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Research Article

On the Lower Bound for the Number of Real Roots of a Random Algebraic Equation

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We estimate a lower bound for the number of real roots of a random alegebraic equation whose random coeffcients are dependent normal random variables.

1. Introduction

Let $N_n(\mathbf{R},\omega)$ be the number of real roots of the random algebraic equation

$$F_n(x,\omega)=\sum_{\nu=0}^{n}a_\nu(\omega)x^\nu=0, \tag{1.1}$$

where the $a_\nu(\omega)$, $\nu=0,1,\dots,n$, are random variables defined on a fixed probability space $(\Omega,\mathcal{A},\Pr)$ assuming real values only.

During the past 40–50 years, the majority of published researches on random algebraic polynomials has concerned the estimation of $N_n(R,\omega)$. Works by Littlewood and Offord [1], Samal [2], Evans [3], and Samal and Mishra [4–6] in the main concerned cases in which the random coefficients $a_\nu(\omega)$ are independent and identically distributed.

For dependent coefficients, Sambandham [7] considered the upper bound for $N_n(\mathbf{R},\omega)$ in the case when the $a_\nu(\omega)$, $\nu=0,1,\dots,n$, are normally distributed with mean zero and joint density function

$$|M|^{1/2}(2\pi)^{-(n+1)/2}\exp\left(-(1/2)\mathbf{a}'M\mathbf{a}\right), \tag{1.2}$$

where M^{-1} is the moment matrix with $\sigma_i=1$, $\rho_{ij}=\rho$, $0<\rho<1$, $(i\neq j)$, $i,j=0,1,\dots,n$ and $\mathbf{a}'$ is the transpose of the column vector $\mathbf{a}$. Also, Uno and Negishi [8] obtained the same result as Sambandham in the case of the moment matrix with $\sigma_i=1$, $\rho_{ij}=\rho_{|i-j|}$,

$(i\neq j)$, $i,j=0,1,\dots,n$, where ρ_j is a nonnegative decreasing sequence satisfying $\rho_1<1/2$ and $\sum_{j=1}^{\infty}\rho_j<\infty$ in (1.2).

The lower bound for $N_n(\mathbf{R},\omega)$ in the case of dependent normally distributed coefficients was estimated by Renganathan and Sambandham [9] and Nayak and Mohanty [10] under the same condition of Sambandham [7]. Uno [11] pointed out the defect in the proofs of the above papers and obtained the result for the lower bound. Additionally, Uno [12] estimated the *strong* result for this particular problem in the sense of Evans [3]. The term *strong* indicates that the estimation for the exceptional set is independent of the degree n.

The object of this paper is to find the lower bound for $N_n(\mathbf{R},\omega)$ when the coefficients are nonidentically distributed dependent normal random variables. We remark that this result is the general form of Uno [11] and that the exceptional set is dependent on the degree n. In this paper, we suppose that the $a_\nu(\omega)$, $\nu=0,1,\dots,n$, have mean zero, and the moment with

$$\rho_{ij}=\begin{cases}1 & (i=j),\\ \rho_{|i-j|} & (1\le |i-j|\le m),\\ 0 & (|i-j|>m), \quad i,j=0,1,\dots,n,\end{cases}\tag{1.3}$$

for a positive integer m, where $0\le\rho_j<1$, $j=1,2,\dots,m$ in (1.2). That is to say we assume the $a_\nu(\omega)'s$ to be m-dependent stationary Gaussian random variables. With Yoshihara ([13, page 29]), we see that this assumption is equivalent to the following two statements for a stationary Gaussian sequence:

(i) $\{a_\nu\}$ is $*$-mixing;

(ii) $\{a_\nu\}$ is ϕ-mixing.

Throughout the paper, we suppose n is sufficiently large. We will follow the line of proof of Samal and Mishra [5].

