname{ess\,sup}_{s\in[0,t]} \|x(s)\|_{L_2}$ is taken with respect to the Lebesgue measure ν. It is proved that the space $\mathcal{L}_1^p$ with the norm $\|\cdot\|_p$ is a Banach space [2, 15].

Consider the following nonlinear stochastic integral equations:

$$\begin{aligned} X(t;w) = h(t,X(t;w)) &+ \sum_{i=1}^{M} \int_0^t f_i(t,s,X(s;w);w)\,ds \\ &+ \sum_{j=1}^{N} \int_0^t g_j(t,s,X(s;w);w)\,d\beta(s;w), \quad t \geq 0, \end{aligned} \tag{2.3}$$

where

(i) $w \in \Omega$, and Ω is the supporting set of a complete probability measure space (Ω,A,P) with A being σ-algebra and P the probability measure;

(ii) $X(t;w)$ is the unknown random process;

(iii) $h(t,x)$ is a map from $R_+ \times R$ into R;

(iv) $f_i(t,s,X;w)$, $i = 1,\ldots,M$, and $g_j(t,s,X;w)$, $j = 1,\ldots,N$, are maps from $R_+ \times R_+ \times R \times \Omega$ into R;

(v) $\beta(t;w)$, where $t \in R$, is a martingale process.

The first part of the stochastic integral (2.3) is to be understood as an ordinary Lebesgue integral with probabilistic characterization, while the second part is an Ito-Doob stochastic integral. With respect to the random process $\beta(t;w)$ we will assume that, for each $t \in R_+$, a minimal σ-algebra $A_t \subset A$ is defined such that $\beta(t;w)$ is measurable with respect to A_t. Further we assume that $\{A_t, t \in R_+\}$ is an increasing family such that

(a) the random process $\{\beta(t;w),\ A_t : t \in R_+\}$ is a real martingale;

(b) there is a real continuous nondecreasing function $F(t)$ such that for $s < t$ we have

$$E\{|\beta(t;w) - \beta(t;w)|^2\} = E\{|\beta(t;w) - \beta(t;w)|^2 \mid A_t\} = F(t) - F(s), \qquad P \text{ a.e.} \tag{2.4}$$

Definition 2.1. A process $X(t;w)$ is said to be a random solution if $\|X(t)\|_{L_2} \in L_1([0,\infty))$ and satisfies the stochastic integral (2.3).

Definition 2.2. A random solution $X(t;w)$ is said to be asymptotically stable in mean-square sense if

$$\lim_{T\to\infty}\int_T^\infty ||X(t)||_{L_2}\,d\nu(t)=0. \tag{2.5}$$

3. Main results

THEOREM 3.1. *Suppose that the functions h, f_i, $i=1,\dots,M$, and g_j, $j=1,\dots,N$, satisfy the following Lipschitz conditions for $X(t;w),Y(t;w)\in\mathcal{L}_1^p$:*

(i) $|h(t,X(t;w))-h(t,Y(t;w))|\le K|X(t;w)-Y(t;w)|$ *P a.s. for* $K>0$;

(ii) $|f_i(t,s,X(s;w);w)-f_i(t,s,Y(s;w);w)|\le a_i(t,s;w)|X(t;w)-Y(t;w)|$ *P a.s.,* $i=1,\dots,M$, *for nonnegative functions* $a_i(t,s;w)$ *belonging to* $L_\infty(\Omega,A,P)$ *with* $\|a_i(t,s)\|=P-\operatorname{ess\,sup}_{w\in\Omega}|a_i(t,s;w)|$, *and* $a_i(t,s;w)$, $i=1,\dots,M$, *are continuous for* $t\in R_+$;

(iii) $|g_j(t,s,X(s;w);w)-g_j(t,s,Y(s;w);w)|\le b_j(t,s;w)|X(t;w)-Y(t;w)|$ *P a.s.,* $j=1,\dots,N$, *for nonnegative functions* $b_j(t,s;w)$ *belonging to* $L_\infty(\Omega,A,P)$, *and* $b_i(t,s;w)$, $i=1,\dots,N$, *are continuous for* $t\in R_+$;

(iv) *let* $Q=K+\sup_{t\in[0,\infty)}\sum_{i=1}^M\int_0^t\|a_i(t,s)\|ds+\sup_{t\in[0,\infty)}(\sum_{j=1}^N\int_0^t\|b_j(t,s)\|ds)^{1/2}$ *be such that* $0<Q<1$.

