SOLUTION OF A TRANSMISSION PROBLEM FOR SEMILINEAR PARABOLIC-HYPERBOLIC EQUATIONS BY THE TIME-DISCRETIZATION METHOD

ABDELFATAH BOUZIANI

Received 7 August 2004; Revised 7 May 2005; Accepted 24 May 2005

We consider a transmission problem for semilinear parabolic-hyperbolic equations. Existence, uniqueness, and continuous dependence of the solution upon the data are proved by using the time-discretization method. Besides, some convergence results of the approximations are established.

1. Introduction

In the present paper, we consider a transmission problem for semilinear parabolic-hyperbolic equations in a multidimensional structure. Precisely, let $\Omega = \Omega_1 \cup \Omega_2$ be an open bounded domain of $\mathbb{R}^n$ with sufficiently smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_2$, Ω_1 and Ω_2 having a common portion of boundary Σ. In the cylinder $Q = \Omega \times (0,T)$, let the unknown function $w = (w^1, w^2)$:

$$w(x,t) = \begin{cases} w^1(x,t), & (x,t) \in Q_1 = \Omega_1 \times (0,T), \\ w^2(x,t), & (x,t) \in Q_2 = \Omega_2 \times (0,T), \end{cases} \tag{1.1}$$

checking the following couple of partial differential equations:

$$\begin{aligned} \frac{\partial w^1}{\partial t} - \Delta w^1 &= f^1(x,t,w^1), \quad \text{in } Q_1, \\ \frac{\partial^2 w^2}{\partial t^2} - \Delta w^2 &= f^2(x,t,w^2), \quad \text{in } Q_2, \end{aligned} \tag{1.2}$$

subject to the initial conditions

$$\begin{gathered} w(x,0) = \left(w^1(x,0), w^2(x,0)\right) = \left(\varphi^1(x), \varphi^2(x)\right), \quad x \in \Omega, \\ \frac{\partial w^2(x,0)}{\partial t} = \psi(x), \quad x \in \Omega_2, \end{gathered} \tag{1.3}$$

Hindawi Publishing Corporation
Journal of Applied Mathematics and Stochastic Analysis
Volume 2006, Article ID 61439, Pages 1–23
DOI 10.1155/JAMSA/2006/61439

the boundary conditions

$$w = (w^1, w^2) = 0, \quad \text{on } \Gamma \times (0,T), \tag{1.4}$$

and the transmission conditions

$$\begin{aligned} w^1 &= w^2, \quad \text{on } \Sigma \times (0,T), \\ \frac{\partial w^1}{\partial \vartheta_1} + \frac{\partial w^2}{\partial \vartheta_2} &= 0, \quad \text{on } \Sigma \times (0,T), \end{aligned} \tag{1.5}$$

where $f^p \varphi^p$, $(p = 1,2)$, ψ are known functions.

The linear case of problem (1.2)–(1.5) has been studied by several authors. We refer the reader, for instance, to [1, 4, 8, 9] and references cited therein. In [8] the authors proved the existence and uniqueness of the solution for a similar problem, by using Oleĭnik's method [10]. However, the author of [9] has employed the variational method to investigate a boundary value problem where the geometrical domain consists of one 3-dimensional part Ω_1 and one 1-dimensional part Ω_2, with a point A common to the two boundaries. The model which is studied by the author consists in a parabolic (resp., hyperbolic) equation in Ω_1 (resp., Ω_2) with transmission conditions at A. The work in [4] (resp., [1]) shows the existence and the uniqueness of the weak (resp., strong) solution, of a transmission problem for parabolic-hyperbolic equations by using the energy-integral method. Lastly, one quotes [5, 6] in which the author has used the energy-integral method to investigate more general problems.

Differently to these works, in the present paper we prove the well-posedness of problem (1.2)–(1.5) via approximation by using the time-discretization method. So, the semilinear parabolic-hyperbolic problem is approximated by a recurrent system of elliptic problems, to be solved at each subsequent time point.

To this end, we introduce two unknown functions:

$$u^1(x,t) := w^1(x,t), \qquad u^2(x,t) := \frac{\partial w^2(x,t)}{\partial t}, \tag{1.6}$$

then

$$w^2(x,t) = \mathfrak{I}_t u^2 + \varphi^2(x), \tag{1.7}$$

where

$$\mathfrak{I}_t u^2 := \int_0^t u^2(\cdot,s)\,ds. \tag{1.8}$$

So, problem (1.2)–(1.5) is seen to be equivalent to the following problem: find the function $u = (u^1, u^2)$ verifying

$$\frac{\partial u^1}{\partial t} - \Delta u^1 = f^1(x,t,u^1), \quad \text{in } Q_1,$$
$$\frac{\partial u^2}{\partial t} - \Delta \Im_t u^2 = f^2(x,t,\Im_t u^2 + \varphi^2) + \Delta\varphi^2, \quad \text{in } Q_2, \tag{1.9}$$

$$u(x,0) = u_0(x) = (\varphi^1(x), \psi(x)), \quad x \in \Omega, \tag{1.10}$$

$$u = (u^1, u^2) = 0, \quad \text{on } \Gamma \times (0,T), \tag{1.11}$$

$$u^1 = u^2, \quad \text{on } \Sigma \times (0,T),$$
$$\frac{\partial u^1}{\partial \vartheta_1} + \frac{\partial \Im_t u^2}{\partial \vartheta_2} + \frac{\partial \varphi^2}{\partial \vartheta_2} = 0, \quad \text{on } \Sigma \times (0,T). \tag{1.12}$$

Hence, instead of looking for the function $w = (w^1, w^2)$, we search for the function $u = (u^1, u^2)$. The solution of problem (1.2)–(1.5) will be given by $w^1 = u^1$ and $w^2 = \Im_t u^2 + \varphi^2$.

In order to solve problem (1.9)–(1.12) by the time-discretization method, we divide the interval I into n subintervals by points $t_j = jh_n$, $j = 0,\dots,n$, where $h_n := T/n$ is a step time. Set, for $j = 1,\dots,n$,

$$u_j = (u_j^1, u_j^2), \quad \text{with } u_j^p := u^p(t_j), \quad \delta u_j^p = \frac{u_j^p - u_{j-1}^p}{h_n} \quad (p = 1,2),$$
$$f_j^1 := f^1(t_j, u_{j-1}^1), \qquad f_j^2 := f^2\left(t_j, h_n \sum_{i=1}^{j-1} u_{j-1}^2 + \varphi^2\right). \tag{1.13}$$

Then, we are conducted to solve successively, for $j = 1,\dots,n$, the following linearized problems.

Starting from

$$u_0 = (u_0^1, u_0^2) = (\varphi^1, \psi), \tag{1.14}$$

find functions $u_j : \Omega \to \mathbb{R}$ such that

$$\delta u_j^1 - \Delta u_j^1 = f_j^1, \quad \text{in } \Omega_1,$$
$$\delta u_j^2 - h_n \Delta \sum_{i=1}^{j} \Delta u_j^2 = f_j^2 + \Delta\varphi^2, \quad \text{in } \Omega_2,$$
$$u_j = (u_j^1, u_j^2) = 0, \quad \text{on } \Gamma, \tag{1.15}$$
$$u_j^1 = u_j^2, \quad \text{on } \Sigma,$$
$$\frac{\partial u_j^1}{\partial \vartheta_1} + h_n \sum_{i=1}^{j} \frac{\partial u_j^2}{\partial \vartheta_2} + \frac{\partial \varphi^2}{\partial \vartheta_2} = 0, \quad \text{on } \Sigma.$$

The plan of this paper is as follows. Section 2 is devoted to some preliminaries: notations, appropriate function spaces, and basic assumptions on the data are introduced in Section 2.1, while in Section 2.2 we give some auxiliary results needed throughout the paper. The variational formulation of the problem and the concept of the solution we are considering are given in Section 2.3. In Section 3, we establish some a priori estimates for the discretized problem. Then, we proceed with some convergence results and prove the existence of the solution in Section 4. Finally, we demonstrate the uniqueness and continuous dependence of the solution upon the data in Section 5.

2. Preliminaries

2.1. Notations and assumptions. First, we introduce some functions obtained from the approximates u_j^p $(p=1,2)$ by piecewise linear interpolation and piecewise constant with respect to the time, respectively:

$$u^{(n)} = \left(u^{1(n)}, u^{2(n)}\right), \qquad \overline{u}^{(n)} = \left(\overline{u}^{1(n)}, \overline{u}^{2(n)}\right), \tag{2.1}$$

with

$$u^{p(n)}(t) = u_{j-1}^p + \delta u_j^p (t - t_{j-1}) \quad (p=1,2),\ t \in [t_{j-1}, t_j],\ j = 1,\dots,n, \tag{2.2}$$

$$\overline{u}^{p(n)}(t) = \begin{cases} u_0^p, & \text{for } t \in [-h_n, 0], \\ u_j^p, & \text{for } t \in (t_{j-1}, t_j],\ (p=1,2),\ j=1,\dots,n, \end{cases} \tag{2.3}$$

$$U^2(t) := \mathfrak{I}_t u^2, \tag{2.4}$$

$$U^{2(n)}(t) := \mathfrak{I}_t \overline{u}^{2(n)}, \tag{2.5}$$

$$\overline{U}^{2(n)}(t) = \begin{cases} h_n u_0^2, & \text{for } t \in [-h_n, 0], \\ h_n \sum_{i=1}^{j} u_i^2, & \text{for } t \in (t_{j-1}, t_j],\ j = 1,\dots,n. \end{cases} \tag{2.6}$$

Moreover, we use the notation

$$\tau_{h_n} \overline{u}^{1(n)}(x,t) = \overline{u}^{1(n)}(x, t-h_n), \tag{2.7a}$$

$$\tau_{h_n} \overline{U}^{2(n)}(x,t) = \overline{U}^{2(n)}(x, t-h_n), \tag{2.7b}$$

then we write

$$\overline{f}^{p(n)}(t,w) := f^p(t_j, w) \quad (p=1,2), \tag{2.8}$$

thus, for $t \in (t_{j-1}, t_j]$, $j = 1, \dots, n$, we have

$$\overline{f}^{1(n)}(t, \tau_{h_n}\overline{u}^{1(n)}) := f^1(t_j, u^1_{j-1}) = f^1_j,$$
$$\overline{f}^{2(n)}(t, \tau_{h_n}\overline{U}^{2(n)} + \varphi^2) := f^2\left(t_j, h_n \sum_{i=1}^{j-1} u^2_{j-1} + \varphi^2\right) = f^2_j. \tag{2.9}$$

