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Research Article

A Family of Non-Gaussian Martingales with Gaussian Marginals

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We construct a family of martingales with Gaussian marginal distributions. We give a weak construction as Markov, inhomogeneous in time processes, and compute their infinitesimal generators. We give the predictable quadratic variation and show that the paths are not continuous. The construction uses distributions G_σ having a log-convolution semigroup property. Further, we categorize these processes as belonging to one of two classes, one of which is made up of piecewise deterministic pure jump processes. This class includes the case where G_σ is an inverse log-Poisson distribution. The processes in the second class include the case where G_σ is an inverse log-gamma distribution. The richness of the family has the potential to allow for the imposition of specifications other than the marginal distributions.

1. Introduction

It is hard to overestimate the importance of the Brownian motion. From the stochastic calculus perspective, the Brownian motion draws its status from the fact that it is a martingale. From the modeling perspective, the Brownian motion has the desirable property of having the Markov property as well as Gaussian marginals. In an attempt to uphold these basic requirements, we construct a rich family of Markov processes which are martingales and have Gaussian marginals. We also exhibit some properties of this family of processes, which has the potential to find applications in many fields, including finance.

Let us first, give a brief survey of related results and ideas found in the literature.

Kellerer gave conditions for the existence of a Markov martingales with given marginals in [1], but offered no explicit construction. He proves the existence of such processes under two conditions: the first one is that the targeted marginal densities, $g(x,t)$, must be increasing in the convex order ($\mathbb{E}[f(X_t)] \geq \mathbb{E}[f(X_s)]$ for $s < t$ and f convex), secondly, that the marginal densities must have means that do not depend on t. Madan and Yor in [2], inspired by the above result, gave three different constructions: a continuous martingale, a time-changed Brownian motion, and a construction that uses Azèma-Yor's solution to the Skorokhod embedding problem.

The continuous martingale approach looks for a process of the form

$$X_t = \int_0^t \sigma(X_s, s)\, dB_s, \tag{1.1}$$

where B_t is a Brownian motion. When applied to the case of $N(0,t)$ marginal densities, this methodology simply produces a Brownian motion. Indeed, writing the forward partial differential equation for these densities, one can see that σ^2 must be identically equal to 1 (see [2]).

In the time change approach, one looks for a process of the form

$$X_t = B_{L_t}, \tag{1.2}$$

where L is an increasing process, assumed to be a Markov process with inhomogeneous independent increments, independent of the Brownian motion B_t. Using the independence of B and L, and the assumption of Gaussian marginals, we have

$$e^{-(\lambda^2/2)t} = \mathbb{E}\big[e^{i\lambda X_t}\big] = \mathbb{E}\big[e^{-(\lambda^2/2)L_t}\big]. \tag{1.3}$$

This implies that $L_t = t$ and $X_t = B_t$.

However, the Skorokhod embedding approach, which we do not review here, yields an example of a discontinuous and time-inhomogeneous Markov non-Gaussian martingale, see [2].

Our approach is different to all of the above and produces a rich family of processes rather than a single process. The richness of the family has the potential to allow for the imposition of specifications other than that of prescribed marginal distributions. Although our method can be extended to include other types of marginal distributions, we choose to focus solely on the Gaussian case. Finally, we comment that all existing approaches yield discontinuous processes (barring the Brownian motion itself), and the question of the existence of a non-Gaussian continuous martingale with Gaussian marginals remains open.

The starting point of our construction is an observation that for any triple (R, Y, ξ) of independent random variables such that R takes values in $(0,1]$, ξ is standard Gaussian and Y is Gaussian with mean zero and variance α^2, the random variable

$$Z = \sigma\Big(\sqrt{R}Y + \alpha\sqrt{1-R}\xi\Big) \tag{1.4}$$

is Gaussian with mean zero and variance $\sigma^2\alpha^2$. However, the joint distribution of (Y,Z) is not bivariate Gaussian, as can be verified by calculating the fourth conditional moment of Z given $Y = 0$. In fact,

$$\mathbb{E}[Z^4 \mid Y = 0] - 3\mathbb{E}[Z^2 \mid Y = 0]^2 = 3\sigma^4\alpha^4(\mathbb{E}[(1-R)^2] - \mathbb{E}[1-R]^2) \tag{1.5}$$

and (Y,Z) is a bivariate Gaussian pair if and only if R is nonrandom. The martingale property of the two-period process (Y,Z) holds if and only if

$$Y = \mathbb{E}[Z \mid Y] = \mathbb{E}[\sigma(\sqrt{R}Y + \alpha\sqrt{1-R}\xi) \mid Y] = \sigma Y\mathbb{E}[\sqrt{R}], \tag{1.6}$$

in other words, if and only if

$$\mathbb{E}[\sqrt{R}] = \frac{1}{\sigma}. \tag{1.7}$$

Furthermore, the conditional distribution of Z given $Y = y$ is

$$F_{Z|Y=y}(dz) = \mathbb{P}[R = 1]\varepsilon_{\sigma y}(dz) + \mathbb{E}[\phi(\sigma\sqrt{R}y, \alpha^2\sigma^2(1-R), z)1_{R<1}]dz, \tag{1.8}$$

where ε_x is the Dirac measure at x and $\phi(\mu,\sigma^2,\cdot)$ denotes the density of the Gaussian distribution with mean μ and variance σ^2.

This construction of a two-step process is extended to that of a continuous time Markov process as described in the following section.

2. A family of non-Gaussian martingales with Gaussian marginals

In this section, we construct a family of Markov martingales, X_t, the marginals of which are Gaussian with mean zero and variance t. The result is stated as Theorem 2.5, the proof of which is broken into several preceding propositions. The process X_t is constructed as an inhomogeneous Markov process with transition function given by (1.8). In other words, it admits the following almost sure representation, see (1.4). For $s < t$,

$$X_t = \sqrt{\frac{t}{s}}\left(\sqrt{R_{s,t}}X_s + \sqrt{s}\sqrt{1-R_{s,t}}\xi_{s,t}\right), \tag{2.1}$$

where X_s, $R_{s,t}$, and $\xi_{s,t}$ are assumed to be independent, $R_{s,t}$ is assumed to take values in $(0,1]$ and to have a distribution that depends on (s,t) only through $\sqrt{t/s}$ for which $\mathbb{E}[\sqrt{R_{s,t}}] = \sqrt{s/t}$. Finally, $\xi_{s,t}$ is assumed to be a standard Gaussian random variable.

As we will shortly discover (see Proposition 2.3), for a family of transition functions given by (1.8) to define a (Markov) process, we will require that the distribution of $R_{s,t}$ generates a so-called log-convolution semigroup.

Definition 2.1. The family of distributions on $(0,+\infty)$ $(G_\sigma)_{\sigma\geq 1}$ is a log-convolution semigroup if $G_1 = \varepsilon_1$ and the distribution of the product of any two independent random variables with distributions G_σ and G_τ is $G_{\sigma\tau}$.

