

Research Article

Unbounded Solutions of a Boundary Value Problem for Abstract n th-Order Differential Equations on an Infinite Interval

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The existence of unbounded nonnegative solutions of a boundary value problem for n th-order differential equations defined on an infinite interval is obtained by means of the Mönch fixed-point theorem. An example is then presented to demonstrate the application of our results.

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1. Introduction

Over the last few years, many researchers have focused their research on the study of boundary value problems for nonlinear differential and integral equations defined on an infinite interval, and various theoretical results have been obtained [1–8]. In [4], the existence of multiple positive solutions of a boundary value problem (BVP) for n th-order nonlinear impulsive integral-differential equations defined on an infinite interval in a Banach space is obtained by means of the fixed-point index theory of completely continuous operators. However, the result requires the use of the measures of noncompactness condition $\alpha(f(t, P_r, \dots, P_r)) = 0$ (where $P_r = \{x \in P : \|x\| \leq r\}$) and the normal and solid cone P in a real Banach space. In [6], by using the Mönch fixed-point theorem, a class of infinite boundary value problems for first-order impulsive differential equations in a Banach space is considered and the existence of positive solutions is obtained, but the solutions are limited to bounded solutions only.

To generalize and further develop the existing results in this field, in this paper we discuss the existence of unbounded solutions for a class of n th-order nonlinear differential equations defined on an infinite interval in a Banach space by using the Mönch fixed-point theorem under certain conditions weaker than those in [4]. The boundary value problem in question is

as follows:

$$\begin{aligned} u^{(n)}(t) &= f(t, u(t), u'(t), \dots, u^{(n-1)}(t)), \\ u^{(i)}(0) &= \beta_i u^{i+1}(0), \quad i = 0, 1, \dots, n-2 \\ u^{(n-1)}(+\infty) &= \beta u^{(n-1)}(0), \end{aligned} \quad (1.1)$$

where $J = [0, +\infty)$, $u^{(n-1)}(+\infty) = \lim_{t \rightarrow +\infty} u^{(n-1)}(t)$, $\beta > 1$, $\beta_i > 0$ ($i = 0, 1, \dots, n-2$), $f \in C[J \times E \times \dots \times E, E]$, in which $(E, \|\cdot\|)$ is a real Banach space.

Let $C[J, E]$ be the space of all continuous functions $u : J \rightarrow E$, and let $L[J, E]$ be the Banach space of all strongly measurable functions $u : J \rightarrow E$ with $\int_J \|u(t)\| dt < +\infty$, equipped with the norm $\|u\|_1 = \int_J \|u(t)\| dt$. Let $C_e[J, E] = \{u \in C[J, E] : e^{-t}\|u(t)\| \rightarrow 0 \text{ for } t \rightarrow +\infty\}$, then it is clear that $C_e[J, E]$ is a Banach space with norm

$$\|u\|_e = \sup_{t \in J} \{e^{-t}\|u(t)\|\}. \quad (1.2)$$

Let $D_e^{n-1}[J, E] = \{u \in C^{n-1}[J, E] : \lim_{t \rightarrow +\infty} e^{-t}\|u^{(i)}(t)\| \rightarrow 0, i = 0, 1, \dots, n-1\}$, then it is also easy to see that $D_e^{n-1}[J, E]$ is a Banach space with norm

$$\|u\|_D = \max \{\|u\|_e, \|u'\|_e, \dots, \|u^{(n-1)}\|_e\}. \quad (1.3)$$

Let P be a cone of the Banach space E , $C_e[J, P] = \{u \in C_e[J, E] : u(t) \geq \theta, t \in J\}$, and $D_e^{n-1}[J, P] = \{u \in D_e^{n-1}[J, E] : u^{(i)}(t) \geq \theta, t \in J, i = 0, 1, \dots, n-1\}$, where θ denotes the zero element of E . Then, it is obvious that $C_e[J, P]$ is a cone in space $C_e[J, E]$, and $D_e^{n-1}[J, P]$ is a cone in space $D_e^{n-1}[J, E]$.