Theorem 1.1. *Let*

$$f_n(x,\omega)=\sum_{\nu=0}^{n}a_\nu(\omega)b_\nu x^\nu=0\tag{1.4}$$

be a random algebraic equation of degree n, where the $a_\nu(\omega)$'s are dependent normally distributed with mean zero, and the moment matrix given by (1.3) and the b_ν, $\nu=0,1,\dots,n$, be positive numbers such that $\lim_{n\to\infty}(k_n/t_n)$ *is finite, where* $k_n=\max_{0\le\nu\le n}b_\nu$ *and* $t_n=\min_{0\le\nu\le n}b_\nu$.

Then for $n>n_0$, the number of real roots of most of the equations $f_n(x,\omega)=0$ is at least $\varepsilon_n\log n$ outside a set of measure at most

$$\frac{\mu}{\varepsilon_n\log n}+\left(\frac{k_n}{t_n}\right)^{\beta}\exp\left(-\frac{\mu'\beta}{\varepsilon_n}\right),\quad \beta>0,\tag{1.5}$$

provided ε_n tends to zero, but $\varepsilon_n\log n$ tends to infinity as n tends to infinity, and μ and μ' are positive constants.

2. Proof of theorem

Let $\{\lambda_n\}$ be any sequence tending to infinity as n tends to infinity and M is the integer defined by

$$M = \left[\alpha^2\lambda_n^2\left(\frac{k_n}{t_n}\right)^2\right] + 1, \tag{2.1}$$

where α is a positive constant and $[x]$ denotes the greatest integer not exceeding x. Let k be the integer determined by

$$M^{2k} \le n < M^{2k+2}. \tag{2.2}$$

We will consider $f_n(x,\omega)$ at the points

$$x_l = \left(1 - \frac{1}{M^{2l}}\right)^{1/2} \tag{2.3}$$

for $l = [k/2]+1, [k/2]+2, \dots, k$.

Let

$$f_n(x_l,\omega) = \sum_1 a_\nu(\omega)b_\nu x_l^\nu + \left(\sum_2 + \sum_3\right)a_\nu(\omega)b_\nu x_l^\nu = U_l(\omega) + R_l(\omega), \quad \text{(say)}, \tag{2.4}$$

where ν ranges from $M^{2l-1}+1$ to M^{2l+1} in $\sum_1$, from 0 to M^{2l-1} in $\sum_2$ and from $M^{2l+1}+1$ to n in $\sum_3$.

The following lemmas are necessary for the proof of the theorem. We will use the fact that each $a_\nu(\omega)$ has marginal frequency function $(2\pi)^{-1/2}\exp(-u^2/2)$.

Lemma 2.1. *For $\alpha_1 > 0$,*

$$\sigma_l > \alpha_1 t_n M^l, \tag{2.5}$$

where

$$\sigma_l^2 = \sum_{i=M^{2l-1}+1}^{M^{2l+1}} b_i^2 x_l^{2i} + 2\sum_{i=M^{2l-1}+1}^{M^{2l+1}-1}\sum_{j=i+1}^{M^{2l+1}} b_i b_j x_l^{i+j}\rho_{j-i}. \tag{2.6}$$

Proof. First, we have

$$\sum_{i=M^{2l-1}+1}^{M^{2l+1}} b_i^2 x_l^{2i} > t_n^2 \sum_{i=M^{2l-1}+1}^{M^{2l}} x_l^{2i} > \left(\frac{B}{A}\right)t_n^2 M^{2l}, \tag{2.7}$$

where A and B are positive constants such that $A > 1$ and $0 < B < 1$.