The following result shows the relationship that exists between log-convolution and convolution semigroups. The proof is straightforward and is left to the reader. Recall that $(K_p)_{p\geq 0}$ is a convolution semigroup if

$$K_0 = \varepsilon_0, \qquad K_p * K_q = K_{p+q}. \tag{2.2}$$

Proposition 2.2. *Let $(G_\sigma)_{\sigma\geq 1}$ be a log-convolution semigroup on $(0,1]$ and, for $\sigma \geq 1$, let R_σ be a random variable with distribution G_σ. If K_p, $p \geq 0$, denotes the distribution of $V_p = -\ln R_{e^p}$, then $(K_p)_{p\geq 0}$ is a convolution semigroup.*

Conversely, let $(K_p)_{p\geq 0}$ be a convolution semigroup and, for $p \geq 0$, let V_p be a random variable with distribution K_p. If G_σ, $\sigma \geq 1$, denotes the distribution of $R_\sigma = e^{-V_{\ln\sigma}}$, then $(G_\sigma)_{\sigma\geq 1}$ is a log-convolution semigroup.

In the next proposition, we check that the Chapman-Kolmogorov equation is satisfied, thus, guarantying the existence of the process X_t. In view of Proposition 2.2 and the well-known Lévy-Khinchin representation, we will later give a simple condition on the family $(G_\sigma)_{\sigma\geq 1}$ for it to generate the desired result (see Theorem 2.5).

Proposition 2.3. *Define, for $x \in \mathbb{R}$, $s > 0$ and $t = \sigma^2 s \geq s$, $P_{s,t}(x,dy)$ as*

$$P_{0,t}(x,dy) = \frac{1}{\sqrt{2\pi}\sqrt{t}} \exp\left(-\frac{(y-x)^2}{2t}\right)dy,$$

$$P_{s,t}(x,dy) = \gamma(\sigma)\varepsilon_{\sigma x}(dy) + \left[\int_{(0,1)} \frac{1}{\sqrt{2\pi}\sqrt{t}\sqrt{1-r}} \exp\left(-\frac{(y-\sigma\sqrt{r}x)^2}{2t(1-r)}\right)G_\sigma(dr)\right]dy, \tag{2.3}$$

where $\gamma(\sigma) = G_\sigma(\{1\})$.

If $(G_\sigma)_{\sigma\geq 1}$ is a log-convolution semigroup on $(0,1]$, then the Chapman-Kolmogorov equations hold. That is, for any $u > t > s > 0$ and any x,

$$\int P_{s,t}(x,dy)P_{t,u}(y,dz) = P_{s,u}(x,dz) \tag{2.4}$$

and, for any $u > t > 0$,

$$\int P_{0,t}(0,dy)P_{t,u}(y,dz) = P_{0,u}(0,dz). \tag{2.5}$$

Proof. Let us first observe that, if R_σ has distribution G_σ, R_τ has distribution G_τ and, R_σ and R_τ are independent, then $R_{\sigma\tau}$ and $R_\sigma R_\tau$ share the same distribution $G_{\sigma\tau}$. Then we have

$$\gamma(\sigma\tau) = \mathbb{P}[R_{\sigma\tau} = 1] = \mathbb{P}[R_\sigma R_\tau = 1] = \mathbb{P}[R_\sigma = 1,\ R_\tau = 1] = \gamma(\sigma)\gamma(\tau) \tag{2.6}$$

and, for any bounded Borel function h,

$$\begin{aligned}
&\int_{(0,1)}\int_{(0,1)} h(ab)G_\sigma(da)G_\tau(db)\\
&\quad= \mathbb{E}\left[h(R_\sigma R_\tau)1_{R_\sigma\neq 1,\ R_\tau\neq 1}\right]\\
&\quad= \mathbb{E}\left[h(R_\sigma R_\tau)1_{R_\sigma R_\tau\neq 1}\right] - \mathbb{E}\left[h(R_\sigma R_\tau)1_{R_\sigma=1,\ R_\tau\neq 1}\right] - \mathbb{E}\left[h(R_\sigma R_\tau)1_{R_\sigma\neq 1,\ R_\tau=1}\right]\\
&\quad= \mathbb{E}\left[h(R_{\sigma\tau})1_{R_{\sigma\tau}\neq 1}\right] - \gamma(\sigma)\mathbb{E}\left[h(R_\tau)1_{R_\tau\neq 1}\right] - \gamma(\tau)\mathbb{E}\left[h(R_\sigma)1_{R_\sigma\neq 1}\right]\\
&\quad= \int_{(0,1)} h(r)\left[G_{\sigma\tau}(dr) - \gamma(\sigma)G_\tau(dr) - \gamma(\tau)G_\sigma(dr)\right].
\end{aligned} \tag{2.7}$$

Next, let $t = \sigma^2 s$, $u = \tau^2 t$, and

$$p_{s,t}(x,y) = \int_{(0,1)} \frac{1}{\sqrt{2\pi}\sqrt{t}\sqrt{1-r}} \exp\left(-\frac{(y-\sigma\sqrt{r}x)^2}{2t(1-r)}\right)G_\sigma(dr), \tag{2.8}$$

so that by (2.4),

$$P_{s,t}(x,dy) = \gamma(\sigma)\varepsilon_{\sigma x}(dy) + p_{s,t}(x,y)dy, \tag{2.9}$$

and for any bounded Borel function h,

$$\begin{aligned}
&\int P_{s,t}(x,dy)\int P_{t,u}(y,dz)h(z)\\
&\quad= \int P_{s,t}(x,dy)\left[\gamma(\tau)h(\tau y) + \int h(z)p_{t,u}(y,z)dz\right]\\
&\quad= \gamma(\sigma)\left[\gamma(\tau)h(\tau\sigma x) + \int h(z)p_{t,u}(\sigma x,z)dz\right]\\
&\qquad+ \int\left[\gamma(\tau)h(\tau y) + \int h(z)p_{t,u}(y,z)dz\right]p_{s,t}(x,y)dy\\
&\quad= \gamma(\sigma\tau)h(\sigma\tau x) + \gamma(\sigma)\int h(z)p_{t,u}(\sigma x,z)dz + \gamma(\tau)\int h(\tau y)p_{s,t}(x,y)dy\\
&\qquad+ \int h(z)\int p_{t,u}(y,z)p_{s,t}(x,y)dy\,dz.
\end{aligned} \tag{2.10}$$

However,

$$\begin{aligned}
&\int p_{s,t}(x,y)p_{t,u}(y,z)dy \\
&\quad = \int \bigg[\int_{(0,1)} \frac{1}{\sqrt{2\pi}\sqrt{t}\sqrt{1-a}} \exp\bigg(-\frac{(y-\sigma\sqrt{a}x)^2}{2t(1-a)} \bigg) G_\sigma(da) \bigg] \\
&\qquad \times \bigg[\int_{(0,1)} \frac{1}{\sqrt{2\pi}\sqrt{u}\sqrt{1-b}} \exp\bigg(-\frac{(z-\tau\sqrt{b}y)^2}{2u(1-b)} \bigg) G_\tau(db) \bigg] dy \\
&\quad = \int_{(0,1)} \int_{(0,1)} \frac{1}{2\pi\sqrt{tu}\sqrt{(1-a)(1-b)}} \\
&\qquad \times \bigg[\int \exp\bigg(-\frac{(y-\sqrt{t/s}\sqrt{a}x)^2}{2t(1-a)} - \frac{(z-\sqrt{u/t}\sqrt{b}y)^2}{2u(1-b)} \bigg) dy \bigg] G_\sigma(da)G_\tau(db).
\end{aligned} \tag{2.11}$$