Let $H^1(\Omega_p)$ be the usual first-order Sobolev space on Ω_p with scalar product $(\cdot,\cdot)_{1,\Omega_p}$ and corresponding norm $\|\cdot\|_{1,\Omega_p}$, let $(\cdot,\cdot)_{0,\Omega_p}$ and $\|\cdot\|_{0,\Omega_p}$ be the scalar product and corresponding norm, respectively, in $L^2(\Omega_p)$, and let

$$H(\Delta, \Omega_p) := \{v^p \in H^1(\Omega_p), \Delta v^p \in L^2(\Omega_p),\ (p = 1,2)\} \tag{2.10}$$

endowed with the norm

$$||v^p||_{H(\Delta,\Omega_p)} := \left(||v^p||^2_{H^1(\Omega_p)} + ||\Delta v^p||^2_{0,\Omega_p}\right)^{1/2}. \tag{2.11}$$

We will also make frequent use of the seminorm

$$|\, v^p \,|_{1,\Omega_p} = ||\nabla v^p||_{0,\Omega_p} = \left(\sum_{i=1}^{n} \left\| \frac{\partial v^p}{\partial x_i} \right\|^2_{0,\Omega_p}\right)^{1/2}. \tag{2.12}$$

Since Ω_p is bounded, there exists a constant $C(\Omega_p)$ such that

$$\forall v^p \in V^p, \quad ||v^p||_{0,\Omega_p} \leqslant C(\Omega_p)\,|\, v^p \,|_{1,\Omega_p} \tag{2.13}$$

(known as the Poincaré-Friedrichs inequality). Therefore, the seminorm $|\cdot|_{1,\Omega_p}$ is a norm over the space V^p, equivalent to the norm $\|\cdot\|_{1,\Omega_p}$.

Let V be the space of functions defined by

$$V := \left\{v = (v^1, v^2)/v^p \in V^p\ (p = 1,2),\ v^1\,|_\Sigma = v^2\,|_\Sigma\right\}, \tag{2.14}$$

where

$$V^p := \{v^p \in H^1(\Omega_p)/v^p = 0 \text{ on } \Gamma_p\} \quad (p = 1,2). \tag{2.15}$$

The space V is equipped with the norm $\|\cdot\|_{1,\Omega_p}$, defined by

$$\|v\|^2_{1,\Omega} = \sum_{p=1}^{2} ||v^p||^2_{1,\Omega_p}. \tag{2.16}$$

We identify $v \in V$ with a function $v : \Omega \to \mathbb{R}$ for which $v|_{\Omega_p} = v^p$, $(p = 1,2)$. Similarly, we introduce the product space $\mathbb{L}^2(\Omega) = L^2(\Omega_1) \times L^2(\Omega_2)$ equipped with the scalar product and the associated norm

$$(u,v)_{0,\Omega} = \sum_{p=1}^{2} (u^p, v^p)_{0,\Omega_p}, \qquad \|u\|_{0,\Omega}^2 = \sum_{p=1}^{2} \|u^p\|_{0,\Omega_p}^2, \tag{2.17}$$

respectively.

Moreover, we will use the standard functional spaces $L^2(I,H)$, $C(I,H)$, $L^\infty(I,H)$, and $C^{0,1}(I,H)$, where H is a Hilbert space. For their properties, we refer the reader, for instance, to [7].

Next we formulate the assumptions which are supposed to hold throughout the paper.

Assumption 2.1. $f^p(t,u^p) : I \times L^2(\Omega_p) \to L^2(\Omega_p)$ is bounded in $L^2(\Omega_p)$ and fulfills the Lipschitz condition

$$\|f^p(t,u^p) - f^p(t',u^{p'})\|_{0,\Omega_p} \leqslant L\Big(|t-t'| + \|u^p - u^{p'}\|_{0,\Omega_p}\Big) \tag{2.18}$$

for all $t, t' \in I$, and $u^p, u^{p'} \in L^2(\Omega_p)$ $(p = 1,2)$.

Assumption 2.2. $\varphi^p \in H(\Delta, \Omega_p)$ $(p = 1,2)$; $\psi \in H(\Delta, \Omega_2)$.

Assumption 2.3. Compatibility conditions: $\varphi^1 = \psi$, and $\partial\varphi^1/\partial\vartheta_1 + h_n(\partial\psi/\partial\vartheta_2) + \partial\varphi^2/\partial\vartheta_2 = 0$, on Σ.

2.2. Auxiliary results. The following results are used in this paper. We list them for convenience.

Lemma 2.4 (an analogue of Gronwall's lemma in continuous form [2]). *Let $f_i(t)$ $(i = 1, 2)$ be real continuous functions on the interval $(0,T)$, $f_3(t) \geqslant 0$ nondecreasing function on t, and $C > 0$. If the inequality*

$$\int_0^t f_1(s)ds + f_2(t) \leqslant f_3(t) + C\int_0^t f_2(s)ds \tag{2.19}$$

fulfills, then the inequality

$$\int_0^t f_1(s)ds + f_2(t) \leqslant f_3(t)e^{Ct} \tag{2.20}$$

holds for all $t \in (0,T)$.

Lemma 2.5 (Gronwall's lemma in discrete form [11]). *Let $\{a_i\}$ be a sequence of real, nonnegative numbers, and A, C, and h_n are positive constants.*

(1) *If the inequality*

$$a_j \leqslant A + Ch_n \sum_{i=1}^{j-1} a_i \tag{2.21}$$

takes place for all $j = 1,\dots,n$, *then the estimate*

$$a_i \leqslant Ae^{C(j-1)h_n} \tag{2.22}$$

holds for all $j = 1,2,\dots,n$.

(2) *If the inequality*

$$a_j \leqslant A + Ch_n \sum_{i=1}^{j} a_i \tag{2.23}$$

fulfills and $h < 1/C$, *then*

$$a_i \leqslant \frac{A}{1-Ch_n} e^{C(j-1)h_n/(1-Ch_n)} \tag{2.24}$$

takes place for all $j = 1,2,\dots,n$.

LEMMA 2.6 [3]. *Let* V, Y *be reflexive Banach spaces and let the imbedding* $V \hookrightarrow Y$ *be compact. If the estimates*

$$\int_I \left\| \frac{du^{(n)}(t)}{dt} \right\|_Y^2 dt \leqslant C, \quad \| \overline{u}^{(n)}(t) \|_V \leqslant C, \quad \forall t \in I \tag{2.25}$$

hold for all $n \geqslant n_0 > 0$, *then there exist* $u \in C(I,Y) \cap L^\infty(I,V)$ *with* $du/dt \in L^2(I,Y)$ *(u is differentiable a.e.* $t \in I$*) and a subsequence* $\{u^{(n_k)}\}$ *of* $\{u^{(n)}\}$ *such that* $u^{(n_k)} \to u$ *in* $C(I,Y)$, $u^{(n_k)}(t) \rightharpoonup u(t)$, $\overline{u}^{(n_k)}(t) \rightharpoonup u(t)$ *in* V *for all* $t \in I$ *and* $du^{(n_k)}(t)/dt \rightharpoonup du(t)/dt$ *in* $L^2(I,Y)$. *Moreover, if* $\|du^{(n)}(t)/dt\|_Y \leqslant C$ *for all* $n \geqslant n_0 > 0$ *and a.e.* $t \in I$, *then* $du/dt \in L^\infty(I,Y)$ *and* $u : I \to Y$ *is Lipschitz continuous, that is,*

$$\| u(t) - u(t') \|_Y \leqslant C |t - t'|, \quad \forall t,t' \in I. \tag{2.26}$$

Moreover, the following identity will be frequently employed:

$$\left(u(t) - w(t), u(t)\right)_{0,\Omega} = \frac{1}{2}\left(\|u(t)\|_{0,\Omega}^2 + \|u(t) - w(t)\|_{0,\Omega}^2 - \|w(t)\|_{0,\Omega}^2 \right), \tag{2.27}$$

for all $t \in I$.

2.3. Variational formulation. Taking the scalar product in $\mathbb{L}^2(\Omega)$ of (1.9) and $v = (v^1, v^2)$, we have

$$\begin{aligned} &\left(\frac{\partial u(\cdot,t)}{\partial t}, v \right)_{0,\Omega} - (\Delta u^1(\cdot,t), v^1)_{0,\Omega_1} - (\Delta \Im_t u^2, v^2)_{0,\Omega_2} \\ &\quad = (f^1(\cdot,t,u^1), v^1)_{0,\Omega_1} + (f^2(\cdot,t,\Im_t u^2 + \varphi^2) + \Delta\varphi^2, v^2)_{0,\Omega_2}. \end{aligned} \tag{2.28}$$

Carrying out some integration by parts and using (1.12), (2.14), and (2.15), we get

$$\begin{aligned}&\left(\frac{\partial u(\cdot,t)}{\partial t},v\right)_{0,\Omega}+\left(\nabla u^1(\cdot,t),\nabla v^1\right)_{0,\Omega_1}+\left(\nabla\mathfrak{I}_t u^2,\nabla v^2\right)_{0,\Omega_2}\\&\quad=\left(f^1(\cdot,t,u^1),v^1\right)_{0,\Omega_1}+\left(f^2(\cdot,t,\mathfrak{I}_t u^2+\varphi^2),v^2\right)_{0,\Omega_2}-\left(\nabla\varphi^2,\nabla v^2\right)_{0,\Omega_2}.\end{aligned}\tag{2.29}$$

Then, we look for a weak solution in the following sense.

Definition 2.7. The weak solution of problem (1.9)–(1.12), intended to be a function $u: I\to\mathbb{L}^2(\Omega)$ verifying the following.

