Hindawi Publishing Corporation
Journal of Applied Mathematics and Stochastic Analysis
Volume 2007, Article ID 59204, 7 pages
doi:10.1155/2007/59204

Research Article

Common Fixed Points of Mappings and Set-Valued Mappings in Symmetric Spaces with Application to Probabilistic Spaces

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Received 15 June 2007; Accepted 24 October 2007

The main purpose of this paper is to give some common fixed point theorems of mappings and set-valued mappings of a symmetric space with some applications to probabilistic spaces. In order to get these results, we define the concept of E-weak compatibility between set-valued and single-valued mappings of a symmetric space.

1. Preliminaries

In this section, we recall some basic definitions from the theory of symmetric spaces. A symmetric function on a set X is a nonnegative real-valued function d on $X \times X$ such that

(1) $d(x,y) = 0$ if and only if $x = y$,

(2) $d(x,y) = d(y,x)$.

Let d be symmetric on a set X, and for $r > 0$ and any $x \in X$, let $B(x,r) = \{y \in X : d(x,y) < r\}$. A topology $t(d)$ on X is given by $U \in t(d)$ if and only if for each $x \in U$, $B(x,r) \subset U$ for some $r > 0$. A symmetric d is semimetric if for each $x \in X$ and each $r > 0$, $B(x,r)$ is a neighborhood of x in the topology $t(d)$. Note that $\lim_{n\to\infty} d(x_n,x) = 0$ if and only if $x_n \to x$ in the topology $t(d)$.

A sequence in X is said to be d-Cauchy sequence if it satisfies the usual metric condition. There are several concepts of completeness in this setting (see [1]).

(i) X is S-complete if for every d-Cauchy sequence (x_n), there exists x in X with $\lim_{n\to\infty} d(x,x_n) = 0$.

(ii) X is d-Cauchy complete if for every d-Cauchy sequence $\{x_n\}$, there exists x in X with $x_n \to x$ in the topology $t(d)$.

In order to unify the notation, we need the following two axioms (W.3) and (W.4) given by Wilson [2] in a symmetric space (X,d):

(W.3) given (x_n), x and y in X, $\lim_{n\to\infty} d(x_n,x) = 0$, and $\lim_{n\to\infty} d(x_n,y) = 0$ imply that $x = y$,

(W.4) given (x_n), (y_n) and x in X, $\lim_{n\to\infty} d(x_n,x) = 0$, and $\lim_{n\to\infty} d(x_n,y_n) = 0$ imply that $\lim_{n\to\infty} d(y_n,x) = 0$.

Finally, a nonempty subset A of a symmetric space (X,d) is said to be

(1) d-closed if $\overline{A}^d = A$, where

$$\overline{A}^d = \{x \in X : d(x,A) = 0\}, \qquad d(x,A) = \inf\{d(x,y) : y \in A\}. \tag{1.1}$$

We denote by $C(X)$ the set of all nonempty d-closed subsets of (X,d).

(2) It is said to be d-bounded if $\delta_d(A) < \infty$, where $\delta_d(A) = \sup\{d(x,y) : x,y \in A\}$.

Let $B(X)$ denote the set of all nonempty d-bounded subsets of X. For $A,B \in B(X)$, we define

$$\delta(A,B) = \sup\{d(a,b) \mid a \in A;\ b \in B\}. \tag{1.2}$$

It follows immediately from this definition that, for all $A,B \in B(X)$, one has

$$\begin{aligned} &\delta(A,B) = \delta(B,A), \\ &\delta(A,B) = 0 \quad \text{if } A = B = \{a\},\ a \in A, \\ &\delta(A,A) = \delta_d(A). \end{aligned} \tag{1.3}$$

2. Main results

2.1. E-weak compatibility.

Definition 2.1. Let $A : X \to 2^X$ be a multivalued mapping and let B be a self-mapping of a symmetric space (X,d). One says that A and B are E-weakly compatible if for each $u \in X$, one has $BAu \subseteq ABu$ whenever $Bu \in Au$.

Examples.

(1) Let $X = [1,+\infty[$. Define $A : X \to 2^X$ and $B : X \to X$ by

$$Ax = [1,2x], \quad Bx = 2x, \quad \forall\, x \in X. \tag{2.1}$$

It is clear that, for each $x \in X$, one has $Bx \in Ax$ and $BAx \subset ABx$. Then A and B are E-weakly compatible.