Next, we reformulate the expression in the exponential term above using the following identity which can be easily checked:

$$\frac{(y-mx)^2}{\mu} + \frac{(z-ny)^2}{\nu} = \frac{(y-k)^2}{\mu\nu/(\nu+\mu n^2)} + \frac{(z-mnx)^2}{\nu+\mu n^2}, \quad k = \frac{\nu mx+\mu nz}{\nu+\mu n^2}. \tag{2.12}$$

Thus,

$$\begin{aligned}
&\int p_{s,t}(x,y)p_{t,u}(y,z)dy \\
&= \int_{(0,1)} \int_{(0,1)} \frac{1}{2\pi\sqrt{tu}\sqrt{(1-a)(1-b)}} \\
&\quad \times \bigg[\int \exp\bigg(-\frac{\big(y-k(s,t,x,z)\big)^2}{2\big(tu(1-a)(1-b)/(1-ab)u\big)} - \frac{\big(z-\sqrt{u/s}\sqrt{ab}x\big)^2}{2(1-ab)u} \bigg) dy \bigg] G_\sigma(da)G_\tau(db) \\
&= \int_{(0,1)} \int_{(0,1)} \frac{1}{\sqrt{2\pi}\sqrt{(1-ab)u}} \exp\bigg(-\frac{\big(z-\sqrt{u/s}\sqrt{ab}x\big)^2}{2(1-ab)u} \bigg) \\
&\quad \times \underbrace{\bigg[\int \frac{\sqrt{(1-ab)u}}{\sqrt{2\pi}\sqrt{tu}\sqrt{(1-a)(1-b)}} \exp\bigg(-\frac{\big(y-k(s,t,x,z)\big)^2}{2tu(1-a)(1-b)/(1-ab)u} \bigg) dy \bigg]}_{=1} G_\sigma(da)G_\tau(db),
\end{aligned} \tag{2.13}$$

where $k(s,t,x,z)$ is some quantity, given by (2.12), that does not depend on the integrating variable y.

It follows that

$$
\begin{aligned}
&\int p_{s,t}(x,y)p_{t,u}(y,z)dy \\
&\quad= \int_{(0,1)}\int_{(0,1)} \frac{1}{\sqrt{2\pi}\sqrt{(1-ab)u}} \exp\left(-\frac{(z-\sigma\tau\sqrt{ab}x)^2}{2(1-ab)u}\right) G_\sigma(da)G_\tau(db) \\
&\quad= \int_{(0,1)}\int_{(0,1)} \phi(\sigma\tau\sqrt{ab}x,(1-ab)u,z)\, G_\sigma(da)G_\tau(db) \\
&\quad= \int_{(0,1)} \phi(\sigma\tau\sqrt{r}x,(1-r)u,z)\, G_{\sigma\tau}(dr) - \gamma(\sigma)\int_{(0,1)} \phi(\sigma\tau\sqrt{r}x,(1-r)u,z)\, G_\tau(dr) \\
&\qquad - \gamma(\tau)\int_{(0,1)} \phi(\sigma\tau\sqrt{r}x,(1-r)u,z)\, G_\sigma(dr) \\
&\quad= p_{s,u}(x,z) - \gamma(\sigma)p_{t,u}(\sigma x,z) - \gamma(\tau)\int_{(0,1)} \phi(\sigma\tau\sqrt{r}x,(1-r)u,z)\, G_\sigma(dr).
\end{aligned}
\tag{2.14}
$$

Therefore, continuing from (2.10),

$$
\begin{aligned}
&\int P_{s,t}(x,dy)\int P_{t,u}(y,dz)h(z) \\
&\quad= \gamma(\sigma\tau)h(\sigma\tau x) + \gamma(\sigma)\int h(z)p_{t,u}(\sigma x,z)dz + \gamma(\tau)\int h(\tau y)p_{s,t}(x,y)dy \\
&\qquad + \int h(z)p_{s,u}(x,z)dz - \gamma(\sigma)\int h(z)p_{t,u}(\sigma x,z)dz \\
&\qquad - \gamma(\tau)\int h(z)\int_{(0,1)} \phi(\sigma\tau\sqrt{r}x,(1-r)u,z)\, G_\sigma(dr)dz \\
&\quad= \gamma(\sigma\tau)h(\sigma\tau x) + \int h(z)p_{s,u}(x,z)dz,
\end{aligned}
\tag{2.15}
$$

where in the last equality, we have used the change of variables $z=\tau y$ to show that

$$
\int h(\tau y)\int_{(0,1)} \phi(\sigma\sqrt{r}x,(1-r)t,y)\, G_\sigma(dr)dy = \int h(z)\int_{(0,1)} \phi(\sigma\tau\sqrt{r}x,(1-r)u,z)\, G_\sigma(dr)dz.
\tag{2.16}
$$

Equation (2.4) immediately follows. Equation (2.5) is shown in a similar way. □

The convolution semigroup K in Proposition 2.2 defines a subordinator (process with positive, independent and stationary increments, i.e., an increasing Lévy process). The proof of the following proposition uses this observation and is a straightforward application of the classical Lévy-Khinchin Theorem on subordinators (see, e.g., [3, Section 1.2]). It is left to the reader.

Proposition 2.4. *Let the family $(G_\sigma)_{\sigma\geq 1}$ be a log-convolution semigroup on $(0,1]$. Define, for R_σ with distribution G_σ, $U_\sigma = -\ln R_\sigma$, and let $L_\sigma(\lambda) = \mathbb{E}[e^{-\lambda U_\sigma}](= \mathbb{E}[e^{\lambda \ln R_\sigma}] = \mathbb{E}[(R_\sigma)^\lambda])$ be the Laplace transform of the (positive) random variable U_σ. Then for any $\sigma \geq 1$, U_σ is infinitely divisible. Moreover,*

$$\ln L_\sigma(\lambda) = -\left[\beta\lambda + \int_0^\infty (1-e^{-\lambda x})\nu(dx)\right]\ln\sigma, \tag{2.17}$$

where the Lévy measure $\nu(dx)$ satisfies $\nu(\{0\}) = 0$ and $\int_0^\infty (1\wedge x)\nu(dx) < \infty$.

Conversely, any function L_σ of the form (2.17) is the λ-moments of a log-convolution semigroup $(G_\sigma)_{\sigma\geq 1}$.

In what follows, we denote by ψ the so-called Laplace exponent of the log-convolution semigroup $(G_\sigma)_{\sigma\geq 1}$:

$$\psi(\lambda) = \beta\lambda + \int_0^\infty (1-e^{-\lambda x})\nu(dx). \tag{2.18}$$

As observed earlier, the requirement that X be a martingale translates into condition (1.7) which in turn, taking $\lambda = 1/2$ in (2.17), reduces to

$$\psi\left(\frac{1}{2}\right) = 1. \tag{2.19}$$

Now we finalize the construction of the process X. Starting from a function ψ of the form (2.18) which satisfies (2.19), we construct the family G_σ and the transition probability function $P_{s,t}(x,dy)$ given in (2.3). We conclude by invoking the Chapman-Kolmogorov existence result (see, e.g., [4, Theorem 1.5]) and state the main theorem of this paper.