(i) $u\in C^{0,1}(I,\mathbb{L}^2(\Omega))\cap L^\infty(I,V)$ and $u^2\in L^2(I,V^2)$.
(ii) $\mathfrak{I}_t u^2\in C^{0,1}(I,L^2(\Omega_2))\cap L^\infty(I,V^2)$.
(iii) u has a strong derivative (a.e. in I) $du/dt\in L^\infty(I,\mathbb{L}^2(\Omega))$.
(iv) $u(0)=u_0=(u_0^1,u_0^2)=(\varphi^1,\psi)$ in $\mathbb{L}^2(\Omega)$.
(v) Identity (2.29) holds for all $v\in V$ and $t\in I$.

We remark that since $u\in C^{0,1}(I,\mathbb{L}^2(\Omega))\subset C^1(I,\mathbb{L}^2(\Omega))$, the condition (iv) makes sense, and according to (i), (ii), (iii) together with Assumption 2.1 each term in the integral identity (2.29) is well defined.

Then, for each $n\geqslant 1$, problem (1.9)–(1.12) may be approximated by the following time-discretized problems.

Starting from the initial conditions

$$u_0=(u_0^1,u_0^2)=(\varphi^1,\psi),\quad \text{in }\Omega,\tag{2.30}$$

find successively for $j=1,\dots,n$, functions $u_j:\Omega\to\mathbb{R}$, verifying the integral identity

$$\begin{aligned}&(\delta u_j,v)_{0,\Omega}+(\nabla u_j^1,\nabla v^1)_{0,\Omega_1}+h_n\left(\nabla\sum_{i=1}^{j}u_i^2,\nabla v^2\right)_{0,\Omega_2}\\&\quad=(f_j^1,v^1)_{0,\Omega_1}+(f_j^2,v^2)_{0,\Omega_2}-(\nabla\varphi^2,\nabla v^2)_{0,\Omega_2},\quad\forall v\in V,\end{aligned}\tag{2.31}$$

which may be rewritten as follows:

$$\begin{aligned}&\frac{1}{h_n}(u_j,v)_{0,\Omega}+(\nabla u_j^1,\nabla v^1)_{0,\Omega_1}+h_n(\nabla u_j^2,\nabla v^2)_{0,\Omega_2}\\&\quad=\frac{1}{h_n}(u_{j-1},v)_{0,\Omega}+(f_j^1,v^1)_{0,\Omega_1}+(f_j^2,v^2)_{0,\Omega_2}\\&\qquad-(\nabla\varphi^2,\nabla v^2)_{0,\Omega_2}-h_n\left(\nabla\sum_{i=1}^{j-1}u_i^2,\nabla v^2\right),\quad\forall v\in V,\end{aligned}\tag{2.32}$$

together with the boundary conditions

$$u_j=(u_j^1,u_j^2)=0,\quad\text{on }\Gamma,\tag{2.33}$$

and the transmission conditions

$$u_j^1 = u_j^2, \quad \text{on } \Sigma,$$
$$\frac{\partial u_j^1}{\partial \vartheta_1} + h_n \sum_{i=1}^{j} \frac{\partial u_i^2}{\partial \vartheta_2} + \frac{\partial \varphi^2}{\partial \vartheta_2} = 0, \quad \text{on } \Sigma. \tag{2.34}$$

Our assumptions permit a successive application of the Lax-Milgram theorem to conclude that the set of linear elliptic transmission problems admits a unique solution $u_j = (u_j^1, u_j^2)$, for $j = 1,\dots,n$.

By using notation from Section 2.1, identity (2.31) may be written in the form

$$\begin{aligned}
&\left(\frac{du^{(n)}}{dt}(t), v\right)_{0,\Omega} + \left(\nabla \overline{u}^{1(n)}(t), \nabla v^1\right)_{0,\Omega_1} + \left(\nabla \overline{U}^{2(n)}(t), \nabla v^2\right)_{0,\Omega_2} \\
&\quad = \left(\overline{f}^{1(n)}\left(t, \tau_{h_n}\overline{u}^{1(n)}\right), v^1\right)_{0,\Omega_1} + \left(\overline{f}^{2(n)}\left(t, \tau_{h_n}\overline{U}^{2(n)} + \varphi^2\right), v^2\right)_{0,\Omega_2} \\
&\qquad - \left(\nabla \varphi^2, \nabla v^2\right)_{0,\Omega_2}, \quad \forall v \in V.
\end{aligned} \tag{2.35}$$

3. A priori estimates for the discretized problem

Now, we derive some a priori estimates.

Lemma 3.1. *Let Assumptions 2.1–2.3 be fulfilled. Then the following estimates:*

$$\|\delta u_j\|_{0,\Omega} \leqslant C_1, \tag{3.1}$$

$$\|u_j\|_{0,\Omega} \leqslant C_2, \tag{3.2}$$

$$|u_j^2|_{1,\Omega_2} \leqslant C_1, \tag{3.3}$$

hold for $j = 1,\dots,n$, where C_1 and C_2 are positive constants independent of h_n and j.

Proof. Taking the difference of identity (2.31) and the same identity written for $j-1$, and setting $v = \delta u_j = (\delta u_j^1, \delta u_j^2)$, we have

$$\begin{aligned}
&(\delta u_j - \delta u_{j-1}, \delta u_j)_{0,\Omega} + (\nabla u_j^1 - \nabla u_{j-1}^1, \nabla \delta u_j^1)_{0,\Omega_1} + h_n (\nabla u_j^2, \nabla \delta u_j^2)_{0,\Omega_2} \\
&\quad = (f_j^1 - f_{j-1}^1, \delta u_j^1)_{0,\Omega_1} + (f_j^2 - f_{j-1}^2, \delta u_j^2)_{0,\Omega_2},
\end{aligned} \tag{3.4}$$

whence, in view of identity (2.27) and some rearrangement, we obtain

$$\begin{aligned}
&\|\delta u_j\|_{0,\Omega}^2 + h_n^2 \|\delta^2 u_j\|_{0,\Omega}^2 + 2h_n |\delta u_j^1|_{1,\Omega_1}^2 + 2|u_j^2|_{1,\Omega_2}^2 \\
&\quad = 2(f_j^1 - f_{j-1}^1, \delta u_j^1)_{0,\Omega_1} + 2(f_j^2 - f_{j-1}^2, \delta u_j^2)_{0,\Omega_2} \\
&\qquad + \|\delta u_{j-1}\|_{0,\Omega}^2 + 2(\nabla u_j^2, \nabla u_{j-1}^2)_{0,\Omega_2}.
\end{aligned} \tag{3.5}$$

Owing to the Cauchy inequality, it comes

$$\begin{aligned}
&\|\delta u_j\|_{0,\Omega}^2 + h_n^2\|\delta^2 u_j\|_{0,\Omega}^2 + 2h_n|\delta u_j^1|_{1,\Omega_1}^2 + |u_j^2|_{1,\Omega_2}^2 \\
&\quad\leqslant \frac{1}{h_n}\Big(\|f_j^1 - f_{j-1}^1\|_{0,\Omega_1}^2 + \|f_j^2 - f_{j-1}^2\|_{0,\Omega_2}^2\Big) + h_n\|\delta u_j\|_{0,\Omega_2}^2 + \|\delta u_{j-1}\|_{0,\Omega}^2 + |u_{j-1}^2|_{1,\Omega_2}^2.
\end{aligned} \tag{3.6}$$

Neglecting the second and third terms in the left-hand side of the last inequality and summing up over $i = 2,\dots,j$, this yields

$$\begin{aligned}
&\|\delta u_j\|_{0,\Omega}^2 + |u_j^2|_{1,\Omega_2}^2 \\
&\quad\leqslant \|\delta u_1\|_{0,\Omega}^2 + |u_1^2|_{1,\Omega_2}^2 + \frac{1}{h_n}\sum_{i=2}^{j}\Big(\|f_i^1 - f_{i-1}^1\|_{0,\Omega_1}^2 + \|f_i^2 - f_{i-1}^2\|_{0,\Omega_2}^2\Big) + h_n\sum_{i=2}^{j}\|\delta u_j\|_{0,\Omega}^2.
\end{aligned} \tag{3.7}$$

According to Assumption 2.1, we get

$$\begin{aligned}
\|f_i^1 - f_{i-1}^1\|_{0,\Omega_1} &= \|f^1(t_i,u_{i-1}^1) - f^1(t_{i-1},u_{i-2}^1)\|_{0,\Omega_1} \\
&\leqslant L\Big(h_n + \|u_{i-1}^1 - u_{i-2}^1\|_{0,\Omega_1}\Big) = Lh_n\Big(1 + \|\delta u_{i-1}^1\|_{0,\Omega_1}\Big), \\
\|f_i^2 - f_{i-1}^2\|_{0,\Omega_2} &= \left\|f^2\left(t_i,h_n\sum_{j=1}^{i-1}u_j^2 + \varphi^2\right) - f^2\left(t_{i-1},h_n\sum_{j=1}^{i-2}u_j^2 + \varphi^2\right)\right\|_{0,\Omega_2} \\
&\leqslant Lh_n\Big(1 + \|u_{i-1}^2\|_{0,\Omega_2}\Big).
\end{aligned} \tag{3.8}$$

Consequently, it follows that

$$\begin{aligned}
&\sum_{i=2}^{j}\Big(\|f_i^1 - f_{i-1}^1\|_{0,\Omega_1}^2 + \|f_i^2 - f_{i-1}^2\|_{0,\Omega_2}^2\Big) \\
&\quad\leqslant 4L^2h_n^2(j-1) + 2L^2h_n^2\sum_{i=2}^{j}\|\delta u_{i-1}^1\|_{0,\Omega_1}^2 + 2L^2h_n^2\sum_{i=2}^{j}\|u_{i-1}^2\|_{0,\Omega_2}^2,
\end{aligned} \tag{3.9}$$

in light of which (3.7) becomes

$$\begin{aligned}
\|\delta u_j\|_{0,\Omega}^2 + |u_j^2|_{1,\Omega_2}^2 &\leqslant \|\delta u_1\|_{0,\Omega}^2 + |u_1^2|_{1,\Omega_2}^2 + 4L^2T \\
&\quad + h_n\sum_{i=2}^{j}\|\delta u_j\|_{0,\Omega}^2 + 2L^2h_n\sum_{i=1}^{j-1}\|\delta u_i^1\|_{0,\Omega_1}^2 + 2L^2h_n\sum_{i=1}^{j-1}\|u_i^2\|_{0,\Omega_2}^2.
\end{aligned} \tag{3.10}$$