Definition 1.1. A function $u \in D_e^{n-1}[J, P] \cap C^n[J, E]$ is called a nonnegative solution of BVP (1.1) if $u(t)$ satisfies (1.1) for $t \in J$.

The rest of the paper is organized as follows. In Section 2, we give some lemmas which provide a theoretical basis for the proof of our main results. The main theorem is presented and proved in Section 3. In Section 4, an example is given to demonstrate the application of our results.

2. Some lemmas

Here we first list some assumptions to be used throughout the rest of the paper.

(H₁) There exist $a(t)$ and $b_i(t) \in C[J, \mathbb{R}_+]$ such that $a(t), e^t b_i(t) \in L[J, \mathbb{R}_+]$ ($i = 0, 1, 2, \dots, n-1$) and

$$\begin{aligned} \|f(t, x_0, x_1, \dots, x_{n-1})\| &\leq a(t) + \sum_{i=0}^{n-1} b_i(t)\|x_i\| \quad \forall t \in J, x_i \in E, \\ \sum_{i=0}^{n-1} b_i^* &< \frac{\beta-1}{\beta^* + \beta'} \quad \beta^* = \max_{0 \leq j \leq n-2} \left\{ \prod_{i=j}^{n-2} \beta_i \right\}, \quad b_i^* = \int_0^{+\infty} e^t b_i(t) dt < +\infty, \quad i = 0, 1, 2, \dots, n-1. \end{aligned} \quad (2.1)$$

(H₂) There exists $l_i(t) \in C[J, \mathbb{R}_+]$ such that $e^t l_i(t) \in L[J, \mathbb{R}_+]$ ($i = 0, 1, \dots, n-1$),

$$\begin{aligned} \alpha(f(t, D_0, D_1, \dots, D_{n-1})) &\leq \sum_{i=0}^{n-1} l_i(t) \alpha(D_i) \quad \forall t \in J, D_i \subset E, \\ \frac{2(\beta^* + \beta)}{\beta - 1} \sum_{i=0}^{n-1} l_i &< 1, \quad l_i = \int_0^{+\infty} e^t l_i(t) dt < +\infty, \quad i = 0, 1, \dots, n-1, \end{aligned} \quad (2.2)$$

where $\alpha(\cdot)$ denotes the Kuratowski measure of noncompactness in E . For details on the definition and properties of the measure of noncompactness, the reader is referred to [9, 10].

In the following, we give various lemmas which are to be used for the proof of the main results to be presented in Section 3.

Lemma 2.1 (see [4]). *Let (H₁) be satisfied. Then $u \in D_e^{n-1}[J, P] \cap C^n[J, E]$ is a solution of BVP (1.1) if and only if $u \in D_e^{n-1}[J, P]$ is a solution of the following integral equation:*

$$\begin{aligned} u(t) &= \frac{1}{\beta - 1} \left[\sum_{j=0}^{n-2} \left(\prod_{i=j}^{n-2} \beta_i \right) \frac{t^j}{j!} + \frac{t^{n-1}}{(n-1)!} \right] \int_0^{+\infty} f(s, u(s), u'(s), \dots, u^{(n-1)}(s)) ds \\ &+ \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s, u(s), u'(s), \dots, u^{(n-1)}(s)) ds. \end{aligned} \quad (2.3)$$

Proof. If $u \in D_e^{n-1}[J, P] \cap C^n[J, E]$ is a solution of BVP (1.1), then by condition (H₁) we have the convergence of the infinite integral

$$\int_0^{+\infty} f(s, u(s), u'(s), \dots, u^{(n-1)}(s)) ds. \quad (2.4)$$

Integrating the first equation in (1.1) from 0 to t , we have

$$u^{(n-1)}(t) - u^{(n-1)}(0) = \int_0^t f(s, u(s), u'(s), \dots, u^{(n-1)}(s)) ds. \quad (2.5)$$