(2) Let $X = \mathbb{N} = \{1,2,\dots\}$. Define $A : X \to 2^X$ and $B : X \to X$ by

$$Ax = \{kx \mid k \in \mathbb{N}\}, \quad Bx = 2x - 1, \quad \forall\, x \in X. \tag{2.2}$$

Clearly, one has $B1 \in A1$ and $BA1 \subset AB1 = \mathbb{N}$. Note that 1 is the unique element u of $\mathbb{N}$ satisfying $Bu \in Au$. Therefore, A and B are E-weakly compatible.

Remark 2.2. If A is a single-valued mapping, then the set ABx consists of a single point. Therefore, E-weak compatibility is reduced to weak compatibility condition given in [3]; that is, two self-mappings A and B of a symmetric space X are said to be weakly compatible if they commute at their coincidence points.

Remark 2.3. In what follows including Section 2.3, we consider a nondecreasing right continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{n\to\infty}\psi^n(t) = 0$, for all $t \in]0,+\infty[$. Under the above properties, ψ satisfies $\psi(t) < t$ for all $t > 0$, and therefore $\psi(0) = 0$.

2.2. Common fixed point results.

Theorem 2.4. *Let $A : X \to C(X)$ be a multivalued mapping and let B be a self-mapping of a d-bounded symmetric space (X,d) satisfying (W.4) such that*
(1) $\delta(Ax,Ay) \le \psi(d(Bx,By))$, *for all* $x \ne y$ *in* X,
(2) *A and B are E-weakly compatible,*
(3) $AX \subset BX$.
If the range of B is an S-complete subspace of X, then A and B have a unique common fixed point.

Proof. Let $x_0 \in X$. Since $Ax_0 \subseteq BX$, choose $x_1 \in X$ such that $Bx_1 \in Ax_0$. Choose $x_2 \in X$ such that $Bx_2 \in Ax_1$. Continuing in this fashion, choose $x_n \in X$ such that $Bx_n \in Ax_{n-1}$. Then we have

$$\begin{aligned} d(Bx_n,Bx_{n+m}) &\le \delta(Ax_{n-1},Ax_{n+m-1}) \\ &\le \psi(d(Bx_{n-1},Bx_{n+m-1})) \\ &\le \psi(\delta(Ax_{n-2},Ax_{n+m-2})) \\ &\le \psi^2(d(Bx_{n-2},Bx_{n+m-2})) \\ &\quad \vdots \\ &\le \psi^n(d(Bx_0,Bx_m)) \le \psi^n(\delta_d(X)). \end{aligned} \tag{2.3}$$

which implies that $\{Bx_n\}$ is a d-Cauchy sequence. Suppose that BX is an S-complete subspace of X, then $\lim_{n\to\infty} d(Bx_n,Bu) = 0$ for some $u \in X$. We claim that $Bu \in Au$. Indeed, we have

$$d(Bx_n,Au) \le \delta(Ax_{n-1},Au) \le \psi(d(Bx_{n-1},Bu)). \tag{2.4}$$

On letting n to infty, we obtain $d(Bx_n,Au) = 0$, and therefore by using (W.4), we obtain $Bu \in \overline{Au}^d = Au$. The E-weak compatibility of A and B implies that $BAu \subseteq ABu$. Since $BAu = \{Ba \mid a \in Au\}$ and $Bu \in Au$, it follows that $BBu \in BAu \subseteq ABu$.

Let us show that Bu is a common fixed point of A and B. Suppose that $BBu \ne Bu$. In view of (1), it follows that

$$d(Bu,BBu) \le \delta(Au,ABu) \le \psi(d(Bu,BBu)) < d(Bu,BBu), \tag{2.5}$$

which gives a contradiction. Therefore, $Bu = BBu \in ABu$ and Bu is a common fixed point of A and B. For uniqueness, suppose that there exists $u,v \in X$ such that $Bu = u \in Au$, $Bv = v \in Av$, and $u \neq v$. In view of (1), we have

$$\begin{aligned} d(u,v) &\leq \delta(Au,Av) \leq \psi(d(Bu,Bv)) \\ &\leq \psi(d(u,v)) < d(u,v), \end{aligned} \tag{2.6}$$

which is a contradiction. Therefore, $u = v$ and the common fixed point is unique. □

When $\psi(t) = kt$, $k \in [0,1[$, we get the following new result.