This inequality can be rewritten in the following way:

$$\begin{aligned}||\delta u_j||_{0,\Omega}^2 + |u_j^2|_{1,\Omega_2}^2 \leqslant ||\delta u_1||_{0,\Omega}^2 + |u_1^2|_{1,\Omega_2}^2 + 4L^2T \\ + (2L^2+1)h_n \sum_{i=1}^{j} ||\delta u_i||_{0,\Omega}^2 + 2L^2 h_n \sum_{i=1}^{j-1} ||u_i^2||_{0,\Omega_2}^2.\end{aligned} \tag{3.11}$$

Thus, owing to inequality (2.13), we get

$$\begin{aligned}||\delta u_j||_{0,\Omega}^2 + |u_j^2|_{1,\Omega_2}^2 \leqslant ||\delta u_1||_{0,\Omega}^2 + |u_1^2|_{1,\Omega_2}^2 + 4L^2T \\ + \max(2L^2+1, 2L^2C_3)h_n \sum_{i=1}^{j} \left(||\delta u_i||_{0,\Omega}^2 + |u_i^2|_{1,\Omega_2}^2\right),\end{aligned} \tag{3.12}$$

where $C_3 = C(\Omega)$. To estimate the first two terms in the right-hand side of (3.12), we consider identity (2.31) for $j = 1$:

$$\begin{aligned}(\delta u_1, v)_{0,\Omega} + (\nabla u_1^1, \nabla v^1)_{0,\Omega_1} + h_n(\nabla u_1^2, \nabla v^2)_{0,\Omega_2} \\ = (f_1^1, v^1)_{0,\Omega_1} + (f_1^2, v^2)_{0,\Omega_2} - (\nabla \varphi^2, \nabla v^2)_{0,\Omega_2}, \quad \forall v \in V,\end{aligned} \tag{3.13}$$

hence

$$\begin{aligned}(\delta u_1, v)_{0,\Omega} + h_n(\nabla \delta u_1^1, \nabla v^1)_{0,\Omega_1} + h_n^2(\nabla \delta u_1^2, \nabla v^2)_{0,\Omega_2} \\ = (f_1^1, v^1)_{0,\Omega_1} + (f_1^2, v^2)_{0,\Omega_2} - (\nabla u_0^1, \nabla v^1)_{0,\Omega_1} \\ - h_n(\nabla u_0^2, \nabla v^2)_{0,\Omega_2} - (\nabla \varphi^2, \nabla v^2)_{0,\Omega_2}, \quad \forall v \in V,\end{aligned} \tag{3.14}$$

so

$$\begin{aligned}(\delta u_1, v)_{0,\Omega} + h_n(\nabla \delta u_1^1, \nabla v^1)_{0,\Omega_1} + h_n^2(\nabla \delta u_1^2, \nabla v^2)_{0,\Omega_2} \\ = (f_1^1, v^1)_{0,\Omega_1} + (f_1^2, v^2)_{0,\Omega_2} - (\nabla \varphi^1, \nabla v^1)_{0,\Omega_1} \\ - h_n(\nabla \psi, \nabla v^2)_{0,\Omega_2} - (\nabla \varphi^2, \nabla v^2)_{0,\Omega_2}, \quad \forall v \in V.\end{aligned} \tag{3.15}$$

However, observe that

$$\begin{aligned}&-(\nabla \varphi^1, \nabla v^1)_{0,\Omega_1} = -\int_{\Gamma_1 \cup \Sigma} \frac{\partial \varphi^1}{\partial \vartheta_1} v^1 d\sigma + (\Delta \varphi^1, v^1)_{0,\Omega_1}, \\ &-h_n(\nabla \psi, \nabla v^2)_{0,\Omega_2} - (\nabla \varphi^2, \nabla v^2)_{0,\Omega_2} \\ &\quad = -\int_{\Gamma_2 \cup \Sigma} \left(h_n \frac{\partial \psi}{\partial \vartheta_2} + \frac{\partial \varphi^2}{\partial \vartheta_2}\right) v^2 d\sigma + h_n(\Delta \psi, v^2)_{0,\Omega_2} + (\Delta \varphi^2, v^2)_{0,\Omega_2}.\end{aligned} \tag{3.16}$$

Therefore, in view of (2.14), (2.15), and Assumption 2.3, we have

$$\begin{aligned}-(\nabla \varphi^1, \nabla v^1)_{0,\Omega_1} - h_n(\nabla \psi, \nabla v^2)_{0,\Omega_2} - (\nabla \varphi^2, \nabla v^2)_{0,\Omega_2} \\ = (\Delta \varphi^1, v^1)_{0,\Omega_1} + h_n(\Delta \psi, v^2)_{0,\Omega_2} + (\Delta \varphi^2, v^2)_{0,\Omega_2}.\end{aligned} \tag{3.17}$$

Substituting (3.17) into (3.15) yields

$$\begin{aligned}&(\delta u_1, v)_{0,\Omega} + h_n(\nabla\delta u_1^1, \nabla v^1)_{0,\Omega_1} + h_n^2(\nabla\delta u_1^2, \nabla v^2)_{0,\Omega_2}\\&\quad = (f_1^1 + \Delta\varphi^1, v^1)_{0,\Omega_1} + (f_1^2 + h_n\Delta\psi + \Delta\varphi^2, v^2)_{0,\Omega_2}, \quad \forall v \in V.\end{aligned} \tag{3.18}$$

Testing the resulting identity with $v = \delta u_1 = (\delta u_1^1, \delta u_1^2)$, we obtain

$$\begin{aligned}&||\delta u_1||_{0,\Omega}^2 + h_n|\delta u_1^1|_{1,\Omega_1}^2 + h_n^2|\delta u_1^2|_{1,\Omega_2}^2\\&\quad\leqslant \Big(||f_1^1||_{0,\Omega_1} + ||\Delta\varphi^1||_{0,\Omega_1}\Big)||\delta u_1^1||_{0,\Omega_1}\\&\qquad + \Big(||f_1^2||_{0,\Omega_2} + h_n\|\Delta\psi\|_{0,\Omega_2} + ||\Delta\varphi^2||_{0,\Omega_2}\Big)||\delta u_1^2||_{0,\Omega_2}.\end{aligned} \tag{3.19}$$

It is easy to see that

$$\begin{aligned}||f_1^1||_{0,\Omega_1} &\leqslant ||f^1(t_1,\varphi^1) - f^1(t_1,0)||_{0,\Omega_1} + ||f^1(t_1,0)||_{0,\Omega_1} \leqslant L||\varphi^1||_{0,\Omega_1} + M,\\ ||f_1^2||_{0,\Omega_2} &\leqslant ||f^2(t_1,h_n\psi+\varphi^2) - f^2(t_1,0)||_{0,\Omega_2} + ||f^2(t_1,0)||_{0,\Omega_2}\\ &\leqslant Lh_n\|\psi\|_{0,\Omega_2} + L||\varphi^2||_{0,\Omega_2} + M,\end{aligned} \tag{3.20}$$

where $M = \max_{t\in I}\|f^p(t,0)\|_{0,\Omega p}$. Then, inserting (3.20) into (3.19), it follows that

$$\begin{aligned}&||\delta u_1||_{0,\Omega}^2 + h_n|\delta u_1^1|_{1,\Omega_1}^2 + h_n^2|\delta u_1^2|_{1,\Omega_2}^2\\&\quad\leqslant \Big(L||\varphi^1||_{0,\Omega_1} + ||\Delta\varphi^1||_{0,\Omega_1} + M\Big)||\delta u_1^1||_{0,\Omega_1}\\&\qquad + \Big(h_n(L\|\psi\|_{0,\Omega_2} + \|\Delta\psi\|_{0,\Omega_2}) + L||\varphi^2||_{0,\Omega_2} + ||\Delta\varphi^2||_{0,\Omega_2} + M\Big)||\delta u_1^2||_{0,\Omega_2},\end{aligned} \tag{3.21}$$

which implies

$$\begin{aligned}&\frac{1}{2}||\delta u_1||_{0,\Omega}^2 + h_n|\delta u_1^1|_{1,\Omega_1}^2 + h_n^2|\delta u_1^2|_{1,\Omega_2}^2\\&\quad\leqslant (\max 5(1,L^2,T^2,L^2T^2)/2)\Big(||\varphi^1||_{H(\Delta,\Omega_1)}^2 + ||\varphi^2||_{H(\Delta,\Omega_2)}^2 + \|\psi\|_{H(\Delta,\Omega_2)}^2\Big) + 4M^2.\end{aligned} \tag{3.22}$$

Hence, omitting the last two terms in the left-hand side of (3.22) yields

$$||\delta u_1||_{0,\Omega} \leqslant C_5, \tag{3.23}$$

where

$$C_5 = \sqrt{\max 5(1,L^2,T^2,L^2T^2)\Big(||\varphi^1||_{H(\Delta,\Omega_1)}^2 + ||\varphi^2||_{H(\Delta,\Omega_2)}^2 + \|\psi\|_{H(\Delta,\Omega_2)}^2\Big) + 8M^2}. \tag{3.24}$$

However, since

$$u_1 = h_n\delta u_1 + u_0, \tag{3.25}$$

then

$$||u_1||_{0,\Omega} \leqslant h_n||\delta u_1||_{0,\Omega} + ||u_0||_{0,\Omega}, \tag{3.26}$$

from which

$$||u_1||_{0,\Omega} \leqslant C_6, \tag{3.27}$$

owing to (3.23), where $C_6 = TC_5 + \|u_0\|_{0,\Omega}$. On the other hand, consider identity (3.18), tested with $v = (v^1, v^2) = (u_1^1, \delta u_1^2)$, which is an element of V,

$$\begin{aligned}(\delta u_1^1, u_1^1)_{0,\Omega_1} + ||\delta u_1^2||^2_{0,\Omega_2} + h_n(\nabla \delta u_1^1, \nabla u_1^1)_{0,\Omega_1} + h_n^2 |\delta u_1^2|^2_{1,\Omega_2} \\ = (f_1^1 + \Delta\varphi^1, u_1^1)_{0,\Omega_1} + (f_1^2 + h_n\Delta\psi + \Delta\varphi^2, \delta u_1^2)_{0,\Omega_2}.\end{aligned} \tag{3.28}$$