Second, we get

$$\begin{aligned}\sum_{i=M^{2l-1}+1}^{M^{2l+1}-1}\sum_{j=i+1}^{M^{2l+1}} b_i b_j x_l^{i+j}\rho_{j-i} &> t_n^2 \sum_{i=M^{2l-1}+1}^{M^{2l}-1}\sum_{j=i+1}^{M^{2l}} x_l^{i+j}\rho_{j-i}\\ &= t_n^2 \frac{x_l^{2(M^{2l-1}+1)}}{1-x_l^2}\left\{\sum_{i=1}^{m}\rho_i x_l^i - \sum_{i=1}^{m}\rho_i x_l^{2(M^{2l}-M^{2l-1})-i}\right\} \geq \left(\frac{B'}{A'}\right)\rho_0 t_n^2 M^{2l},\end{aligned} \tag{2.8}$$

where $\rho_0 = \sum_{j=1}^{m}\rho_j$ and A' and B' are positive constants satisfying $A' > 1$ and $0 < B' < 1$. So we get

$$\sigma_l^2 \geq \alpha_1^2 t_n^2 M^{2l}, \tag{2.9}$$

where α_1 is a positive constant, as required. □

Lemma 2.2. *Let*

$$\Pr\left(\left\{\omega; \left|\sum_2 a_\nu(\omega) b_\nu x_l^\nu\right| > \lambda_n \tilde{\sigma}_l\right\}\right) < \sqrt{\frac{2}{\pi}}\frac{e^{-\lambda_n^2/2}}{\lambda_n}, \tag{2.10}$$

where

$$\tilde{\sigma}_l^2 = \sum_{i=0}^{M^{2l-1}} b_i^2 x_l^{2i} + 2\sum_{i=0}^{M^{2l-1}-1}\sum_{j=i+1}^{M^{2l-1}} b_i b_j x_l^{i+j}\rho_{j-i}. \tag{2.11}$$

Proof. We get

$$\Pr\left(\left\{\omega; \left|\sum_2 a_\nu(\omega) b_\nu x_l^\nu\right| > \lambda_n \tilde{\sigma}_l\right\}\right) = \sqrt{\frac{2}{\pi}}\int_{\lambda_n}^{\infty} e^{-u^2/2}du < \sqrt{\frac{2}{\pi}}\frac{e^{-\lambda_n^2/2}}{\lambda_n}. \tag{2.12}$$

□

Lemma 2.3. *Let*

$$\Pr\left(\left\{\omega; \left|\sum_3 a_\nu(\omega) b_\nu x_l^\nu\right| > \lambda_n \tilde{\tilde{\sigma}}_l\right\}\right) < \sqrt{\frac{2}{\pi}}\frac{e^{-\lambda_n^2/2}}{\lambda_n}, \tag{2.13}$$

where

$$\tilde{\tilde{\sigma}}_l^2 = \sum_{i=M^{2l+1}+1}^{n} b_i^2 x_l^{2i} + 2\sum_{i=M^{2l+1}+1}^{n-1}\sum_{j=i+1}^{n} b_i b_j x_l^{i+j}\rho_{j-i}. \tag{2.14}$$

The proof is similar to that of Lemma 2.2.

Lemma 2.4. *For a fixed l,*

$$\Pr\left(\{\omega; |R_l(\omega)| < \sigma_l\}\right) > 1 - 2\sqrt{\frac{2}{\pi}}\frac{1}{\lambda_n}e^{-\lambda_n^2/2}. \tag{2.15}$$

Proof. By Lemmas 2.2 and 2.3, we get, for a given l,

$$|R_l(\omega)| < \lambda_n(\tilde{\sigma}_l + \tilde{\tilde{\sigma}}_l) \tag{2.16}$$

outside a set of measure at most $2(2/\pi)^{1/2}\lambda_n^{-1}\exp(-\lambda_n^2/2)$. Again, we have

$$\begin{gathered} \sum_{i=0}^{M^{2l-1}} b_i^2 x_l^{2i} \le 2k_n^2 M^{2l-1}, \\ \sum_{i=0}^{M^{2l-1}-1} \sum_{j=i+1}^{M^{2l-1}} b_i b_j x_l^{i+j} \rho_{j-i} \le k_n^2 \sum_{i=1}^{m} \rho_i \sum_{j=1}^{M^{2l-1}-(i-1)} x_l^{2j+i-2} \le \rho_0 k_n^2 M^{2l-1}. \end{gathered} \tag{2.17}$$