Theorem 2.5. *Let the family $(G_\sigma)_{\sigma\geq 1}$ form a log-convolution semigroup with Laplace exponent*

$$\psi(\lambda) = \beta\lambda + \int_0^\infty (1-e^{-\lambda x})\nu(dx). \tag{2.20}$$

Assume that $\psi(1/2) = 1$. Then the coordinate process starting at zero, hereby denoted $(X_t)_{t\geq 0}$, is a Markov martingale with respect to its natural filtration $(\mathcal{F}_t)_{t\geq 0}$ and with transition probabilities $P_{s,t}(x,dy)$ given by (2.3). Furthermore, the marginal distributions of X_t are Gaussian with mean zero and variance t and, for $0 < s < t$,

$$X_t = \sqrt{\frac{t}{s}}\left(\sqrt{R_{s,t}}X_s + \sqrt{s}\sqrt{1-R_{s,t}}\,\xi_{s,t}\right), \tag{2.21}$$

where $R_{s,t}$ and $\xi_{s,t}$ are independent of each other and of $\mathcal{F}_s$, $R_{s,t}$ has distribution $G_{\sqrt{t/s}}$ and $\xi_{s,t}$ is standard Gaussian.

3. Path properties

As a martingale X_t admits a càdlàg version. In the sequel, we assume that X_t itself is càdlàg.

THEOREM 3.1. *The process X_t is continuous in probability*

$$\forall c > 0, \quad \lim_{s\to t} \mathbb{P}\big[\,|X_t - X_s| > c\big] = 0. \tag{3.1}$$

Proof. Using Lemma 3.2 below, we write

$$\mathbb{P}\big[\,|X_t - X_s| > c\big] \le \frac{1}{c^2}\mathbb{E}\big[(X_t - X_s)^2\big] = \frac{1}{c^2}\big[t - t^{1-\delta}s^{\delta} + t^{1-\delta}s^{\delta} - s\big] = \frac{t-s}{c^2}. \tag{3.2}$$

□

LEMMA 3.2. *Let $\delta = \psi(1)/2$ so that $L_\sigma(1) = \sigma^{-2\delta}$. Then*

$$\mathbb{E}\big[(X_t - X_s)^2 \mid X_s\big] = t - t^{1-\delta}s^{\delta} + t^{1-\delta}s^{-1+\delta}X_s^2 - X_s^2. \tag{3.3}$$

Proof. Using representation (2.21), we see that, with $\sigma = \sqrt{t/s}$,

$$\begin{aligned}
\mathbb{E}\big[(X_t - X_s)^2 \mid X_s\big] &= \mathbb{E}\big[\mathbb{E}\big[(X_t - X_s)^2 \mid X_s, R_{s,t}\big] \mid X_s\big] \\
&= t\mathbb{E}\big[1 - R_{s,t}\big] + \mathbb{E}\Big[\Big(\sigma\sqrt{R_{s,t}} - 1\Big)^2\Big]X_s^2 \\
&= t\big(1 - L_\sigma(1)\big) + \big(\sigma^2\mathbb{E}[R_{s,t}] - 1\big)X_s^2 \\
&= t\big(1 - L_\sigma(1)\big) + \big(\sigma^2 L_\sigma(1) - 1\big)X_s^2 \\
&= t - s\sigma^{2-2\delta} + \sigma^{2-2\delta}X_s^2 - X_s^2.
\end{aligned} \tag{3.4}$$

□

THEOREM 3.3. *The (predictable) quadratic variation of X_t is*

$$\langle X, X\rangle_t = \delta t + (1-\delta)\int_0^t \frac{X_s^2}{s}\,ds, \tag{3.5}$$

where $\delta = \psi(1)/2$. Furthermore, it can be obtained as a limit,

$$\langle X, X\rangle_t = \lim_{n\to\infty}\sum_{k=0}^{n-1}\mathbb{E}\Big[\big(X_{t_{k+1}} - X_{t_k}\big)^2 \mid X_{t_k}\Big] \tag{3.6}$$

in L^2, where $t_0 < t_1 < \cdots < t_n$ is a subdivision of $[0,t]$.

Proof. First note that X_t is a square integrable martingale on any finite interval $[0,T]$. In fact, since $\mathbb{E}[X_t^2] = t$, $\sup_{t\le T}\mathbb{E}[X_t^2] = T$.

To obtain the first statement, we show that $X_t^2 - \delta t - (1-\delta)\int_0^t (X_u^2/u)\,du$ is a martingale, thus establishing that $\delta t + (1-\delta)\int_0^t (X_s^2/s)\,ds$ is the predictable quadratic variation of X_t (see [5, Theorem 4.2]).

Now, with $\sigma = \sqrt{t/s}$,

$$\mathbb{E}[X_t^2 \mid \mathcal{F}_s] = \mathbb{E}[t(1-R_{s,t}) + \sigma^2 R_{s,t} X_s^2 \mid X_s] = t(1-L_\sigma(1)) + \sigma^2 L_\sigma(1) X_s^2. \tag{3.7}$$

Since $L_\sigma(1) = \sigma^{-2\delta} = s^\delta t^{-\delta}$, we find

$$\mathbb{E}[X_t^2 \mid \mathcal{F}_s] = t - t^{1-\delta}s^\delta + t^{1-\delta}s^{-1+\delta}X_s^2. \tag{3.8}$$

It follows that

$$\begin{aligned}
&\mathbb{E}\left[(1-\delta)\int_0^t \frac{X_u^2}{u}du \mid \mathcal{F}_s\right] \\
&\quad = (1-\delta)\int_0^s \frac{X_u^2}{u}du + (1-\delta)\int_s^t \left(1 - u^{-\delta}s^\delta + u^{-\delta}s^{-1+\delta}X_s^2\right)du \\
&\quad = (1-\delta)\int_0^s \frac{X_u^2}{u}du + (1-\delta)(t-s) - s^\delta\left(1 - s^{-1}X_s^2\right)\left(t^{1-\delta} - s^{1-\delta}\right) \\
&\quad = (1-\delta)\int_0^s \frac{X_u^2}{u}du + (1-\delta)(t-s) - s^\delta t^{1-\delta} + s + t^{1-\delta}s^{-1+\delta}X_s^2 - X_s^2, \\
&\mathbb{E}\left[X_t^2 - \delta t - (1-\delta)\int_0^t \frac{X_u^2}{u}du \mid \mathcal{F}_s\right] \\
&\quad = t - t^{1-\delta}s^\delta + t^{1-\delta}s^{-1+\delta}X_s^2 - \delta t - (1-\delta)\int_0^s \frac{X_u^2}{u}du - (1-\delta)(t-s) \\
&\qquad + s^\delta t^{1-\delta} - s - t^{1-\delta}s^{-1+\delta}X_s^2 + X_s^2 \\
&\quad = t - \delta t - (1-\delta)\int_0^s \frac{X_u^2}{u}du - (1-\delta)(t-s) - s + X_s^2 \\
&\quad = X_s^2 - \delta s - (1-\delta)\int_0^s \frac{X_u^2}{u}du.
\end{aligned} \tag{3.9}$$