Applying identity (2.27) to the third term in the left-hand side, hence (3.28) becomes after some rearrangement

$$\begin{aligned}&||\delta u_1^2||^2_{0,\Omega_2} + \frac{1}{2}|u_1^1|^2_{1,\Omega_1} + |u_1^2|^2_{1,\Omega_2} \\ &\quad\leqslant \frac{1}{2}|\varphi^1|^2_{1,\Omega_1} + |\psi|^2_{1,\Omega_2} + (-\delta u_1^1 + f_1^1 + \Delta\varphi^1, u_1^1)_{0,\Omega_1} + (f_1^2 + h_n\Delta\psi + \Delta\varphi^2, \delta u_1^2)_{0,\Omega_2},\end{aligned} \tag{3.29}$$

from which it comes

$$||\delta u_1^2||^2_{0,\Omega_2} + |u_1|^2_{1,\Omega} \leqslant C_7, \tag{3.30}$$

where

$$C_7 = \max(3, 3L^2, 3L^2T, T)\left(||\varphi^1||^2_{H(\Delta,\Omega_1)} + ||\varphi^2||^2_{H(\Delta,\Omega_2)} + \|\psi\|^2_{H(\Delta,\Omega_2)}\right) + 5M^2 + 3C_6^2 + C_5^2. \tag{3.31}$$

Consequently,

$$|u_1^2|^2_{1,\Omega_2} \leqslant C_7. \tag{3.32}$$

Inserting (3.23) and (3.32) into (3.12) yields

$$||\delta u_j||^2_{0,\Omega} + |u_j^2|^2_{1,\Omega_2} \leqslant C_8 + \max(2L^2 + 1, 2L^2C_3)h_n \sum_{i=1}^{j-1}\left(||\delta u_i||^2_{0,\Omega} + |u_i^2|^2_{1,\Omega_2}\right), \tag{3.33}$$

with $C_8 := C_5^2 + C_7$.

Thanks to Lemma 2.5, we obtain

$$||\delta u_j||^2_{0,\Omega} + |u_j^2|^2_{1,\Omega_2} \leqslant C_8 \exp\left(\max(2L^2 + 1, 2L^2C_3)(j-1)h_n\right) \leqslant C_9, \quad \forall j = 1, 2, \dots, n, \tag{3.34}$$

where

$$C_9 = C_8 \exp\big(\max\big(2L^2+1, 2L^2C_3\big)T\big), \tag{3.35}$$

from which we get estimates (3.1) and (3.3), with $C_1 = C_9^{1/2}$. However, since

$$u_j = h_n \sum_{i=1}^{j} \delta u_j + u_0, \tag{3.36}$$

then

$$||u_j||_{0,\Omega} \leqslant h_n \sum_{i=1}^{j} ||\delta u_j||_{0,\Omega} + ||u_0||_{0,\Omega}, \tag{3.37}$$

whence

$$||u_j||_{0,\Omega} \leqslant j h_n C_1 + ||u_0||_{0,\Omega}, \tag{3.38}$$

from which we obtain estimate (3.2) with $C_2 = C_1 T + \|u_0\|_{0,\Omega}$. This achieves the proof of Lemma 3.1. □

As a consequence of Lemma 3.1, we have the following results.

Corollary 3.2. *The functions $u^{(n)}$ and $\overline{u}^{(n)}$ obey the estimates*

$$||u^{(n)}(t)||_{1,\Omega} \leqslant C_2, \quad \forall t \in I, \tag{3.39a}$$

$$||\overline{u}^{(n)}(t)||_{1,\Omega} \leqslant C_2, \quad \forall t \in I, \tag{3.39b}$$

$$\int_I ||\overline{u}^{2(n)}(t)||^2_{0,\Omega_2}\, dt \leqslant C_2^2 T, \quad \forall t \in I, \tag{3.40a}$$

$$|\overline{U}^{2(n)}(t)|_{1,\Omega_2} \leqslant C_1 T, \quad \forall t \in I, \tag{3.40b}$$

$$\left|\left|\frac{du^{(n)}(t)}{dt}\right|\right|_{0,\Omega} \leqslant C_1, \quad \textit{a.e. in } I, \tag{3.41}$$

$$\begin{gathered} ||\overline{u}^{p(n)}(t) - u^{p(n)}(t)||_{0,\Omega_p} \leqslant C_1 h_n, \quad \forall t \in I\ (p=1,2), \\ ||\overline{u}^{p(n)}(t) - \tau_{h_n}\overline{u}^{p(n)}(t)||_{0,\Omega_p} \leqslant C_1 h_n, \quad \forall t \in I\ (p=1,2), \end{gathered} \tag{3.42}$$

$$\begin{gathered} ||\overline{U}^{2(n)}(t) - U^{2(n)}(t)||_{0,\Omega_2} \leqslant C_2 h_n, \quad \forall t \in I, \\ ||\overline{U}^{2(n)}(t) - \tau_{h_n}\overline{U}^{2(n)}(t)||_{0,\Omega_2} \leqslant C_2 h_n, \quad \forall t \in I, \end{gathered} \tag{3.43}$$

where $C_{10} = C_1 T$, C_1 and C_2 are the same constants given in Lemma 3.1.

Proof. Estimates (3.39) are an immediate consequence of (3.1), with the same constant, while estimate (3.40a) follows directly from (3.39b). As for estimate (3.40b), we have, in light of (2.6)

$$\overline{U}^{2(n)} = h_n \sum_{i=1}^{j} u_i^2, \quad \forall t \in (t_{j-1}, t_j],\ j = 1, \ldots, n, \tag{3.44}$$

so, according to (3.3), we have

$$|\,\overline{U}^{2(n)}\,|_{1,\Omega_2} \leqslant h_n \sum_{i=1}^{j} |\,u_i^2\,|_{1,\Omega_2} \leqslant C_1 h_n j \leqslant C_{10}. \tag{3.45}$$

However, it follows from (2.2) that

$$\frac{du^{(n)}(t)}{dt} = \left(\frac{du^{1(n)}(t)}{dt}, \frac{du^{2(n)}(t)}{dt}\right) = (\delta u_j^1, \delta u_j^2) = \delta u_j, \tag{3.46}$$

for all, $t \in (t_{j-1}, t_j]$ $j = 1, \ldots, n$. Hence, thanks to (3.1), it is easy to get

$$\left\|\frac{du^{(n)}(t)}{dt}\right\|_{0,\Omega} \leqslant C_1, \quad \text{for a.e. } t \in I, \tag{3.47}$$

from which we have

$$\int_I \left\|\frac{du^{(n)}(t)}{dt}\right\|_{0,\Omega}^2 dt \leqslant C_{11}, \tag{3.48}$$

with $C_{11} = C_1^2 T$.

Next, observe that

$$\begin{aligned} \overline{u}^{p(n)}(t) - u^{p(n)}(t) &= (t_j - t)\delta u_j^p, \quad \forall t \in (t_{j-1}, t_j]\ (j = 1, \ldots, n), \\ \overline{u}^{p(n)}(t) - \tau_{h_n}\overline{u}^{p(n)}(t) &= u_j^p - u_{j-1}^p, \quad \forall t \in (t_{j-1}, t_j]\ (j = 1, \ldots, n), \end{aligned} \tag{3.49}$$

in view of (2.2)-(2.3), and (2.3), (2.7a), respectively. Consequently, we have

$$\begin{aligned} \|\overline{u}^{p(n)}(t) - u^{p(n)}(t)\|_{0,\Omega_p} &\leqslant C_1 h_n, \quad \forall t \in I, \\ \|\overline{u}^{p(n)}(t) - \tau_{h_n}\overline{u}^{p(n)}(t)\|_{0,\Omega_p} &\leqslant C_1 h_n, \quad \forall t \in I, \end{aligned} \tag{3.50}$$

in view of (3.1). On the other hand, it comes from (2.5), (2.6), and (2.3) for $p = 2$:

$$\begin{aligned} \overline{U}^{2(n)}(t) - U^{2(n)}(t) &= h_n \sum_{i=1}^{j} u_i^2 - \int_0^t \overline{u}^{2n}(s)\,ds \\ &= h_n \sum_{i=1}^{j} u_i^2 - \sum_{i=1}^{j-1} \int_{(i-1)h_n}^{ih_n} \overline{u}^{2(n)}(s)\,ds - \int_{(i-1)h_n}^{t} \overline{u}^{2(n)}(s)\,ds = (t_j - t)u_j^2, \quad \forall t \in (t_{j-1}, t_j], \end{aligned} \tag{3.51}$$

therefore

$$||\overline{U}^{2(n)}(t) - U^{2(n)}(t)||_{0,\Omega_2} \leqslant C_2 h_n, \quad \forall t \in I, \tag{3.52}$$

according to (3.2). Similarly, it follows from (2.6) and (2.7b), for $t \in (t_{j-1}, t_j]$,

$$\overline{U}^{2(n)}(t) - \tau_{h_n}\overline{U}^{2(n)}(t) = h_n \sum_{i=1}^{j} u_i^2 - h_n \sum_{i=1}^{j-1} u_i^2 = h_n u_j^2, \tag{3.53}$$

so

$$||\overline{U}^{2(n)}(t) - \tau_{h_n}\overline{U}^{2(n)}(t)||_{0,\Omega_2} \leqslant h_n ||u_j^2||_{0,\Omega_2} \leqslant C_2 h_n, \tag{3.54}$$

which completes the proof of Corollary 3.2. □

Moreover, we need two other estimates.