Hence we get, for a positive constant α_2,

$$\tilde{\sigma}_l^2 \le \alpha_2^2 k_n^2 M^{2l-1}. \tag{2.18}$$

Similarly, we have

$$\tilde{\tilde{\sigma}}_l^2 \le \alpha_3^2 k_n^2 M^{2l-1} \tag{2.19}$$

for a positive constant α_3. Therefore, we obtain, outside the exceptional set,

$$|R_l(\omega)| < \lambda_n(\alpha_2+\alpha_3)k_n M^{l-(1/2)} < \left(\frac{\alpha_2+\alpha_3}{\alpha_1}\frac{k_n}{t_n}\lambda_n\sigma_l\right)/M^{1/2} < \sigma_l, \tag{2.20}$$

by Lemma 2.1 and (2.1). □

Let us define random events E_p, F_p by

$$\begin{aligned} E_p &= \{\omega;\ U_{3p}(\omega) \ge \sigma_{3p}, U_{3p+1}(\omega) < -\sigma_{3p+1}\}, \\ F_p &= \{\omega;\ U_{3p}(\omega) < -\sigma_{3p}, U_{3p+1}(\omega) \ge \sigma_{3p+1}\}. \end{aligned} \tag{2.21}$$

It can be easily seen that

$$\Pr(E_p \cup F_p) = \delta_p \quad (\text{say}) > \delta, \tag{2.22}$$

where $\delta > 0$ is a certain constant. Let η_p be a random variable such that

$$\eta_p = \begin{cases} 1 & \text{on } E_p \cup F_p, \\ 0 & \text{elsewhere.} \end{cases} \tag{2.23}$$

Then we get

$$E(\eta_p) = \delta_p, \qquad V(\eta_p) = \delta_p - \delta_p^2. \tag{2.24}$$

Let q be the total number of pairs (U_{3p}, U_{3p+1}) for which

$$\left[\frac{k}{2}\right] + 1 \le 3p < 3p+1 \le k, \tag{2.25}$$

q must be at least equal to $[k/3]-[([k/2]+1)/3]-1$. Take

$$\eta=\sum\eta_p, \tag{2.26}$$

where the summation is taken over all the q pairs. Applying Tschebyscheff inequality, we have, for $0<\varepsilon<\delta$,

$$\Pr(\{|\eta-E(\eta)|\geq q\varepsilon\})\leq\frac{V(\eta)}{q^2\varepsilon^2}\leq\frac{\sum\delta_p}{q^2\varepsilon^2}\leq\frac{1}{q\varepsilon^2}, \tag{2.27}$$

since for n sufficiently large, $\operatorname{Cov}(\eta_i,\eta_j)=0\ (i\neq j)$. But

$$q\geq\left[\frac{k}{3}\right]-\left[\frac{[k/2]+1}{3}\right]-1\geq\frac{k}{3}-1-\left(\frac{(k/2)+1}{3}\right)-1=\frac{1}{6}(k-14)\geq\mu_1k, \tag{2.28}$$

where μ_1 is a positive constant. Therefore, outside a set of measure at most μ_2/k,

$$|\eta-E(\eta)|<q\varepsilon, \tag{2.29}$$

that is,

$$\eta-E(\eta)>-q\varepsilon \tag{2.30}$$

or

$$\eta>E(\eta)-q\varepsilon=\sum\delta_p-q\varepsilon>q(\delta-\varepsilon)\geq\mu_3k, \tag{2.31}$$

where μ_2 and μ_3 are positive constants. Thus we have proved that outside a set of measure at most μ_2/k, either $U_{3p}\geq\sigma_{3p}$ and $U_{3p+1}<-\sigma_{3p+1}$, or $U_{3p}<-\sigma_{3p}$ and $U_{3p+1}\geq\sigma_{3p+1}$ for at least μ_3k values of l.