Since $\langle X, X\rangle_t$ is continuous and

$$\begin{aligned}
\mathbb{E}[\langle X, X\rangle_t^2] &= \delta^2 t^2 + 2\delta(1-\delta)t^2 + (1-\delta)^2\mathbb{E}\left[\left(\int_0^t \frac{X_s^2}{s}ds\right)^2\right] \\
&\leq \delta^2 t^2 + 2\delta(1-\delta)t^2 + (1-\delta)^2 t\int_0^t \frac{\mathbb{E}[X_s^4]}{s^2}ds \\
&< +\infty,
\end{aligned} \tag{3.10}$$

it follows, by application of the dominated convergence theorem, that

$$\langle X, X\rangle_t = \lim_{n\to\infty}\sum_{k=0}^{n-1}\mathbb{E}[\langle X, X\rangle_{t_{k+1}} - \langle X, X\rangle_{t_k} \mid \mathcal{F}_{t_k}] \tag{3.11}$$

in L^2, where $t_0 < t_1 < \cdots < t_n$ is a subdivision of $[0,t]$. The second statement of the theorem now follows from the fact that

$$\mathbb{E}\left[\left(X_{t_{k+1}} - X_{t_k}\right)^2 \mid X_{t_k}\right] = \mathbb{E}\left[\langle X,X\rangle_{t_{k+1}} - \langle X,X\rangle_{t_k} \mid \mathcal{F}_{t_k}\right]. \tag{3.12}$$

□

The next result states that the only continuous process that can be constructed in the way described in Section 2 is the Brownian motion.

Theorem 3.4. *The process X_t is quasi-left-continuous. It is continuous if and only if $G_\sigma \equiv \varepsilon_{\sigma^{-2}}$ (i.e., $R_{s,t} \equiv s/t$), in which case X_t is a standard Brownian motion.*

Proof. The quasi-left continuity of X_t immediately follows from the continuity of $\langle X,X\rangle_t$ (see [5, Theorem I.4.2, page 38]). Obviously, if $R_{s,t} \equiv s/t$ so that X_t is a Brownian motion, then it must be continuous. Conversely, if X_t is continuous, then, Itô's formula for $e^{i\lambda X_t}$ gives

$$e^{i\lambda X_t} = 1 + M_t - \frac{\lambda^2}{2}\int_0^t e^{i\lambda X_s}\, d\langle X,X\rangle_s, \tag{3.13}$$

where $M_t = \int_0^t i\lambda e^{i\lambda X_s}\, dX_s$ is a true martingale. In fact,

$$\begin{aligned}
\mathbb{E}\left[\,\left|\langle M,M\rangle_t\right|\,\right] &= \mathbb{E}\left[\,\left| -\int_0^t \lambda^2 e^{i2\lambda X_s}\, d\langle X,X\rangle_s \right|\,\right] \\
&= \mathbb{E}\left[\,\left| -\int_0^t \lambda^2 e^{i2\lambda X_s}\left(\delta ds + (1-\delta)\frac{X_s^2}{s}ds\right)\right|\,\right] \\
&\le \delta\lambda^2 t + (1-\delta)\lambda^2 \int_0^t \frac{\mathbb{E}\left[X_s^2\right]}{s}ds \\
&= \delta\lambda^2 t + (1-\delta)\lambda^2 t \\
&= \lambda^2 t.
\end{aligned} \tag{3.14}$$

Taking expectations in (3.13), we obtain that $\theta(\lambda,t) = \mathbb{E}[e^{i\lambda X_t}] = e^{-\lambda^2 t/2}$ must satisfy

$$\begin{aligned}
\theta(\lambda,t) &= 1 - \frac{\lambda^2}{2}\left[\delta\int_0^t \theta(\lambda,s)ds + (1-\delta)\int_0^t \mathbb{E}\left[X_s^2 e^{i\lambda X_s}\right]\frac{ds}{s}\right] \\
&= 1 - \frac{\lambda^2}{2}\left[\delta\int_0^t \theta(\lambda,s)ds - (1-\delta)\int_0^t \frac{\partial^2\theta}{\partial\lambda^2}(\lambda,s)\frac{ds}{s}\right].
\end{aligned} \tag{3.15}$$

Differentiating in t, we get that $\theta(\lambda,t)$ must satisfy

$$-\frac{\lambda^2}{2}\theta(\lambda,t) = -\frac{\lambda^2}{2}\left[\delta\theta(\lambda,t) - \frac{1-\delta}{t}\frac{\partial^2\theta}{\partial\lambda^2}(\lambda,t)\right], \tag{3.16}$$

that is,

$$-\frac{\lambda^2}{2} = -\frac{\lambda^2}{2}\left[\delta - (1-\delta)(\lambda^2 t - 1)\right]. \tag{3.17}$$

This, of course, can only occur if $\delta = 1$, which corresponds to $L_\sigma(1) = \sigma^{-2}$ and $R_{s,t}$ being nonrandom equal to s/t. □

X_t being quasi-left-continuous, $\Delta X_T = 0$ (*a.s.*) for every finite predictable time T. In particular, X_t does not have any fixed points of discontinuity. One of the aims of the constructions given in the following section is to describe the jumps of the process X_t.

4. Explicit constructions

Before we engage in the explicit construction of the processes outlined in the previous sections, let us observe that they fall into one of two classes according to whether or not $G_\sigma(\{1\})$ is nil, uniformly in $\sigma > 1$.

Indeed, if R_σ has distribution G_σ, then

$$\begin{aligned} L_\sigma(\lambda) &= \mathbb{E}\left[(R_\sigma)^\lambda\right] = \mathbb{P}\left[R_\sigma = 1\right] + \mathbb{E}\left[(R_\sigma)^\lambda, R_\sigma < 1\right], \\ \gamma(\sigma) &= \mathbb{P}\left[R_\sigma = 1\right] = \lim_{\lambda\uparrow\infty} L_\sigma(\lambda) = \lim_{\lambda\uparrow\infty} \exp\left(-\psi(\lambda)\ln\sigma\right). \end{aligned} \tag{4.1}$$

That is, uniformly in $\sigma > 1$,

$$\gamma(\sigma) = 0 \Longleftrightarrow \lim_{\lambda\uparrow\infty}\psi(\lambda) = +\infty. \tag{4.2}$$

4.1. The case $\gamma(\sigma) > 0$. In this section, we apply our construction to the case where $\gamma(\sigma) = G_\sigma(\{1\}) > 0$. The processes thus obtained are piecewise deterministic pure jump processes in the sense that between any two consecutive jumps, the process behaves according to a deterministic function. Examples of such processes include the case where G_σ is an inverse log-Poisson distribution.

The interpretation of these processes as piecewise deterministic pure jump processes requires the computation of the infinitesimal generator.