Lemma 3.3. *Under Assumptions 2.1, the following estimates take place:*

$$\begin{aligned} &\left| \left(\overline{f}^{1(n_k)}(t, \tau_{h_{n_k}} \overline{u}^{1(n_k)}), v^1\right)_{0,\Omega_1} + \left(\overline{f}^{2(n_k)}\left(t, \tau_{h_{n_k}} \overline{U}^{2(n_k)} + \varphi^2\right), v^2\right)_{0,\Omega_2} \right| \\ &\qquad \leqslant C_{12} \|v\|_{0,\Omega}, \quad \forall v \in V,\ \forall t \in I, \end{aligned} \tag{3.55}$$

$$\begin{aligned} &\left| \left(\nabla \overline{u}^{1(n_k)}(t), \nabla v^1\right)_{0,\Omega_1} + \left(\nabla \overline{U}^{2(n_k)}(t), \nabla v^2\right)_{0,\Omega_2} \right| \\ &\qquad \leqslant C_{13} \|v\|_{1,\Omega}, \quad \forall v \in V,\ \text{a.e. } t \in I. \end{aligned} \tag{3.56}$$

Proof. Observe that

$$\begin{aligned} &\left| \left(\overline{f}^{1(n_k)}(t, \tau_{h_{n_k}} \overline{u}^{1(n_k)}), v^1\right)_{0,\Omega_1} + \left(\overline{f}^{2(n_k)}\left(t, \tau_{h_{n_k}} \overline{U}^{2(n_k)} + \varphi^2\right), v^2\right)_{0,\Omega_2} \right| \\ &\quad \leqslant \left\| \overline{f}^{1(n_k)}(t, \tau_{h_{n_k}} \overline{u}^{1(n_k)}) \right\|_{0,\Omega_1} ||v^1||_{0,\Omega_1} + \left\| \overline{f}^{2(n_k)}\left(t, \tau_{h_{n_k}} \overline{U}^{2(n_k)} + \varphi^2\right) \right\|_{0,\Omega_2} ||v^2||_{0,\Omega_2} \\ &\quad \leqslant ||f_j^1||_{0,\Omega_1} ||v^1||_{0,\Omega_1} + ||f_j^2||_{0,\Omega_2} ||v^2||_{0,\Omega_2}, \end{aligned} \tag{3.57}$$

thanks to the Schwarz inequality and (2.9). By virtue of Assumption 2.1, we have

$$\begin{aligned} ||f_j^1||_{0,\Omega_1} &\leqslant ||f^1(t_j, u_{j-1}^1) - f^1(t_j, 0)||_{0,\Omega_1} + ||f^1(t_j, 0)||_{0,\Omega_1} \\ &\leqslant L||u_{j-1}^1||_{0,\Omega_1} + M, \\ ||f_j^2||_{0,\Omega_2} &\leqslant \left\| f^2\left(t_j, h_{n_k} \sum_{i=1}^{j-1} u_i^2 + \varphi^2\right) - f^2(t_j, 0) \right\|_{0,\Omega_2} + ||f^2(t_j, 0)||_{0,\Omega_2} \\ &\leqslant L h_{n_k} \sum_{i=1}^{j-1} ||u_i^2||_{0,\Omega_2} + L||\varphi^2||_{0,\Omega_2} + M, \end{aligned} \tag{3.58}$$

from where, according to estimate (3.2), we obtain

$$\begin{gathered}||f_j^1||_{0,\Omega_1} \leqslant LC_2 + M, \\ ||f_j^2||_{0,\Omega_2} \leqslant L\Big(TC_2 + ||\varphi^2||_{0,\Omega_2}\Big) + M.\end{gathered} \tag{3.59}$$

Hence, substituting (3.59) into (3.57), we obtain (3.55), where $C_{12} = \sqrt{2}\max(LC_2 + M, L(TC_2 + \|\varphi^2\|_{0,\Omega_2}) + M)$.

On the other hand, it follows from the integral identity (2.35) that

$$\begin{aligned}&\left|(\nabla\overline{u}^{1(n_k)}(t), \nabla v^1)_{0,\Omega_1} + \Big(\nabla\overline{U}^{2(n_k)}(t), \nabla v^2\Big)_{0,\Omega_2}\right| \\ &\quad = \Big(\overline{f}^{1(n_k)}(t, \tau_{h_{n_k}}\overline{u}^{1(n_k)}), v\Big)_{0,\Omega_1} + \Big(\overline{f}^{2(n_k)}\Big(t, \tau_{h_{n_k}}\overline{U}^{2(n_k)} + \varphi^2\Big), v^2\Big)_{0,\Omega_2} \\ &\qquad - \left(\frac{du^{(n_k)}}{dt}(t), v\right)_{0,\Omega} - (\nabla\varphi^2, \nabla v^2)_{0,\Omega_2}, \quad \forall v \in V, \text{ a.e. } t \in I,\end{aligned} \tag{3.60}$$

whence, owing to the Schwarz inequality and estimates (3.41) and (3.55), it comes that

$$\begin{aligned}&\left|(\nabla\overline{u}^{1(n_k)}(t), \nabla v^1)_{0,\Omega_1} + \Big(\nabla\overline{U}^{2(n_k)}(t), \nabla v^2\Big)_{0,\Omega_2}\right| \\ &\quad \leqslant (C_{12} + C_1)\|v\|_{0,\Omega} + |\varphi^2|_{1,\Omega_2}|v^2|_{1,\Omega_2}, \quad \forall v \in V, \text{ a.e. } t \in I,\end{aligned} \tag{3.61}$$

from which we obtain (3.56), where $C_{13} = C_{12} + C_1 + |\varphi^2|_{1,\Omega_2}$. □

4. Convergence and existence results

First, owing to (3.39b), (3.41), and the continuous imbedding $V \hookrightarrow \mathbb{L}^2(\Omega)$, we obtain, on the basis of Lemma 2.6, that there exist

$$u \in L^\infty(I, V) \cap C^{0,1}(I, \mathbb{L}^2(\Omega)), \tag{4.1}$$

with

$$\frac{du}{dt} \in L^\infty(I, \mathbb{L}^2(\Omega)) \quad (u \text{ is differentiable } \textit{a.e.} \text{ in } I), \tag{4.2}$$

and subsequences $\{u^{(n_k)}\}$ and $\{\overline{u}^{(n_k)}\}$ of $\{u^{(n)}\}$ and $\{\overline{u}^{(n)}\}$, respectively, such that

$$u^{(n_k)} \longrightarrow u, \quad \text{in } C(I, \mathbb{L}^2(\Omega)), \tag{4.3}$$

$$u^{(n_k)}(t), \quad \overline{u}^{(n_k)}(t) \rightharpoonup u(t), \quad \text{in } V \ \forall t \in I, \tag{4.4}$$

$$\frac{du^{(n_k)}}{dt} \rightharpoonup \frac{du}{dt}, \quad \text{in } L^2(I, \mathbb{L}^2(\Omega)). \tag{4.5}$$

Similarly, by virtue of (3.40), and the continuous imbedding $V^2 \hookrightarrow L^2(\Omega_2)$, we deduce that there exist

$$\mathfrak{I}_t u^2 \in L^\infty(I, V^2) \cap C^{0,1}(I, L^2(\Omega_2)), \tag{4.6}$$

with

$$u^2 \in L^\infty(I, L^2(\Omega_2)), \tag{4.7}$$

and subsequences $\{U^{2(n_k)}\}$ and $\{\overline{U}^{2(n_k)}\}$ of $\{U^{2(n)}\}$ and $\{\overline{U}^{2(n)}\}$, respectively, verifying

$$U^{2(n_k)} \longrightarrow U^2, \quad \text{in } C(I, L^2(\Omega_2)), \tag{4.8}$$

$$U^{2(n_k)}(t), \quad \overline{U}^{2(n_k)}(t) \rightharpoonup U^2(t), \quad \text{in } V^2 \ \forall t \in I, \tag{4.9}$$

$$\overline{u}^{2(n_k)} \rightharpoonup u^2, \quad \text{in } L^2(I, V^2). \tag{4.10}$$

Therefore, the first three points of Definition 2.7 are already verified. Besides, since by definition $u^{(n_k)}(0) = (u^{1(n_k)}(0), u^{2(n_k)}(0)) = (u_0^1, u_0^2) = u_0$, it then follows from (4.3) that $u(0) = u_0$ holds in $\mathbb{L}^2(\Omega)$, thus point (iv) of Definition 2.7 is fulfilled. As for point (v), we integrate identity (2.35) written for n_k, over $(0,t) \subset I$ by taking into account that if $u^{(n_k)}(0) = u_0$, we get

$$\begin{aligned}
&(u^{(n_k)}(t) - u_0, v)_{0,\Omega} + \int_0^t \Big((\nabla \overline{u}^{1(n_k)}(s), \nabla v^1)_{0,\Omega_1} + \Big(\nabla \overline{U}^{2(n_k)}(s), \nabla v^2\Big)_{0,\Omega_2}\Big) ds \\
&\quad = \int_0^t \Big(\Big(\overline{f}^{1(n_k)}(s, \tau_{h_{n_k}} \overline{u}^{1(n_k)}), v^1\Big)_{0,\Omega_1} + \Big(\overline{f}^{2(n_k)}\Big(s, \tau_{h_{n_k}} \overline{U}^{2(n_k)} + \varphi^2\Big), v^2\Big)_{0,\Omega_2}\Big) ds \\
&\quad\quad - t(\nabla \varphi^2, \nabla v^2)_{0,\Omega_2}, \quad \forall t \in I, \ \forall v \in V.
\end{aligned} \tag{4.11}$$

In order to investigate the behavior of (4.11) as n_k tends to infinity, we prove some convergence statements. First, due to (4.3), we have

$$(u^{(n_k)}(t), v)_{0,\Omega} \xrightarrow[n_k \to \infty]{} (u(t), v)_{0,\Omega}, \quad \forall t \in I. \tag{4.12}$$

Furthermore, in view of (4.4) and (4.9), we deduce that

$$\begin{aligned}
&(\nabla \overline{u}^{1(n_k)}(t), \nabla v^1)_{0,\Omega_1} + \Big(\nabla \overline{U}^{2(n_k)}(t), \nabla v^2\Big)_{0,\Omega_2} \\
&\quad \xrightarrow[n_k \to \infty]{} (\nabla u^1(t), \nabla v^1)_{0,\Omega_1} + (\nabla U^2(t), \nabla v^2)_{0,\Omega_2},
\end{aligned} \tag{4.13}$$

from which together with estimate (3.56), we may apply the Lebesgue theorem of dominated convergence to obtain

$$\begin{aligned}
&\int_0^t \Big((\nabla \overline{u}^{1(n_k)}(s), \nabla v^1)_{0,\Omega_1} + \Big(\nabla \overline{U}^{2(n_k)}(s), \nabla v^2\Big)_{0,\Omega_2}\Big) ds \\
&\quad \xrightarrow[n_k \to \infty]{} \int_0^t \Big((\nabla u^1(s), \nabla v^1)_{0,\Omega_1} + (\nabla U^2(s), \nabla v^2)_{0,\Omega_2}\Big) ds.
\end{aligned} \tag{4.14}$$