Define

$$\zeta_p=\begin{cases}0 & \text{if } |R_{3p}|<\sigma_{3p},\ |R_{3p+1}|<\sigma_{3p+1},\\ 1 & \text{elsewhere.}\end{cases} \tag{2.32}$$

Let $\xi_p=\eta_p-\eta_p\zeta_p$. If $\xi_p=1$, there is a root of the polynomial in the interval (x_{3p},x_{3p+1}). Hence the number of real roots in the interval $(x_{[k/2]+1},x_k)$ must exceed $\sum\xi_p$, where the summation is taken over all the q pairs. Now, by using Lemma 2.4, we have

$$\begin{aligned}E\Big(\sum\eta_p\zeta_p\Big)&=\sum E(\eta_p\zeta_p)\leq\sum E(\zeta_p)=\sum\Pr(\zeta_p=1)\\&\leq\sum\{\Pr(|R_{3p}|\geq\sigma_{3p})+\Pr(|R_{3p+1}|\geq\sigma_{3p+1})\}\\&<\mu_4(k+1)\frac{1}{\lambda_n}e^{-\lambda_n^2/2},\end{aligned} \tag{2.33}$$

where μ_4 is a constant. Hence we have, for $\beta>0$,

$$\Pr\left(\left\{\sum\eta_p\zeta_p>\mu_4(k+1)\lambda_n^{\beta}\frac{1}{\lambda_n}e^{-\lambda_n^2/2}\right\}\right)<\frac{E(\sum\eta_p\zeta_p)}{\mu_4(k+1)\lambda_n^{\beta-1}e^{-\lambda_n^2/2}}<\frac{1}{\lambda_n^{\beta}}. \tag{2.34}$$

So we get

$$\sum \eta_p \zeta_p \le \mu_4 (k+1) \lambda_n^{\beta-1} e^{-\lambda_n^2/2}, \tag{2.35}$$

except for a set of measure at most $1/\lambda_n^{\beta}$. Therefore, we have, outside a set of measure at most $\mu_2/k + 1/\lambda_n^{\beta}$,

$$N_n > \sum \xi_p > \mu_3 k - \mu_4 (k+1) \lambda_n^{\beta-1} e^{-\lambda_n^2/2} \ge k(\mu_3 - \varepsilon_1), \tag{2.36}$$

where $0 < \varepsilon_1 < \mu_3$ (since $\mu_4 \lambda_n^{\beta-1} \exp(-\lambda_n^2/2)$ tends to zero as n tends to infinity). But it follows from (2.1) and (2.2) that

$$\begin{gathered} \mu_5 \left(\frac{k_n}{t_n}\right)^2 \lambda_n^2 \le M \le \mu_6 \left(\frac{k_n}{t_n}\right)^2 \lambda_n^2, \\ \frac{\mu_7 \log n}{\log((k_n/t_n)\lambda_n)} \le k \le \frac{\mu_8 \log n}{\log((k_n/t_n)\lambda_n)}, \end{gathered} \tag{2.37}$$

where μ_i, $i = 5,6,7,8$, are constants. Hence we get outside the exceptional set

$$N_n > \frac{\mu_9 \log n}{\log((k_n/t_n)\lambda_n)}, \tag{2.38}$$

where μ_9 is a constant.

Taking $\lambda_n = (t_n/k_n) \exp(\mu_9/\varepsilon_n)$, we obtain

$$N_n > \varepsilon_n \log n \tag{2.39}$$

outside a set of measure at most

$$\frac{\mu}{\varepsilon_n \log n} + \left(\frac{k_n}{t_n}\right)^{\beta} \exp\left(-\frac{\mu' \beta}{\varepsilon_n}\right), \tag{2.40}$$

where μ and μ' are constants. This completes the proof of the theorem.

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