Proposition 4.1. *Let G_σ be a log-convolution semigroup. Assume that $\gamma(\sigma) = G_\sigma(\{1\}) > 0$, γ is differentiable at* 1 *and* $\lim_{\lambda\downarrow 0}\psi(\lambda) = 0$. *Then the infinitesimal generator of X_t on the set of C_0^2-functions is given by*

$$\begin{aligned} A_0 f(x) &= \frac{1}{2} f''(x), \\ A_s f(x) &= \frac{x}{2s} f'(x) + \frac{-\gamma'(1)}{2s}\int\left[f(x+z) - f(x)\right] \\ &\quad\times \int_{(0,1)} \phi\left((\sqrt{r}-1)x, s(1-r), z\right)\overline{G}(dr)\,dz \quad \text{for } s > 0, \end{aligned} \tag{4.3}$$

where

$$\overline{G}(dr) = \lim_{\sigma\downarrow 1} \frac{G_\sigma(dr \cap (0,1))}{G_\sigma((0,1))} \tag{4.4}$$

is a probability measure on $(0,1)$, *and the limit is understood in the weak sense.*

Thus the process X starts off as a Brownian motion ($A_0 f(x) = (1/2)f''(x)$) and, when in x at time s, drifts at the rate of $x/(2s)$, and jumps at the rate of $-\gamma'(1)/(2s)$. The size of the jump from x has density $\int_{(0,1)} \phi((\sqrt{r}-1)x, s(1-r), z)\overline{G}(dr)$, the mean of which is $\int_{(0,1)}(\sqrt{r}-1)\overline{G}(dr)x$ (see [6, Section 4.2] for a detailed study of Markov jump processes). In other words, while in positive territory, X_t continuously drifts upwards and has jumps that tend to be negative. In negative region, the reverse occurs; X_t drifts downwards and has (on average) positive jumps.

Proof. First note that the conditional moment generating function of U_σ given $U_\sigma > 0$ is

$$L_\sigma^*(\lambda) = \frac{L_\sigma(\lambda) - \gamma(\sigma)}{1-\gamma(\sigma)} \tag{4.5}$$

and converges to

$$\lim_{\sigma\downarrow 1} L_\sigma^*(\lambda) = 1 + \frac{\psi(\lambda)}{\gamma'(1)}. \tag{4.6}$$

By the (Laplace) continuity theorem, if $\lim_{\lambda\downarrow 0}\psi(\lambda) = 0$, then there exists a probability measure on $(0,1)$, $\overline{G}(dr)$, such that

$$\overline{G}(dr) = \lim_{\sigma\downarrow 1} \frac{G_\sigma(dr\cap(0,1))}{G_\sigma((0,1))}. \tag{4.7}$$

Next, with $\sigma = \sqrt{t/s}$,

$$\begin{aligned}
&\frac{1}{t-s}(\mathbb{E}[f(X_t) \mid X_s = x] - f(x)) \\
&\quad = \frac{1}{s}\left[\frac{f(\sigma x)\gamma(\sigma) - f(x)}{\sigma^2 - 1} + \frac{1}{\sigma^2-1}\int f(y)\int_{(0,1)} \phi(\sigma\sqrt{r}x, t(1-r), y)G_\sigma(dr)dy\right] \\
&\quad = \frac{1}{s}\left[\frac{f(\sigma x)\gamma(\sigma) - f(x)}{\sigma^2 - 1} + \frac{1-\gamma(\sigma)}{\sigma^2-1}\int f(y)\int_{(0,1)} \phi(\sigma\sqrt{r}x, t(1-r), y)\frac{G_\sigma(dr)}{1-\gamma(\sigma)}dy\right].
\end{aligned} \tag{4.8}$$

Letting σ decrease to 1, we see that

$$\begin{aligned}
A_s f(x) &= \lim_{t\downarrow s}\frac{1}{t-s}(\mathbb{E}[f(X_t) \mid X_s = x] - f(x)) \\
&= \frac{1}{s}\left[\frac{xf'(x) + \gamma'(1)f(x)}{2} - \frac{\gamma'(1)}{2}\int f(y)\int_{(0,1)}\phi(\sqrt{r}x, s(1-r), y)\overline{G}(dr)dy\right] \\
&= \frac{x}{2s}f'(x) + \frac{-\gamma'(1)}{2s}\int [f(y) - f(x)]\int_{(0,1)}\phi(\sqrt{r}x, s(1-r), y)\overline{G}(dr)dy \\
&= \frac{x}{2s}f'(x) + \frac{-\gamma'(1)}{2s}\int [f(x+z) - f(x)]\int_{(0,1)}\phi((\sqrt{r}-1)x, s(1-r), z)\overline{G}(dr)dz.
\end{aligned} \tag{4.9}$$

□

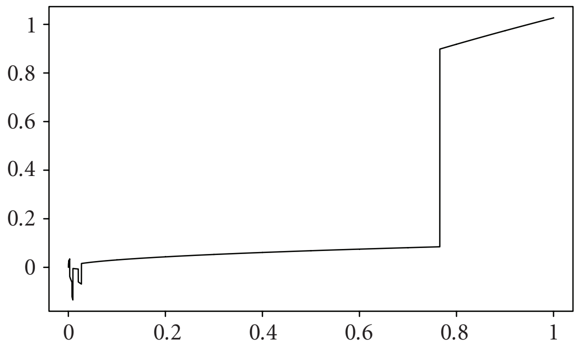

Figure 4.1

Note that the domain of A_s can be extended to include functions that do not vanish at infinity, such as $f(x) = x^2$. Indeed by Theorem 3.3, $g_s(x) = \delta + (1-\delta)x^2/s$ solves the martingale problem for $f(x) = x^2$.

The next proposition immediately follows from the observation that the process X does not jump between times s and t if and only if $X_u = \sqrt{u/s}X_s$ for $u \in (s,t)$.

Proposition 4.2. *Let T_s denote the first jump time after $s > 0$. Then, for any $t > s$,*

$$\mathbb{P}[T_s > t] = \gamma(\sigma), \tag{4.10}$$

where $\sigma = \sqrt{t/s}$.

4.2. The Poisson case $\gamma(\sigma) = \sigma^{-c}$. In this case, $\beta = 0$, $\nu(dx) = c\delta_1(dx)$ with $c = 1/(1-e^{-1/2})$, and $\psi(\lambda) = c(1-e^{-\lambda})$ (see [3, Section 1.2]). In other words, $U_\sigma = -\ln R_\sigma$ has a Poisson distribution with mean $c\ln\sigma$.

The assumptions of Proposition 4.1 are clearly satisfied with $\gamma(\sigma) = \sigma^{-c}$, $\gamma'(1) = -c$, $\lim_{\sigma\downarrow 1} L_\sigma^*(\lambda) = e^{-\lambda}$, and $\overline{G}(dr) = \varepsilon_{e^{-1}}(dr)$, so that X_t has infinitesimal generator

$$A_s f(x) = \frac{x}{2s} f'(x) + \frac{c}{2s}\int [f(x+z) - f(x)]\phi(-x/c, s(1-e^{-1}), z)\,dz. \tag{4.11}$$

It jumps at the rate of $c/2s$ with a size distributed as a Gaussian random variable with mean $-x/c$ and variance $s(1-e^{-1})$. Figure 4.1 shows a simulation of a path of such a process.