By virtue of $u^{(n-1)}(+\infty) = \beta u^{(n-1)}(0)$, let $t \rightarrow +\infty$ in (2.5), we get

$$u^{(n-1)}(0) = \frac{1}{\beta - 1} \int_0^{+\infty} f(s, u(s), u'(s), \dots, u^{(n-1)}(s)) ds. \quad (2.6)$$

From the second equation in (1.1), we have

$$u^{(i)}(0) = \beta_i u^{(i+1)}(0) = \beta_i \beta_{i+1} u^{(i+2)}(0) = \beta_i \beta_{i+1} \beta_{i+2} u^{(i+3)}(0) = \prod_{j=i}^{n-2} \beta_j u^{(n-1)}(0). \quad (2.7)$$

Substituting (2.6) into (2.7) yields for $i = 0, 1, \dots, n - 2$,

$$u^{(i)}(0) = \frac{1}{\beta - 1} \prod_{j=i}^{n-2} \beta_j \int_0^{+\infty} f(s, u(s), u'(s), \dots, u^{(n-1)}(s)) ds. \quad (2.8)$$

From (2.5) and (2.6), we have

$$u^{(n-1)}(t) = \frac{1}{\beta - 1} \int_0^{+\infty} f(s, u(s), u'(s), \dots, u^{(n-1)}(s)) ds + \int_0^t f(s, u(s), u'(s), \dots, u^{(n-1)}(s)) ds. \quad (2.9)$$

Integrating (2.9) from 0 to t and using (2.8) for $i = n - 2$, we have

$$\begin{aligned} & u^{(n-2)}(t) \\ &= u^{(n-2)}(0) + \frac{t}{\beta - 1} \int_0^{+\infty} f(s, u(s), u'(s), \dots, u^{(n-1)}(s)) ds + \int_0^t ds \int_0^s f(w, u(w), \dots, u^{(n-1)}(w)) dw \\ &= \left(\frac{\beta_{n-2}}{\beta - 1} + \frac{t}{\beta - 1} \right) \int_0^{+\infty} f(s, u(s), u'(s), \dots, u^{(n-1)}(s)) ds + \int_0^t (t - s) f(s, u(s), u'(s), \dots, u^{(n-1)}(s)) ds. \end{aligned} \quad (2.10)$$

It is not difficult to show by mathematical induction that u satisfies (2.3). Conversely, if $u \in D_e^{n-1}[J, P]$ is a solution of (2.3), then direct differentiation of (2.3) gives

$$\begin{aligned} u^{(i)}(t) &= \frac{1}{\beta - 1} \left[\sum_{j=i}^{n-2} \left(\prod_{s=j}^{n-2} \beta_s \right) \frac{t^{j-i}}{(j-i)!} + \frac{t^{n-i-1}}{(n-i-1)!} \right] \int_0^{+\infty} f(s, u(s), u'(s), \dots, u^{(n-1)}(s)) ds \\ &\quad + \frac{1}{(n-i-1)!} \int_0^t (t-s)^{n-i-1} f(s, u(s), u'(s), \dots, u^{(n-1)}(s)) ds, \quad i = 0, 1, \dots, n-2, \end{aligned} \quad (2.11)$$

$$\begin{aligned} u^{(n-1)}(t) &= \frac{1}{\beta - 1} \int_0^{+\infty} f(s, u(s), u'(s), \dots, u^{(n-1)}(s)) ds + \int_0^t f(s, u(s), u'(s), \dots, u^{(n-1)}(s)) ds, \\ u^{(n)}(t) &= f(t, u(t), u'(t), \dots, u^{(n)}(t)) \quad \forall t \in J. \end{aligned} \quad (2.12)$$

Consequently, $u \in C^n[J, E]$, and by (2.11) and (2.12), it is easy to see that $u(t)$ satisfies (1.1). The proof of Lemma 2.1 is completed. \square

We now consider an operator A defined by

$$(Au)(t) = \frac{1}{\beta - 1} \left[\sum_{j=0}^{n-2} \left(\prod_{s=j}^{n-2} \beta_s \right) \frac{t^j}{j!} + \frac{t^{n-1}}{(n-1)!} \right] \int_0^{+\infty} f(s, u(s), u'(s), \dots, u^{(n-1)}(s)) ds$$

$$+ \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s, u(s), u'(s), \dots, u^{(n-1)}(s)) ds \quad \forall t \in J. \quad (2.13)$$

By Lemma 2.1, $u \in D_e^{n-1}[J, P] \cap C^n[J, E]$ is a solution of BVP (1.1) if and only if u is a fixed point of the operator A in $u \in D_e^{n-1}[J, P]$.