Corollary 2.5. *Let $A : X \to C(X)$ be a multivalued mapping and let B be a self-mapping of a d-bounded symmetric space (X,d) satisfying (W.4) such that*
(1) $\delta(Ax,Ay) \leq kd(Bx,By)$, $k \in [0,1[$, *for all* $x \neq y$ *in* X,
(2) *A and B are E-weakly compatible,*
(3) $AX \subset BX$.
If the range of B is an S-complete subspace of X, then A and B have a unique common fixed point.

Remark 2.6. When A is a single-valued mapping, Corollary 2.5 is reduced to a generalization of [4, Theorem 2.1] which in turn generalizes [1, Theorem 1].

Also letting $B = Id_X$ (resp., $A = Id_X$) in Theorem 2.4, we get the following new results.

Corollary 2.7. *Let $A : X \to C(X)$ be a multivalued mapping of a d-bounded S-complete symmetric space (X,d) satisfying (W.4) such that*

$$\delta(Ax,Ay) \leq \psi(d(x,y)), \quad \forall\, x \neq y \text{ in } X. \tag{2.7}$$

Then A has a unique fixed point.

Corollary 2.8. *Let B be a subjective self-mapping of a d-bounded symmetric space (X,d) satisfying (W.4) such that*

$$d(x,y) \leq \psi(d(Bx,By)), \quad \forall\, (x,y) \in X^2. \tag{2.8}$$

If the range of B is an S-complete subspace of X, then B has a unique fixed point.

2.3. Application. A distribution function f is a nondecreasing left continuous real-valued function f defined on the set of real numbers, with $\inf f = 0$ and $\sup f = 1$.

Definition 2.9. Let X be a set and$\mathfrak{I}$ a function defined on $X \times X$ such that $\mathfrak{I}(x,y) = F_{x,y}$ is a distribution function. Consider the following conditions:

(i) $F_{x,y}(0) = 0$ for all $x, y \in X$,

(ii) $F_{x,y} = H$ if and only if $x = y$, where H denotes the distribution function defined by

$$H(x) = \begin{cases} 0, & \text{if } x \le 0, \\ 1, & \text{if } x > 0, \end{cases} \tag{2.9}$$

(iii) $F_{x,y} = F_{y,x}$,

(iv) if $F_{x,y}(\epsilon) = 1$ and $F_{y,z}(\delta) = 1$, then $F_{x,z}(\epsilon + \delta) = 1$.

If $\mathfrak{I}$ satisfies (i) and (ii), then it is called a PPM structure on X, and the pair $(X, \mathfrak{I})$ is called a PPM space. An $\mathfrak{I}$ satisfying (iii) is said to be symmetric. A symmetric PPM structure $\mathfrak{I}$ satisfying (iv) is a probabilistic metric structure, and the pair $(X, \mathfrak{I})$ is a probabilistic metric space.

Let $(X, \mathfrak{I})$ be a symmetric PPM space. For $\epsilon, \lambda > 0$ and x in X, let $N_x(\epsilon, \lambda) = \{y \in X : F_{x,y}(\epsilon) > 1 - \lambda\}$. A T_1 topology $t(\mathfrak{I})$ on X is defined as follows:

$$t(\mathfrak{I}) = \{U \subseteq X \mid \text{ for each } x \in U, \text{ there exists } \epsilon > 0, \text{ such that } N_x(\epsilon, \epsilon) \subseteq U\}. \tag{2.10}$$

Recall that a sequence $\{x_n\}$ is called a fundamental sequence if $\lim_{n\to\infty} F_{x_n,x_m}(t) = 1$ for all $t > 0$. The space $(X, \mathfrak{I})$ is called F-complete if for every fundamental sequence $\{x_n\}$ there exists x in X such that $\lim_{n\to\infty} F_{x_n,x}(t) = 1$, for all $t > 0$. Recently, in [1], it was proved that each symmetric PPM space admits a compatible symmetric function as follows.

Theorem 2.10 (see [1]). *Let $(X, \mathfrak{I})$ be a symmetric PPM space. Let $d : X \times X \to \mathbb{R}^+$ be a function defined as follows:*

$$d(x,y) = \begin{cases} 0, & \text{if } y \in N_x(t,t) \ \forall\, t > 0, \\ \sup\{t : y \notin N_x(t,t),\ 0 < t < 1\}, & \text{otherwise.} \end{cases} \tag{2.11}$$

Then,

(1) *$d(x,y) < t$ if and only if $F_{x,y}(t) > 1 - t$;*

(2) *d is compatible and symmetric for $t(\mathfrak{I})$;*

(3) *$(X, \mathfrak{I})$ is F-complete if and only if (X, d) is S-complete.*

Definition 2.11. Let $(X, \mathfrak{I})$ be a symmetric PPM space and A a nonempty subset of X. One says that A is $\mathfrak{I}$-closed if $\overline{A}^{\mathfrak{I}} = A$, where

$$\overline{A}^{\mathfrak{I}} = \left\{x \in X : \sup_{a \in A} F_{x,a}(t) = 1,\ \forall t > 0\right\}. \tag{2.12}$$

One denotes by $C_{\mathfrak{I}}(X)$ the set of all nonempty $\mathfrak{I}$-closed subsets of X.