On the other hand, by virtue of (2.9), Assumption 2.1, and estimates (3.42) at $p = 1$, it follows, for all $t \in (t_{j-1}, t_j]$ $(j = 1, \dots, n)$, that

$$\begin{aligned} &\left\| \overline{f}^{1(n_k)}(t, \tau_{h_{n_k}} \overline{u}^{1(n_k)}) - f^1(t, u^1(t)) \right\|_{0,\Omega_1} \\ &\quad = \| f^1(t_j, \tau_{h_{n_k}} \overline{u}^{1(n_k)}) - f^1(t, u^1(t)) \|_{0,\Omega_1} \\ &\quad \leqslant L\Big(|t_j - t| + \| \tau_{h_{n_k}} \overline{u}^{1(n_k)} - u^1(t) \|_{0,\Omega_1} \Big) \\ &\quad \leqslant L\Big(h_{n_k} + \| \tau_{h_{n_k}} \overline{u}^{1(n_k)} - \overline{u}^{1(n_k)}(t) \|_{0,\Omega_1} + \| \overline{u}^{1(n_k)}(t) - u^{1(n_k)}(t) \|_{0,\Omega_1} \\ &\qquad\quad + \| u^{1(n_k)}(t) - u^1(t) \|_{0,\Omega_1} \Big) \\ &\quad \leqslant L\Big((1 + 2C_1) h_{n_k} + \| u^{1(n_k)} - u^1 \|_{C(I, L^2(\Omega_1))} \Big). \end{aligned} \tag{4.15}$$

Hence, passing to the limit when n_k tends to infinity by taking into account (4.3), we get

$$\left\| \overline{f}^{1(n_k)}(t, \tau_{h_{n_k}} \overline{u}^{1(n_k)}) - f^1(t, u^1(t)) \right\|_{0,\Omega_1} \xrightarrow[n_k \to \infty]{} 0, \quad \forall t \in I. \tag{4.16}$$

Similarly, we have

$$\begin{aligned} &\left\| \overline{f}^{2(n_k)}\left(s, \tau_{h_{n_k}} \overline{U}^{2(n_k)} + \varphi^2\right) - f^2(s, U^2(t) + \varphi^2) \right\|_{0,\Omega_2} \\ &\quad \leqslant L\bigg(h_{n_k} + \left\| \tau_{h_{n_k}} \overline{U}^{2(n_k)} - \overline{U}^{2(n_k)}(t) \right\|_{0,\Omega_2} + \left\| \overline{U}^{2(n_k)}(t) - U^{2(n_k)}(t) \right\|_{0,\Omega_2} \\ &\qquad\quad + \| U^{2(n_k)}(t) - U^2(t) \|_{0,\Omega_2} \bigg) \\ &\quad \leqslant L\Big((1 + 2C_2) h_{n_k} + \| U^{2(n_k)}(t) - U^2(t) \|_{C(I, L^2(\Omega_2))} \Big), \end{aligned} \tag{4.17}$$

in view of Assumption 2.1 and estimates (3.43). Then, owing to the continuous imbedding $V^2 \hookrightarrow L^2(\Omega_2)$ and limit relation (4.8), we obtain

$$\left\| \overline{f}^{2(n_k)}\left(t, \tau_{h_{n_k}} \overline{U}^{2(n_k)} + \varphi^2\right) - f^2(t, U^2(t) + \varphi^2) \right\|_{0,\Omega_2} \xrightarrow[n_k \to \infty]{} 0, \quad \forall t \in I. \tag{4.18}$$

Therefore, by virtue of estimates (3.55), and limit relations (4.16) and (4.18), the Lebesgue theorem of dominated convergence implies that

$$\begin{aligned}&\int_0^t \Big(\big(\overline{f}^{1(n_k)}(s,\tau_{h_{n_k}}\overline{u}^{1(n_k)}),v^1\big)_{0,\Omega_1}+\big(\overline{f}^{2(n_k)}\big(s,\tau_{h_{n_k}}\overline{U}^{2(n_k)}+\varphi^2\big),v^2\big)_{0,\Omega_2}\Big)ds\\&\quad\xrightarrow[n_k\to\infty]{}\int_0^t\Big(\big(f^1(s,u^1(s)),v^1\big)_{0,\Omega_1}+\big(f^2(s,\Im_s u^2+\varphi^2),v^2\big)_{0,\Omega_2}\Big)ds,\end{aligned}\tag{4.19}$$

for all $v\in V$ and all $t\in I$.

Now we are ready to pass to the limit as $n_k\to\infty$ in (4.11). Observing (4.12), (4.14), and (4.19), then (4.11) passes into the following integral identity:

$$\begin{aligned}&(u(t),v)_{0,\Omega}+\int_0^t(\nabla u^1(s),\nabla v^1)_{0,\Omega_1}ds+\int_0^t(\nabla U^2(s),\nabla v^2)_{0,\Omega_2}ds\\&\quad=\int_0^t\Big(\big(f^1(s,u^1(s)),v^1\big)_{0,\Omega_1}+\big(f^2(s,\Im_s u^2+\varphi^2),v^2\big)_{0,\Omega_2}\Big)ds\\&\qquad+(u_0,v)_{0,\Omega}-t(\nabla\varphi^2,\nabla v^2)_{0,\Omega_2},\end{aligned}\tag{4.20}$$

for all $t\in I$, for all $v\in V$. Since $u:I\to V$ is strongly differentiable for a.e. $t\in I$, then the differentiation of the above identity with respect to t leads to the desired identity (2.35) thanks to the equality

$$\frac{d}{dt}(u(t),v)_{0,\Omega}=\left(\frac{d}{dt}u(t),v\right)_{0,\Omega},\quad \text{a.e. } t\in I,\ \forall v\in V.\tag{4.21}$$

Thus, we have proved the following theorem.

Theorem 4.1. *Let Assumptions 2.1–2.3 be fulfilled. Then, problem (1.9)–(1.12) admits at most a solution in the sense of Definition 2.7, verifying* $u\in L^\infty(I,V)\cap C^{0,1}(I,\mathbb{L}^2(\Omega))$ *with* $u^2\in L^2(I,V^2)$, $du/dt\in L^\infty(I,\mathbb{L}^2(\Omega))$, *and* $\Im_t u^2\in L^\infty(I,V^2)\cap C^{0,1}(I,L^2(\Omega_2))$.

A subsequence $\{u^{(n_k)}\}$ [$\{\overline{u}^{(n_k)}\}$, *resp.) of* $\{u^{(n)}\}$ *and* ($\{\overline{u}^{(n)}\}$, *resp.) converges to the solution* u *in the following sense:*

(i) $u^{(n_k)}\to u$ *in* $C(I,\mathbb{L}^2(\Omega))$;

(ii) $u^{(n_k)}(t)\rightharpoonup u(t)$ *in* V *for all* $t\in I$;

(iii) $\overline{u}^{(n_k)}(t)\rightharpoonup u(t)$ *in* V *for all* $t\in I$;

(iv) $U^{2(n_k)}\to\Im_t u^2$ *in* $C(I,L^2(\Omega_2))$;

(v) $U^{2(n_k)}(t)$, $\overline{U}^{2(n_k)}(t)\rightharpoonup\Im_t u^2$ *in* V^2 *for all* $t\in I$;

(vi) $\overline{u}^{2(n_k)}\rightharpoonup u^2$ *in* $L^2(I,L^2(\Omega_2))$;

(vii) $du^{(n_k)}/dt\rightharpoonup du/dt$ *in* $L^2(I,\mathbb{L}^2(\Omega))$,

as n_k *tends to infinity.*

5. Continuous dependence and uniqueness

We, first, prove the continuous dependence of the solution upon the data. Then, the uniqueness is an immediate corollary of it. To this end, we subtract the integral identity (2.29) written for u^{**} from (2.29) written for u^*, and setting $v=u^*(t)-u^{**}(t)$ in the obtained identity, we find for *a.e.* $t \in I$,

$$\begin{aligned}
&\left(\frac{d}{dt}(u^*(t) - u^{**}(t)), u^*(t) - u^{**}(t)\right)_{0,\Omega} \\
&\quad + (\nabla u^{1*}(t) - \nabla u^{1**}(t), \nabla u^{1*}(t) - \nabla u^{1**}(t))_{0,\Omega_1} \\
&\quad + (\nabla(\Im_t u^{2*} + \varphi^{2*}) - \nabla(\Im_t u^{2**} + \varphi^{2**}), \nabla u^{2*}(t) - \nabla u^{2**}(t))_{0,\Omega_2} \\
&= (f^{1*}(t, u^{1*}(t)) - f^{1**}(t, u^{1**}(t)), u^{1*}(t) - u^{1**}(t))_{0,\Omega_1} \\
&\quad + (f^{2*}(t, \Im_t u^{2*} + \varphi^{2*}) - f^{2**}(t, \Im_t u^{2**} + \varphi^{2**}), u^{2*}(t) - u^{2**}(t))_{0,\Omega_2},
\end{aligned} \tag{5.1}$$

whence

$$\begin{aligned}
&2|u^{1*}(t) - u^{1**}(t)|^2_{1,\Omega_1} + \frac{d}{dt}||u^*(t) - u^{**}(t)||^2_{0,\Omega} + \frac{d}{dt}|(\Im_t u^{2*} + \varphi^{2*}) - (\Im_t u^{2**} + \varphi^{2**})|^2_{1,\Omega_2} \\
&\quad \leqslant 2||f^{1*}(t, u^{1*}(t)) - f^{1**}(t, u^{1**}(t))||_{0,\Omega_1} ||u^{1*}(t) - u^{1**}(t)||_{0,\Omega_1} \\
&\qquad + 2||f^{2*}(t, \Im_t u^{*2} + \varphi^{*2}) - f^{2**}(t, \Im_s u^{**2} + \varphi^{**2})||_{0,\Omega_2} \times ||u^{2*}(t) - u^{2**}(t)||_{0,\Omega_2}.
\end{aligned} \tag{5.2}$$