Lemma 2.2 (see [10, 11, 13]). *Let E be a Banach space, let $J_0 = [a, b]$ be a finite interval, and let $H \subset C[J_0, E]$ be a countable set. Assume that there exists $\rho \in L[J_0, \mathbb{R}_+]$ such that $\|u(t)\| \leq \rho(t)$, $t \in J_0$, $u \in H$. Then $\alpha(\{u(t) : u \in H\}) \in L[J_0, \mathbb{R}_+]$ and*

$$\alpha\left(\left\{\int_{J_0} u(t) dt : u \in H\right\}\right) \leq 2 \int_{J_0} \alpha(\{u(t) : u \in H\}) dt. \quad (2.14)$$

Lemma 2.3. *Let E be a Banach space, and $H \subset C[J, E]$. If H is a countable set and there exists $\rho \in L[J, \mathbb{R}_+]$ such that $\|u(t)\| \leq \rho(t)$, $t \in J$, $u \in H$, then $\alpha(\{u(t) : u \in H\})$ is integrable on J , and*

$$\alpha\left(\left\{\int_0^{+\infty} u(t) dt : u \in H\right\}\right) \leq 2 \int_0^{+\infty} \alpha(\{u(t) : u \in H\}) dt. \quad (2.15)$$

Proof. By $\|u(t)\| \leq \rho(t)$ for any $u \in H$ and $t \in J$, we get $u(t) \in L[J, E]$ for all $u \in H$ and $\alpha(\{u(t) : u \in H\}) \leq 2\rho(t)$. As $\rho(t) \in L[J, \mathbb{R}_+]$, $\alpha(\{u(t) : u \in H\})$ is integrable in J . For any $\varepsilon > 0$, from $\rho(t) \in L[J, \mathbb{R}_+]$, there exists $T_0 > 0$ such that $\int_t^{+\infty} \rho(s) ds < \varepsilon$ for any $t > T_0$. So for any $t > T_0$ and $u \in H$, we have

$$\left\| \int_0^t u(s) ds - \int_0^{+\infty} u(s) ds \right\| = \left\| \int_t^{+\infty} u(s) ds \right\| < \int_t^{+\infty} \rho(s) ds < \varepsilon. \quad (2.16)$$

Let

$$S(t) = \left\{ \int_0^t u(s) ds : u \in H \right\}, \quad T = \left\{ \int_0^{+\infty} u(s) ds : u \in H \right\}. \quad (2.17)$$

By (2.16) for any $t > T_0$ and $u \in H$, we obtain

$$d\left(\int_0^t u(s) ds, T\right) = \inf_{u \in H} \left\{ \left\| \int_0^t u(s) ds - \int_0^{+\infty} u(s) ds \right\| \right\} \leq \left\| \int_0^t u(s) ds - \int_0^{+\infty} u(s) ds \right\| < \varepsilon, \quad (2.18)$$

where d denotes the distance of a point to a set. Using the same method, we have

$$d\left(\int_0^{+\infty} u(s)ds, S(t)\right) < \varepsilon, \quad t > T_0, u \in H. \quad (2.19)$$

So

$$\sup_{u \in H} d\left(\int_0^t u(s)ds, T\right) \leq \varepsilon, \quad \sup_{u \in H} d\left(\int_0^{+\infty} u(s)ds, S(t)\right) \leq \varepsilon. \quad (2.20)$$