Then there exists a unique solution $X\in\mathcal{L}_1^p$ *to (2.3).*

Proof. For processes $X,Y\in\mathcal{L}_1^p$, define the process $GX-GY$ by

$$\begin{aligned}GX(t;w)-GY(t;w)&=h(t,X(t;w))-h(t,y(t;w))\\&\quad+\sum_{i=1}^M\int_0^t[f_i(t,s,X(s;w);w)-f_i(t,s,Y(s;w);w)]ds\\&\quad+\sum_{j=1}^N\int_0^t[g_j(t,s,X(s;w);w)-g_j(t,s,Y(s;w);w)]d\beta(s;w).\end{aligned} \tag{3.1}$$

By assumptions on $\beta(t;w)$ and for $X\in\mathcal{L}_1^p$, one can get the following estimate:

$$\sum_{j=1}^N\left\|\int_0^t b_j(t,s)X(s)d\beta(s)\right\|_{L_2}\le\left(\sum_{j=1}^N\int_0^t||b_j(t,s)||\,||X(s)||_{L_2}dF(s)\right)^{1/2}. \tag{3.2}$$

Let

$$K(t)=\sum_{j=1}^N\int_0^t[g_j(t,s,X(s;w);w)-g_j(t,s,Y(s;w);w)]d\beta(s;w). \tag{3.3}$$

Now, from hypothesis (iii) and (3.2), we have

$$\begin{aligned}
&\int_0^\infty p(t)\nu-\operatorname*{ess\,sup}_{s\in[0,t]}\|K(s)\|_{L_2}d\nu(t)\\
&\quad=\int_0^\infty p(t)\nu-\operatorname*{ess\,sup}_{s\in[0,t]}\bigg\|\sum_{j=1}^N\int_0^s|g_j(s,\tau;X(\tau;w);w)\\
&\qquad\qquad -g_j(s,\tau;X(\tau;w);w)\,|\,d\beta(\tau;w)\bigg\|_{L_2}d\nu(t)\\
&\quad\le\int_0^\infty p(t)\nu-\operatorname*{ess\,sup}_{s\in[0,t]}\bigg\|\sum_{j=1}^N\int_0^s|\,b_j(s,\tau;w)\,|\,X(\tau;w)-Y(\tau;w)\,|\,d\beta(\tau;w)\bigg\|_{L_2}d\nu(t)\\
&\quad\le\int_0^\infty p(t)\nu-\operatorname*{ess\,sup}_{s\in[0,t]}\bigg(\sum_{j=1}^N\int_0^s\|b_j(s,\tau)\|^2\|X(\tau)-Y(\tau)\|_{L_2}^2dF(s)\bigg)^{1/2}d\nu(t)\\
&\quad\le\int_0^\infty p(t)\nu-\operatorname*{ess\,sup}_{s\in[0,t]}\nu-\operatorname*{ess\,sup}_{\tau\in[0,s]}\|X(\tau)-Y(\tau)\|_{L_2}\bigg(\sum_{j=1}^N\int_0^s\|b_j(s,\tau)\|^2dF(s)\bigg)^{1/2}d\nu(t)\\
&\quad\le\int_0^\infty\sup_{t\in[0,\infty)}\bigg(\sum_{j=1}^N\int_0^t\|b_j(t,s)\|^2dF(s)\bigg)^{1/2}p(t)\nu-\operatorname*{ess\,sup}_{s\in[0,t]}\|X(s)-Y(s)\|_{L_2}d\nu(t)\\
&\quad\le\sup_{t\in[0,\infty)}\bigg(\sum_{j=1}^N\int_0^t\|b_j(t,s)\|^2dF(s)\bigg)^{1/2}\int_0^\infty p(t)\nu-\operatorname*{ess\,sup}_{s\in[0,t]}\|X(s)-Y(s)\|_{L_2}d\nu(t).
\end{aligned}\tag{3.4}$$