Remark 2.12. Let $(X, \mathfrak{I})$ be a symmetric PPM space and let $C_{\mathfrak{I}}(X)$ be the set of all nonempty $\mathfrak{I}$-closed subsets of X. It is not hard to see that if d is a compatible symmetric function for $t(\mathfrak{I})$, then $C_{\mathfrak{I}}(X) = C(X)$, where $C(X)$ is the set of all nonempty d-closed subsets of (X, d). For $A, B \in C_{\mathfrak{I}}(X)$, set $D_{A,B}(\epsilon) = \inf_{a \in A, b \in B} F_{a,b}(\epsilon)$,nbsp;$\epsilon > 0$.

Remark 2.13. Note that condition (W.4), defined earlier, is equivalent to the following condition:

(P.4) $\lim_{n\to\infty} F_{x_n,x}(t) = 1$ and $\lim_{n\to\infty} F_{x_n,y_n}(t) = 1$ imply that $F_{y_n,x}(t) = 1$, for all $t > 0$.

As an application of our main Theorem 2.4, we have the following result.

Theorem 2.14. *Let $(X,\mathfrak{I})$ be a symmetric PPM space that satisfies (P.4) and d a compatible symmetric function for $t(\mathfrak{I})$. Let $A : X \to C_{\mathfrak{I}}(X)$ be a multivalued mapping and let $B : \to X$ be a self-mapping of X such that*

(1) *$F_{Bx,By}(t) > 1 - t$ implies that $D_{Ax,Ay}(\psi(t)) > 1 - \psi(t)$, for all $t > 0$, for all $x \neq y$ in X,*
(2) *A and B are E-weakly compatible,*
(3) *$AX \subset BX$.*

If the range of B is an F-complete subspace of X, then A and B have a unique common fixed point.

Proof. Note that (X,d) is d-bounded and BX is an S-complete subspace of X. Also $d(x,y) < t$ if and only if $F_{x,y}(t) > 1 - t$. Let $\epsilon > 0$ be given, and set $t = d(Bx,By) + \epsilon$. Then $d(Bx,By) < t$ gives $F_{Bx,By}(t) > 1 - t$, and therefore $D_{Ax,Ay}(\psi(t)) > 1 - \psi(t)$.

We claim that $\delta(Ax,Ay) \leq \psi(d(Bx,By))$. Indeed, from $D_{Ax,Ay}(\psi(t)) > 1 - \psi(t)$, it follows that

$$\inf_{a\in Ax, b\in Ay} F_{a,b}(\psi(t)) > 1 - \psi(t) \Longrightarrow \forall (a,b) \in Ax \times Ay,\ F_{a,b}(\psi(t)) > 1 - \psi(t), \tag{2.13}$$

which implies that for all $(a,b) \in Ax \times Ay$, $d(a,b) < \psi(t)$, and therefore $\delta(Ax,Ay) < \psi(t) = \psi(d(Bx,By) + t)$. On letting ϵ be 0 (since $\epsilon > 0$ was arbitrary), we have $\delta(Ax,Ay) \leq \psi(d(Bx,By))$. Now apply Theorem 2.4. □

For $\psi(t) = kt$, $k \in [0,1[$, Theorem 2.14 is reduced to the following new result.

Corollary 2.15. *Let $(X,\mathfrak{I})$ be a symmetric PPM space that satisfies (P.4) and d a compatible symmetric function for $t(\mathfrak{I})$. Let $A : X \to C_{\mathfrak{I}}(X)$ be a multivalued mapping and let $B : \to X$ be a self-mapping of X such that*

(1) *$F_{Bx,By}(t) > 1 - t$ implies that $D_{Ax,Ay}(kt) > 1 - kt$, $k \in [0,1[$, for all $t > 0$, for all $x \neq y$ in X,*
(2) *A and B are E-weakly compatible,*
(3) *$AX \subset BX$.*

If the range of B is an F-complete subspace of X, then A and B have a unique common fixed point.

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