Hence, for any $t > T_0$, we have

$$\begin{aligned} d_h\left(\left\{\int_0^t u(s)ds : u \in H\right\}, \left\{\int_0^{+\infty} u(s)ds : u \in H\right\}\right) \\ = \max\left\{\sup_{u \in H} d\left(\int_0^t u(s)ds, T\right), \sup_{u \in H} d\left(\int_0^{+\infty} u(s)ds, S(t)\right)\right\} \leq \varepsilon, \end{aligned} \quad (2.21)$$

where $d_h(D_1, D_2)$ denotes the Hausdorff distance of the sets D_1 and D_2 , that is,

$$d_h(D_1, D_2) = \max\left\{\sup_{x \in D_1} d(x, D_2), \sup_{x \in D_2} d(x, D_1)\right\}. \quad (2.22)$$

So, by (2.21), we obtain for any $t > T_0$,

$$\begin{aligned} \left|\alpha\left(\left\{\int_0^t u(s)ds : u \in H\right\}\right) - \alpha\left(\left\{\int_0^{+\infty} u(s)ds : u \in H\right\}\right)\right| \\ \leq 2d_h\left(\left\{\int_0^t u(s)ds : u \in H\right\}, \left\{\int_0^{+\infty} u(s)ds : u \in H\right\}\right) \leq 2\varepsilon. \end{aligned} \quad (2.23)$$

Hence, we have

$$\lim_{t \rightarrow +\infty} \alpha\left(\left\{\int_0^t u(s)ds : u \in H\right\}\right) = \alpha\left(\left\{\int_0^{+\infty} u(s)ds : u \in H\right\}\right). \quad (2.24)$$

By Lemma 2.3 for any $t \in [0, +\infty)$, we have

$$\alpha\left(\left\{\int_0^t u(s)ds : u \in H\right\}\right) \leq 2 \int_0^t \alpha(\{u(s) : u \in H\})ds. \quad (2.25)$$

Hence, from (2.24) and (2.25), we have

$$\begin{aligned} \alpha\left(\left\{\int_0^{+\infty} u(s)ds : u \in H\right\}\right) \\ = \lim_{t \rightarrow +\infty} \alpha\left(\left\{\int_0^t u(s)ds : u \in H\right\}\right) \leq 2 \lim_{t \rightarrow +\infty} \int_0^t \alpha(\{u(s) : u \in H\})ds = 2 \int_0^{+\infty} \alpha(\{u(s) : u \in H\})ds. \end{aligned} \quad (2.26)$$

Therefore, (2.15) is satisfied. The proof of Lemma 2.3 is completed. \square

Remark 2.4. Lemma 2.3 generalizes Lemma 2.2 from a finite interval to an infinite interval, and it plays an important role in studying the differential equations defined on an infinite interval. It should be emphasized that Lemma 2.3 has no counterpart in the existing literature.

Lemma 2.5 (see [2, Lemma 7]). *Let (H_1) be satisfied, and let $V = \{u_m\} \subset D_e^{n-1}[J, E]$ be a countable bounded set. Then*

$$\alpha_D(AV) = \max \left\{ \sup_{t \in J} [e^{-t} \alpha((AV)^{(i)}(t))] : i = 0, 1, \dots, n-1 \right\}, \quad (2.27)$$

where $\alpha_D(\cdot)$ denotes the Kuratowski measure of noncompactness in $D_e^{n-1}[J, E]$, $(AV)^{(i)} = \{(Au_m)^{(i)}(t) : u_m \in V, m = 1, 2, \dots\}$ ($i = 0, 1, 2, \dots, n-1$).