Let

$$L(t)=\sum_{i=1}^M\int_0^t[f_i(t,s,X(s;w);w)-f_i(t,s,Y(s;w);w)]ds.\tag{3.5}$$

Then, by hypothesis (ii), we have

$$\begin{aligned}
&\int_0^\infty p(t)\nu-\operatorname*{ess\,sup}_{s\in[0,t]}\|L(s)\|_{L_2}d\nu(t)\\
&\quad\le\int_0^\infty p(t)\nu-\operatorname*{ess\,sup}_{s\in[0,t]}\sum_{i=1}^M\int_0^s\|a_i(s,\tau)\|\,\|X(\tau)-Y(\tau)\|_{L_2}d\tau\,d\nu(t)\\
&\quad\le\sup_{t\in[0,\infty)}\sum_{i=1}^M\int_0^t\|a_i(t,s)\|ds\int_0^\infty p(t)\nu-\operatorname*{ess\,sup}_{s\in[0,t]}\|X(s)-Y(s)\|_{L_2}d\nu(t).
\end{aligned}\tag{3.6}$$

Combining the above three inequalities and (i), we get

$$\begin{aligned}
\|GX-GY\|_p &= \int_0^\infty p(t)\nu-\operatorname*{ess\,sup}_{s\in[0,t]}\|(GX)(s)-(GY)(s)\|_{L_2}d\nu(t)\\
&\le K\int_0^\infty p(t)\nu-\operatorname*{ess\,sup}_{s\in[0,t]}\|X(s)-Y(s)\|_{L_2}d\nu(t)\\
&\quad+\int_0^\infty p(t)\nu-\operatorname*{ess\,sup}_{s\in[0,t]}\|L(s)\|_{L_2}d\nu(t)\\
&\quad+\int_0^\infty p(t)\nu-\operatorname*{ess\,sup}_{s\in[0,t]}\|K(s)\|_{L_2}d\nu(t)\le Q\|X-Y\|_p,
\end{aligned}\tag{3.7}$$

which proves that G is a continuous function. Further, G is a contraction mapping, since $Q<1$, and therefore by the Banach fixed point theorem, there exists a unique X such that $GX=X$, which is the solution of (2.3). □

Remark 3.2. Let $h(t,X(t;w))\in D([0,\infty))$. By Theorem 3.1 the solution $X(t;w)$ to (2.3) belongs to $D([0,\infty))$, satisfying

$$\lim_{T\to\infty}\int_T^\infty p(t)\nu-\operatorname*{ess\,sup}_{s\in[0,t]}\|X(s)\|_{L_2}d\nu(t)=0.\tag{3.8}$$

Remark 3.3. If $p(t)=1$ for $t\in R_+$, then the random solution of (2.3) is asymptotically stable in the sense of Definition 2.2.

Next we consider the stochastic integral equation of the form

$$X(t;w)=h(t,X(t;w))+\sum_{j=1}^N\int_0^t g_j(t-s,X(t-s;w);w)e(s;w)ds,\quad t\ge 0,\tag{3.9}$$

which is equivalent to the following equation:

$$X(t;w)=h(t,X(t;w))+\sum_{j=1}^N\int_0^t g_j(s,X(s;w);w)e(t-s;w)ds,\tag{3.10}$$

where $e(t-s;w)\in L_\infty(\Omega,A,P)$.

Theorem 3.4. *Suppose that for $X(t;w),Y(t;w)\in\mathscr{L}_1^p$,*
(i) *$|h(t,X(t;w))-h(t,Y(t;w))|\le K|X(t;w)-Y(t;w)|$ P a.s. for $K>0$;*
(ii) *$|g_j(s,X(s;w);w)-g_j(s,Y(s;w);w)|\le b_j(s;w)|X(s;w)-Y(s;w)|$ P a.s. $j=1,\dots,N$, where $b_j(s;w)\in L_\infty(\Omega,A,P)$;*
(iii) *let $M=K+\sup_{t\in[0,\infty)}\sum_{j=1}^N\int_0^t\|b_j(s)\|\|e(t-s)\|ds$ be such that $0<M<1$.*
Then there exists a unique solution $X\in\mathscr{L}_1^p$ to (3.9) such that

$$\lim_{T\to\infty}\int_T^\infty p(t)\nu-\operatorname*{ess\,sup}_{s\in[0,t]}\|X(s)\|_{L_2}d\nu(t)=0.\tag{3.11}$$

Remark 3.5. If $p(t) = 1$ for $t \in R_+$, then the random solution to (3.9) is asymptotically stable in the sense that

$$\limsup_{t\to\infty} \frac{||x(t)||_{L_2}}{u(t)} \leq K, \quad K > 0, \tag{3.12}$$

whenever $\int_0^\infty u(t)d\nu(t) < \infty$. Hence we conclude that exponential stability is a particular case of this result.

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