Then integrating (5.2) over $(0,t)$ and applying the Cauchy inequality, it follows that

$$\begin{aligned}
&2\int_0^t |u^{1*}(s) - u^{1**}(s)|^2_{1,\Omega_1} ds + ||u^*(t) - u^{**}(t)||^2_{0,\Omega} + |(\Im_t u^{2*} + \varphi^{2*}) - (\Im_t u^{2**} + \varphi^{2**})|^2_{1,\Omega_2} \\
&\quad \leqslant \int_0^t ||f^{1*}(s, u^{1*}(s)) - f^{1**}(s, u^{1**}(s))||^2_{0,\Omega_1} ds \\
&\qquad + \int_0^t ||f^{2*}(s, \Im_s u^{*2} + \varphi^{*2}) - f^{2**}(s, \Im_s u^{**2} + \varphi^{**2})||^2_{0,\Omega_2} ds \\
&\qquad + \int_0^t ||u^*(s) - u^{**}(s)||^2_{0,\Omega} ds.
\end{aligned} \tag{5.3}$$

According to Lemma 2.4, (5.3) becomes

$$\begin{aligned}
&||u^*(t) - u^{**}(t)||^2_{0,\Omega} + 2\int_0^t |u^{1*}(s) - u^{1**}(s)|^2_{1,\Omega_1} ds + |(\Im_t u^{2*} + \varphi^{2*}) - (\Im_t u^{2**} + \varphi^{2**})|^2_{1,\Omega_2} \\
&\quad \leqslant e^T \int_0^t \Big(||f^{1*}(s, u^{1*}(s)) - f^{1**}(s, u^{1**}(s))||^2_{0,\Omega_1} \\
&\qquad\qquad + ||f^{2*}(s, \Im_s u^{*2} + \varphi^{*2}) - f^{2**}(s, \Im_s u^{**2} + \varphi^{**2})||^2_{0,\Omega_2}\Big) ds.
\end{aligned} \tag{5.4}$$

Consequently, we can state the following theorem.

THEOREM 5.1. *Assume that (f^*, u^{0*}) and (f^{**}, u^{0**}) satisfy Assumptions 2.1–2.3. Let $u^*(x,t)$ and $u^{**}(x,t)$ be two solutions of problem (1.9)–(1.12) corresponding, respectively,* *to the above data. Then the estimate*

$$\begin{aligned} &\|u^*(t)-u^{**}(t)\|_{0,\Omega}^2 \\ &\quad\leqslant C\bigg(\int_0^t \Big(\|f^{1*}(s,u^{1*}(s))-f^{1**}(s,u^1(s))\|_{0,\Omega_1}+\|f^{2*}(s,\Im_s u^{2*}+\varphi^{2*}) \\ &\qquad\qquad -f^{2**}(s,\Im_s u^{2**}+\varphi^{2**})\|_{0,\Omega_2}\Big)ds+\|u^{0*}-u^{0**}\|_{0,\Omega}^2\bigg) \end{aligned} \tag{5.5}$$

takes place for all $t\in I$, where C is a positive constant depending only on the known data.

This theorem leads to the following result.

Corollary 5.2. *Under assumptions of Theorem 5.1, the weak solution of problem (1.9)–(1.12) is unique.*

Proof. Assume that $\hat{u}$ and $\tilde{u}$ are two weak solutions of (1.9)–(1.12). Taking the difference of the integral identities (2.29) corresponding to $\hat{u}$ and $\tilde{u}$ tested with $v=\hat{u}-\tilde{u}$ and performing a similar calculation to that for Theorem 5.1, we obtain

$$\begin{aligned} &2\int_0^t |\hat{u}^1(s)-\tilde{u}^1(s)|_{1,\Omega_1}^2 ds+\|\hat{u}(t)-\tilde{u}(t)\|_{0,\Omega}^2+|\Im_t\hat{u}^2-\Im_t\tilde{u}^2|_{1,\Omega_2}^2 \\ &\quad\leqslant \int_0^t \|f^1(s,\hat{u}^1(s))-f^1(s,\tilde{u}^1(s))\|_{0,\Omega_1}^2 ds \\ &\qquad+\int_0^t \|f^2(s,\Im_s\hat{u}^2+\varphi^2)-f^2(s,\Im_s\tilde{u}^2+\varphi^2)\|_{0,\Omega_2}^2 ds \\ &\qquad+\int_0^t \|\hat{u}(s)-\tilde{u}(s)\|_{0,\Omega}^2 ds. \end{aligned} \tag{5.6}$$

Employing Assumption 2.1 and omitting the first term in the left-hand side of (5.6), we find

$$\begin{aligned} &\|\hat{u}(t)-\tilde{u}(t)\|_{0,\Omega}^2+|\Im_t\hat{u}^2-\Im_t\tilde{u}^2|_{1,\Omega_2}^2 \\ &\quad\leqslant L^2\int_0^t \|\hat{u}^1(s)-\tilde{u}^1(s)\|_{0,\Omega_1}^2 ds+\int_0^t \|\Im_s\hat{u}^2-\Im_s\tilde{u}^2\|_{0,\Omega_2}^2 ds+\int_0^t \|\hat{u}(s)-\tilde{u}(s)\|_{0,\Omega}^2 ds, \end{aligned} \tag{5.7}$$

from which, by using inequality (2.13), we get

$$\begin{aligned} &\|\hat{u}(t)-\tilde{u}(t)\|_{0,\Omega}^2+|\Im_t\hat{u}^2-\Im_t\tilde{u}^2|_{1,\Omega_2}^2 \\ &\quad\leqslant \max\left(L^2+1,C(\Omega)\right)\bigg(\int_0^t \|\hat{u}(s)-\tilde{u}(s)\|_{0,\Omega}^2 ds+\int_0^t |\Im_s\hat{u}^2-\Im_s\tilde{u}^2|_{1,\Omega_2}^2 ds\bigg). \end{aligned} \tag{5.8}$$

Thanks to Lemma 2.4, we deduce that $\hat{u}=\tilde{u}$, which completes the proof. □

Remark 5.3. By standard arguments we see that the uniqueness of the solution implies that all our convergence results for subsequences (4.3)–(4.10) take place for the whole sequences corresponding to a subdivision with arbitrary $h_n\to 0$, too.

Remark 5.4. Our results can be extended to the following differential equations:

$$\begin{aligned} &\frac{\partial w^1}{\partial t} + A^{(1)}w^1 + S^{(1)}w^1 = f^1(x,t,w^1), \quad \text{in } Q_1, \\ &\frac{\partial^2 w^2}{\partial t^2} + A^{(2)}w^2 + S^{(2)}w^2 = f^2(x,t,w^2), \quad \text{in } Q_2, \end{aligned} \tag{5.9}$$

with corresponding boundary and transmission conditions [5, 6], where $A^{(p)} = \sum_{|\alpha|\leqslant 2m} a_\alpha^{(p)}(x)D^\alpha$ is a positive elliptic operator in $\overline{\Omega_p}$ for each $t \in I$, $S^{(p)} = \partial^p/\partial t^p + \varrho(x)\partial^{p-1}/\partial t^{p-1} + \sum_{|\mu|\leqslant m} \sigma_\mu^{(p)}(x)D^\mu$ $(p = 1,2)$. Here $D = (D_1,\dots,D_n)$, $D_k = -i\partial/\partial x_k$, $\alpha = (\alpha_1,\dots,\alpha_n)$, $\mu = (\mu_1,\dots,\mu_n)$, $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$, and $|\mu| = |\mu_1| + \cdots + |\mu_n|$.

References

[1] A. Bouziani, *Solution forte d'un problème de transmission parabolique-hyperbolique pour une structure pluridimensionnelle* [*Strong solution of a parabolic-hyperbolic jump problem for a multidimensional structure*], Académie Royale de Belgique. Bulletin de la Classe des Sciences. 6e Série **7** (1996), no. 7–12, 369–386 (1997) (French).

[2] L. Gårding, *Cauchy's Problem For Hyperbolic Equations*, Lecture Notes, University of Chicago, Chicago, 1957.

[3] J. Kačur, *Method of Rothe in Evolution Equations*, Teubner-Texte zur Mathematik, vol. 80, BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1985.

[4] V. I. Korzjuk, *A conjugacy problem for equations of hyperbolic and parabolic types*, Differentsial' nye Uravneniya **4** (1968), no. 10, 1854–1866 (Russian).

[5] V. I. Korzjuk, *A mixed problem for certain nonstationary equations with discontinuous coefficients*, Differentsial' nye Uravneniya **6** (1970), no. 2, 343–357 (Russian).

[6] ______, *The problem of the conjugacy of certain nonstationary differential equations of order* $2m$, Differentsial' nye Uravneniya **7** (1971), no. 4, 750–753 (Russian).

[7] A. Kufner, O. John, and S. Fučík, *Function Spaces*, Monographs and Textbooks on Mechanics of Solids and Fluids; Mechanics: Analysis, Noordhoff, Leyden; Academia, Prague, 1977.

[8] O. A. Ladyzhenskaya and L. I. Stupyalis, *Problème de conjugaison de l'équation ondulatoire et de l'équation de conductibilité thermique*, Differentsial' nye Uravneniya **1** (1965), no. 4, 38–46 (French).

[9] J.-L. Lions, *Un exemple de problème aux limites couplé parabolique-hyperbolique pour une structure pluri-dimensionnelle* [*An example of a coupled parabolic-hyperbolic boundary value problem for a pluridimensional structure*], Calcolo **22** (1985), no. 1, 7–15 (French).

[10] O. A. Oleĭnik, *Boundary-value problems for linear equations of elliptic parabolic type with discontinuous coefficients*, Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya **25** (1961), no. 1, 3–20 (Russian).

[11] K. Rektorys, *The Method of Discretization in Time and Partial Differential Equations*, Mathematics and Its Applications (East European Series), vol. 4, D. Reidel, Dordrecht, 1982.

Abdelfatah Bouziani: Department of Mathematics, The Larbi Ben M'hidi University Centre, P.O. Box 565, Oum El-Bougui 04000, Algeria

E-mail addresses: af_bouziani@hotmail.com ; aefbouziani@yahoo.fr