Lemma 2.6. *If condition (H_1) is satisfied, then the operator A is continuous and*

$$A(D_e^{n-1}[J, P]) \subset D_e^{n-1}[J, P]. \quad (2.28)$$

Proof. For any $u \in D_e^{n-1}[J, P]$, we have

$$\begin{aligned} (Au)(t) &= \frac{1}{\beta-1} \left[\sum_{j=0}^{n-2} \left(\prod_{s=j}^{n-2} \beta_s \right) \frac{t^j}{j!} + \frac{t^{n-1}}{(n-1)!} \right] \int_0^{+\infty} f((s), u(s), u'(s), \dots, u^{(n-1)}(s)) ds \\ &\quad + \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s, u(s), u'(s), \dots, u^{(n-1)}(s)) ds, \end{aligned} \quad (2.29)$$

$$\begin{aligned} (Au)^{(i)}(t) &= \frac{1}{\beta-1} \left[\sum_{j=i}^{n-2} \left(\prod_{s=j}^{n-2} \beta_s \right) \frac{t^{j-i}}{(j-i)!} + \frac{t^{n-i-1}}{(n-i-1)!} \right] \int_0^{+\infty} f(s, u(s), \dots, u^{(n-1)}(s)) ds \\ &\quad + \frac{1}{(n-i-1)!} \int_0^t (t-s)^{n-i-1} f((s), u(s), \dots, u^{(n-1)}(s)) ds, \quad i = 1, \dots, n-1, \end{aligned}$$

$$(Au)^{(n)}(t) = f(t, u(t), u'(t), \dots, u^{(n-1)}(t)). \quad (2.30)$$

So, by (2.30) and condition (H_1) for $i = 0, 1, \dots, n-1$, we have

$$\begin{aligned} e^{-t} \|(Au)^{(i)}(t)\| &\leq \frac{\beta^* + \beta}{\beta-1} e^{-t} \sum_{j=i}^{n-1} \frac{t^{j-i}}{(j-i)!} \int_0^{+\infty} \|f((s), u(s), u'(s), \dots, u^{(n-1)}(s))\| ds \\ &\leq \frac{\beta^* + \beta}{\beta-1} e^{-t} \sum_{j=i}^{n-1} \frac{t^{j-i}}{(j-i)!} \left(a^* + \sum_{i=0}^{n-1} b_i^* \|u\|_D \right), \end{aligned} \quad (2.31)$$

where $a^* = \int_0^{+\infty} a(t) dt$. Hence, $\lim_{t \rightarrow +\infty} e^{-t} \|(Au)^{(i)}(t)\| = 0$. Therefore, (2.28) is satisfied.

Now we show that the operator A is continuous. Let $u_m, \bar{u} \in D_e^{n-1}[J, P]$, $\|u_m - \bar{u}\|_D \rightarrow 0$. Then $r = \sup_m \|u_m\|_D < +\infty$ and $\|\bar{u}\| \leq r$. By the integrability of $a(t)$ and $e^t b_i(t)$ ($i = 0, 1, \dots, n-1$) for any $\varepsilon > 0$, there exists $T > 0$ such that

$$\frac{\beta^* + \beta}{\beta - 1} \left[2 \int_T^{+\infty} a(t) dt + 2rn \int_T^{+\infty} e^t b_i(t) dt \right] < \frac{\varepsilon}{2}. \quad (2.32)$$

On the other hand, by the continuity of f , it is easy to see that

$$\int_0^T \|f(s, u_m(s), \dots, u_m^{(n-1)}(s)) - f(s, \bar{u}(s), \dots, \bar{u}^{(n-1)}(s))\| ds \rightarrow 0, \quad m \rightarrow +\infty. \quad (2.33)$$

Hence, for the above $\varepsilon > 0$, there exists a natural number m_0 such that for any $m > m_0$,

$$\frac{\beta^* + \beta}{\beta - 1} \int_0^T \|f(s, u_m(s), \dots, u_m^{(n-1)}(s)) - f(s, \bar{u}(s), \dots, \bar{u}^{(n-1)}(s))\| ds < \frac{\varepsilon}{2}. \quad (2.34)$$