Furthermore, the law of the first jump time after s is given by

$$\mathbb{P}[T_s > t] = \gamma\left(\sqrt{\frac{t}{s}}\right) = \frac{s^{c/2}}{t^{c/2}}. \tag{4.12}$$

In other words, T_s is Pareto distributed (with location parameter s and scale parameter $c/2 \sim 1.27$). In particular,

$$\mathbb{E}[T_s] = \frac{cs}{c-2}, \qquad \mathbb{E}[T_s^2] = \infty. \tag{4.13}$$

4.3. The case $\gamma(\sigma) = 0$. We give the infinitesimal generator for functions of a specific type, which include polynomials. But for specific cases, such as the gamma case, the generator is given for much wider class of functions.

PROPOSITION 4.3. *Assume that $\beta = 0$ so that*

$$\psi(\lambda) = \int_0^\infty (1 - e^{-\lambda x})\nu(dx). \tag{4.14}$$

Let f be a C^1-function with the following property: there exist a function N_f and a (signed) finite measure M_f such that

$$\begin{gathered} f(\sigma e^{-u/2}x + \sigma\sqrt{s}\sqrt{1-e^{-u}}z) = N_f(\sigma)\int_0^\infty e^{-\lambda u} M_f(s,x,z,d\lambda), \quad u > 0, \\ \lim_{\sigma\downarrow 1} N_f(\sigma) = 1. \end{gathered} \tag{4.15}$$

Then, for any $s > 0$,

$$A_s f(x) = \frac{x}{2s} f'(x) + \frac{1}{2s}\int [f(x+y) - f(x)] \int_0^{+\infty} \phi((e^{-\omega/2} - 1)x, s(1 - e^{-\omega}), y)\nu(d\omega)dy. \tag{4.16}$$

Proof. Let

$$C_\sigma f(u) = C_\sigma f(s,x,z,u) = f(\sigma e^{-u/2}x + \sigma\sqrt{s}\sqrt{1-e^{-u}}z) \tag{4.17}$$

and $t = \sigma^2 s$. Then, since $\gamma(\sigma) = 0$, U_σ is almost surely strictly positive and

$$\begin{aligned} &\frac{1}{t-s}(\mathbb{E}[f(X_t) \mid X_s = x] - f(x)) \\ &\quad = \frac{1}{s}\frac{1}{\sigma^2 - 1}\int (\mathbb{E}[f(\sigma e^{-U_\sigma/2}x + \sqrt{t}\sqrt{1-e^{-U_\sigma}}z)] - f(x))\phi(z)dz \\ &\quad = \frac{1}{s}\frac{1}{\sigma^2 - 1}\left\{\int (\mathbb{E}[C_\sigma f(U_\sigma)] - C_\sigma f(0))\phi(z)dz + (f(\sigma x) - f(x))\right\} \\ &\quad = \frac{1}{s}\frac{1}{\sigma^2 - 1}\left\{\int \mathbb{E}[N_f(\sigma)\int_0^\infty (e^{-\lambda U_\sigma} - 1)M_f(d\lambda)]\phi(z)dz + (f(\sigma x) - f(x))\right\} \\ &\quad = \frac{1}{s}\frac{1}{\sigma^2 - 1}\left\{N_f(\sigma)\int\int_0^\infty (e^{-\psi(\lambda)\ln\sigma} - 1)M_f(d\lambda)\phi(z)dz + (f(\sigma x) - f(x))\right\} \\ &\quad = \frac{1}{s}\frac{1}{\sigma + 1}\left\{\frac{N_f(\sigma)\ln\sigma}{\sigma - 1}\int\int_0^\infty \frac{e^{-\psi(\lambda)\ln\sigma} - 1}{\ln\sigma} M_f(d\lambda)\phi(z)dz + \frac{f(\sigma x) - f(x)}{\sigma - 1}\right\}. \end{aligned} \tag{4.18}$$

Taking the limit as $\sigma \downarrow 1$ (i.e., $t \downarrow s$), we get

$$A_s f(x) = \frac{x}{2s} f'(x) - \frac{1}{2s} \int \int_0^\infty \psi(\lambda) M_f(s,x,z,d\lambda)\phi(z)dz. \tag{4.19}$$

Since

$$\psi(\lambda) = \int_0^\infty (1 - e^{-\lambda\omega})\nu(d\omega),$$

$$\begin{aligned} A_s f(x) &= \frac{x}{2s} f'(x) - \frac{1}{2s} \int \int_0^\infty \left[\int_0^\infty (1 - e^{-\lambda\omega})\nu(d\omega) \right] M_f(s,x,z,d\lambda)\phi(z)dz \\ &= \frac{x}{2s} f'(x) - \frac{1}{2s} \int \int_0^\infty \left[\int_0^\infty (1 - e^{-\lambda\omega}) M_f(s,x,z,d\lambda) \right] \nu(d\omega)\phi(z)dz \\ &= \frac{x}{2s} f'(x) + \frac{1}{2s} \int \int_0^\infty \left[f(e^{-\omega/2}x + \sqrt{s}\sqrt{1-e^{-\omega}}z) - f(x) \right] \nu(d\omega)\phi(z)dz, \end{aligned} \tag{4.20}$$

and the proof is completed by a change of variables in z. □

Lemma 4.4. *Let* $f(x) = x^n$, *then*

$$f(\sigma e^{-u/2}x + \sigma\sqrt{s}\sqrt{1-e^{-u}}z) = \sigma^n \int_0^\infty e^{-\lambda u} M_f(s,x,z,d\lambda), \tag{4.21}$$

where

$$M_f(s,x,z,d\lambda) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{n!}{k!j!(n-k-j)!} (-1)^j x^k s^{(n-k)/2} z^{n-k} (\varepsilon_{k/2} * m_j)(d\lambda) \tag{4.22}$$

and $m_j(d\lambda)$ *is the* j*-order convolution of the probability measure*

$$m(d\lambda) = \frac{1}{2\sqrt{\pi}} \sum_{n=1}^{+\infty} \frac{\Gamma(n-1/2)}{n!} \varepsilon_n(d\lambda). \tag{4.23}$$

Proof. First, write the Taylor series of the (analytic on $(0,1)$) function $1-\sqrt{1-x}$,

$$1 - \sqrt{1-x} = \frac{1}{2} \sum_{n=1}^{+\infty} \frac{\Gamma(n-1/2)}{n!\Gamma(1/2)} x^n. \tag{4.24}$$

It immediately follows that

$$1 - \sqrt{1-e^{-u}} = \frac{1}{2} \sum_{n=1}^{+\infty} \frac{\Gamma(n-1/2)}{n!\Gamma(1/2)} e^{-nu} = \int_0^\infty e^{-\lambda u} m(d\lambda), \tag{4.25}$$

where $m(d\lambda) = (1/2\sqrt{\pi})\sum_{n=1}^{+\infty}(\Gamma(n-1/2)/n!)\varepsilon_n(d\lambda)$ is a probability measure. Now,

$$\begin{aligned}
&f(\sigma e^{-u/2}x+\sigma\sqrt{s}\sqrt{1-e^{-u}}z)\\
&\quad=\sigma^n\sum_{k=0}^{n}\binom{n}{k}e^{-ku/2}x^k s^{(n-k)/2}(1-e^{-u})^{(n-k)/2}z^{n-k}\\
&\quad=\sigma^n\sum_{k=0}^{n}\binom{n}{k}e^{-ku/2}x^k s^{(n-k)/2}\left[1-\left(1-\sqrt{1-e^{-u}}\right)\right]^{n-k}z^{n-k}\\
&\quad=\sigma^n\sum_{k=0}^{n}\sum_{j=0}^{n-k}\frac{n!}{k!j!(n-k-j)!}(-1)^j x^k s^{(n-k)/2}z^{n-k}e^{-ku/2}\left(1-\sqrt{1-e^{-u}}\right)^j.
\end{aligned}\tag{4.26}$$

The proof is ended by observing that

$$e^{-ku/2}\left(1-\sqrt{1-e^{-u}}\right)^j=\int_0^\infty e^{-\lambda u}\left(\varepsilon_{k/2}*m_j\right)(d\lambda).\tag{4.27}$$

□

The following theorem is now proven.