Thus, from (2.13), (2.32), (2.34), and (H_1) for any $m > m_0$, we obtain

$$\begin{aligned} \|Au_m - A\bar{u}\|_D &\leq \frac{\beta^* + \beta}{\beta - 1} \int_0^{+\infty} \|f(s, u_m(s), \dots, u_m^{(n-1)}(s)) - f(s, \bar{u}(s), \dots, \bar{u}^{(n-1)}(s))\| ds \\ &\leq \frac{\varepsilon}{2} + \frac{\beta^* + \beta}{\beta - 1} \left[2 \int_T^{+\infty} a(t) dt + \sum_{i=0}^{n-1} (\|u_m\|_D + \|\bar{u}\|_D) \int_T^{+\infty} e^t b_i(t) dt \right] \\ &\leq \frac{\varepsilon}{2} + \frac{\beta^* + \beta}{\beta - 1} \left[2 \int_T^{+\infty} a(t) dt + 2rn \int_T^{+\infty} e^t b_i(t) dt \right] < \varepsilon. \end{aligned} \quad (2.35)$$

Therefore, the continuity of A is proved. So, the proof of Lemma 2.6 is completed. \square

Lemma 2.7 ([10, 12] (Mönch)). *Let V be a closed and convex subset of E and $x_0 \in V$. Assume that operator $A : V \rightarrow V$ has the following property:*

$$C \subset V \text{ countable, } C \subset \overline{\text{co}}(\{x_0\} \cup A(V)) \implies V \text{ is relatively compact.} \quad (2.36)$$

Then A has a fixed point in V .

3. Main result

In this section, we present the main results we obtained.

Theorem 3.1. *Suppose (H_1) and (H_2) hold. Then BVP (1.1) has at least one nonnegative solution.*

Proof. Choose

$$R > a^* \left(\frac{\beta - 1}{\beta^* + \beta} - \sum_{i=0}^{n-1} b_i^* \right)^{-1}, \quad (3.1)$$

and let $V = \{u \in D_e^{n-1}[J, P] : \|u\|_D \leq R\}$. Obviously, V is a bounded convex closed set. For any $u \in V$, by (2.28), (H_1) and the definition of the norm $\|\cdot\|_e$, we have

$$\|(Au)^{(i)}\|_e = \sup_{t \in J} \{e^{-t} \|(Au)^{(i)}(t)\|\} \leq \frac{\beta^* + \beta}{\beta - 1} \left(a^* + \sum_{j=0}^{n-1} b_j^* \|u\|_D \right) \leq R, \quad i = 0, 1, \dots, n-1, \quad (3.2)$$

from which and the definition of the norm $\|\cdot\|_D$, we obtain

$$\|Au\|_D = \max \{ \|(Au)^{(i)}\|_e : 0 \leq i \leq n-1 \} \leq R. \quad (3.3)$$

Therefore, by Lemma 2.6, we conclude that A is a continuous operator from V to V .

Next, we prove that C is a relatively compact set if $C \subset V$ is a countable set satisfying $C \subset \overline{\text{co}}(\{u\} \cup AC)$ for some $u \in V$. From this and the properties of the Kuratowski measure of noncompactness in $D_e^{n-1}[J, E]$, we have

$$\alpha_D(C) \leq \alpha_D(AC). \quad (3.4)$$

On the other hand, by (2.28), Lemmas 2.2–2.5, and (H_2) for any $t \in J$ and $0 \leq i \leq n-1$, we have

$$\begin{aligned} e^{-t} \alpha((AC)^{(i)}(t)) &\leq \frac{\beta^* + \beta}{\beta - 1} 2 \int_0^{+\infty} \alpha(f(s, C(s), C'(s), \dots, C^{(n-1)}(s))) ds \\ &\leq \frac{2(\beta^* + \beta)}{\beta - 1} \sum_{j=0}^{n-1} l_j \alpha_D(C). \end{aligned} \quad (3.5)$$

Thus, by (3.4), (3.5), and Lemma 2.5, we get

$$\alpha_D(C) \leq \alpha_D(AC) \leq \left[\frac{2(\beta^* + \beta)}{\beta - 1} \sum_{i=0}^{n-1} l_i \right] \alpha_D(C). \quad (3.6)$$

Hence, by (H_2) , we have $\alpha_D(C) = 0$, that is, C is a relatively compact set in $D_e^{n-1}[J, E]$. Therefore, by Lemma 2.7, we conclude that A has at least one fixed point in $V \subset D_e^{n-1}[J, P]$, that is, BVP (1.1) has at least one solution in $D_e^{n-1}[J, P] \cap C^n[J, E]$. \square

4. An example

Example 4.1. Consider the following boundary value problem for a second-order differential equation defined on an infinite interval:

$$\begin{aligned} u_n''(t) &= e^{-t} + e^{-14t} u_n(t) + e^{-16t} u_{n+1}'(t), \quad t \in J, \\ u_n(0) &= u_{n+1}'(0), \\ u_n'(+\infty) &= 2u_n'(0), \quad n = 1, 2, 3, \dots \end{aligned} \quad (4.1)$$

5. Conclusion

BVP (4.1) has at least one nonnegative solution.

Proof. Let

$$E = l^1 = \left\{ u = (u_1, u_2, \dots, u_n, \dots) : \sum_{n=0}^{\infty} \|u_n\| < \infty \right\} \quad (5.1)$$

with norm $\|u\| = \sum_{n=0}^{\infty} \|u_n\|$. $P = \{u = (u_1, u_2, \dots, u_n, \dots) \in l^1 : u_n \geq 0, n = 1, 2, \dots\}$ is a cone in a Banach space E and BVP (4.1) is in the form of BVP (1.1) in E with $n = 2$. In this situation, $J = [0, \infty)$, $u = (u_1, \dots, u_n, \dots)$, $v = (v_1, \dots, v_n, \dots)$, and $f = (f_1, \dots, f_n, \dots)$ in which

$$f_n(t, u, v) = e^{-t} + e^{-14t}u_n(t) + e^{-16t}u'_{n+1}(t). \quad (5.2)$$

It is clear that $f \in C[J \times E \times E, E]$. For any $t \in J$, $u, v \in P$, we have

$$|f_n(t, u, v)| \leq e^{-t} + e^{-14t}u_n(t) + e^{-16t}u'_{n+1}(t). \quad (5.3)$$

So we get

$$\|f(t, u, v)\| \leq a(t) + b_0(t)\|u\| + b_1(t)\|v\|, \quad (5.4)$$

where $a_0(t) = e^{-t}$, $b_0(t) = e^{-14t}$, $b_1(t) = e^{-16t}$. From (4.1) and (5.3), we have

$$\begin{aligned} a^* &= \int_0^{+\infty} e^{-t} dt = 1, & b_0^* &= \int_0^{+\infty} e^t e^{-14t} dt = \frac{1}{13}, \\ b_1^* &= \int_0^{+\infty} e^t e^{-16t} dt = \frac{1}{15}, & \beta^* &= \beta_0 = 1. \end{aligned} \quad (5.5)$$

So, we have

$$\frac{\beta - 1}{\beta^* + \beta} = \frac{2 - 1}{1 + 2} = \frac{1}{3} > \frac{1}{13} + \frac{1}{15} = b_0^* + b_1^*. \quad (5.6)$$

Hence, the condition (H_1) is satisfied. By (5.3) for any bounded sets $D_1, D_2 \subset l^1$, we get

$$\alpha(f(t, D_1, D_2)) \leq l_1(t)\alpha(D_1) + l_2(t)\alpha(D_2), \quad (5.7)$$

where $l_1(t) = e^{-14t}$, $l_2(t) = e^{-16t}$. So

$$\begin{aligned} l_1 &= \int_0^{+\infty} e^t l_1(t) dt = \frac{1}{13} < +\infty, & l_2 &= \int_0^{+\infty} e^t l_2(t) dt = \frac{1}{15} < +\infty, \\ \frac{2(\beta^* + \beta)}{\beta - 1} (l_1 + l_2) &= \left(\frac{1}{13} + \frac{1}{15} \right) \times 6 < 1. \end{aligned} \quad (5.8)$$

Hence, the condition (H_2) is satisfied. Therefore, our conclusion follows from Theorem 3.1. \square

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