Theorem 4.5. *Assume that $\beta = 0$. For any polynomial f and any $s > 0$,*

$$\begin{aligned}
&A_s f(x)\\
&\quad=\frac{x}{2s}f'(x)+\frac{1}{2s}\int\left[f(x+y)-f(x)\right]\int_0^{+\infty}\phi\left(\left(e^{-\omega/2}-1\right)x,s\left(1-e^{-\omega}\right),y\right)\nu(d\omega)dy.
\end{aligned}\tag{4.28}$$

4.4. The gamma case $\gamma(\sigma) = 0$. Here, $\beta = 0$, $\nu(dx) = ax^{-1}e^{-bx}dx$ with $a = 1/\ln(1+(1/2b))$ and $\psi(\lambda) = a\ln(1+(\lambda/b))$ (see [3, Section 1.2]), that is, U_σ has a gamma distribution with density

$$h_\sigma(u)=\frac{b^{a\ln\sigma}}{\Gamma(a\ln\sigma)}u^{a\ln\sigma-1}e^{-bu},\quad u>0,\tag{4.29}$$

and R_σ has an inverse log-gamma distribution with density

$$g_\sigma(r)=\frac{b^{a\ln\sigma}}{\Gamma(a\ln\sigma)}(-\ln r)^{a\ln\sigma-1}r^{b-1},\quad 0<r<1.\tag{4.30}$$

See Figure 4.2 for a simulation of such a process.

In this case, it is possible to compute the generator for a much wider class of functions.

Proposition 4.6. *Let G_σ be the log-convolution semigroup of the inverse log-gamma distributions. Then (4.28) holds for any bounded function with bounded first derivative.*

Proof. In the proof of Proposition 4.3, we write

$$\frac{1}{t-s}\left(\mathbb{E}[f(X_t)\mid X_s=x]-f(x)\right)=\frac{1}{s}\frac{1}{\sigma^2-1}\int\left(\mathbb{E}[C_\sigma f(U_\sigma)]-f(x)\right)\phi(z)dz,\tag{4.31}$$

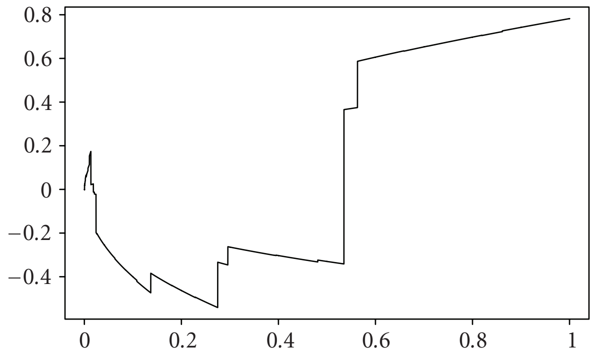

Figure 4.2

where $\sigma = \sqrt{t/s}$. Denote by $\theta(u)$ the quantity $e^{-u/2}x + \sqrt{s}\sqrt{1-e^{-u}}z$. Then, inserting $\mathbb{E}[C_1 f(U_\sigma)] = \mathbb{E}[f(\theta(U_\sigma))]$, we get

$$\begin{aligned} &\frac{1}{t-s}\left(\mathbb{E}[f(X_t) \mid X_s = x] - f(x)\right) \\ &\quad = \frac{1}{s}\frac{1}{\sigma+1}\int\left\{\frac{\mathbb{E}[C_\sigma f(U_\sigma)] - \mathbb{E}[C_1 f(U_\sigma)]}{\sigma-1} + \frac{\mathbb{E}[C_1 f(U_\sigma)] - f(x)}{\sigma-1}\right\}\phi(z)dz. \end{aligned} \tag{4.32}$$

Since

$$\frac{C_\sigma f(U_\sigma) - C_1 f(U_\sigma)}{\sigma-1} = \frac{f(\sigma\theta(U_\sigma)) - f(\theta(U_\sigma))}{\sigma-1} = \theta(U_\sigma)f'(\eta_\sigma), \tag{4.33}$$

for some η_σ between $\theta(U_\sigma)$ and $\sigma\theta(U_\sigma)$. θ and f' being bounded, we obtain that

$$\lim_{\sigma\downarrow 1}\int\frac{\mathbb{E}[C_\sigma f(U_\sigma)] - \mathbb{E}[C_1 f(U_\sigma)]}{\sigma-1}\phi(z)dz = xf'(x). \tag{4.34}$$

To compute the limit of the second term in (4.32), we use Lemma 4.7, which shows that

$$\begin{aligned} &\lim_{\sigma\downarrow 1}\int\frac{\mathbb{E}[C_1 f(U_\sigma)] - f(x)}{\sigma-1}\phi(z)dz \\ &\quad = a\int\int_0^\infty\frac{f(e^{-u/2}x + \sqrt{s}\sqrt{1-e^{-u}}z) - f(x)}{u}e^{-bu}du\phi(z)dz \\ &\quad = \int[f(x+y) - f(x)]\int_0^\infty\phi(x(e^{-u/2}-1), s(1-e^{-u}), y)a\frac{e^{-bu}}{u}du\,dy. \end{aligned} \tag{4.35}$$

□

Note that since $\nu((0,\infty)) = +\infty$, $\int_0^\infty\phi(x(e^{-u/2}-1), s(1-e^{-u}), y)\nu(du)$ cannot be rescaled to produce a density for the jumps of the process.

Lemma 4.7. *Let V_p have a gamma distribution with density*

$$h_p(v) = \frac{b^p}{\Gamma(p)} v^{p-1} e^{-bv}, \quad v > 0. \tag{4.36}$$

Let g be such that $g(v)/v$ is bounded. Then

$$\lim_{p \downarrow 0} \frac{1}{p} \mathbb{E}[g(V_p)] = \int_0^\infty \frac{g(v)}{v} e^{-bv} dv. \tag{4.37}$$

Proof. First observe that

$$\frac{1}{p} \mathbb{E}[g(V_p)] = \frac{1}{b} \mathbb{E}\left[\frac{g(V_{p+1})}{V_{p+1}}\right]. \tag{4.38}$$

Taking the limit as $p \downarrow 0$, we obtain by dominated convergence

$$\lim_{p \downarrow 0} \frac{1}{p} \mathbb{E}[g(V_p)] = \frac{1}{b} \mathbb{E}\left[\frac{g(V_1)}{V_1}\right] = \int_0^\infty \frac{g(v)}{v} e^{-bv} dv. \tag{4.39}$$

□

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