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Research Article

On Modelling Long Term Stock Returns with Ergodic Diffusion Processes: Arbitrage and Arbitrage-Free Specifications

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We investigate the arbitrage-free property of stock price models where the local martingale component is based on an ergodic diffusion with a specified stationary distribution. These models are particularly useful for long horizon asset-liability management as they allow the modelling of long term stock returns with heavy tail ergodic diffusions, with tractable, time homogeneous dynamics, and which moreover admit a complete financial market, leading to unique pricing and hedging strategies. Unfortunately the standard specifications of these models in literature admit arbitrage opportunities. We investigate in detail the features of the existing model specifications which create these arbitrage opportunities and consequently construct a modification that is arbitrage free.

1. Introduction

Ever since the fundamental work of Black and Scholes, there has been extensive work in the literature on alternative stock price models. In a continuous time setting, these include, for example, jump diffusions, Levy processes, stochastic volatility models, and regime switching models. Extensive references can be found in Cont and Tankov [1] and Fouque et al. [2]. These models and tools have proven invaluable for long term asset liability management, in particular with applications to insurance and pensions. These include the modelling and pricing of long term embedded guarantees (see, e.g., Sherris [3], Bauer et al. [4], Milevsky and Salisbury [5], Zaglauer and Bauer [6]) and also with the study of optimal asset allocation problems (see, e.g., Cairns [7], Gerber and Shiu [8], Stamos [9]). Extensive references to the vast literature can also be found in the monographs of Hardy [10], Schmidli [11], Milevsky [12], and Møller and Steffensen [13].

In this paper we investigate the arbitrage-free property of the class of Brownian based stock price models where the local martingale component of the (log) stock returns is assumed to be an ergodic diffusion. This class of models was first investigated by Bibby and Sørensen [14] and Rydberg [15, 16] (henceforth "BSR") who reported good fit of their models to financial data. The use of ergodic diffusions imply that the marginal distributions over a long horizon will be approximately equal to that of the specified stationary distribution, such as the Student-t (Bibby and Sørensen [14] provide additional discussion of this property). The dynamics of these models are time homogeneous and, as it is based on diffusions, tractable. The financial market under these models will be complete, and hence the valuation of options and guarantees can be performed without requiring extra assumptions regarding the market price of risk. In contrast, most alternative stock price models admit incomplete markets, with no unique pricing of options and guarantees available in general.

A significant drawback of the ergodic diffusion approach however was also noted by BSR (in particular, Bibby and Sørensen [14] and Rydberg [15]) who showed that no standard equivalent local martingale measure can exist for the ergodic diffusion model with a (generalized) hyperbolic ergodic distribution model they considered, and so the model they considered is only arbitrage free up to a stopping time. By the fundamental theorem of asset pricing (Harrison and Kreps [17], Harrison and Pliska [18], Delbaen and Schachermayer [19]) it follows that these models permit arbitrage. Furthermore the discussions of this issue in BSR for the (generalized) hyperbolic class of models further suggest that this feature may perhaps be present in many ergodic diffusion-based models in general.

Following on from the previous discussion in literature, in this paper we provide a detailed proof that any ergodic diffusion process used as a stock return model, and as specified in literature, will admit arbitrage. We further analyze in detail the cause for these arbitrage opportunities and consequently propose a modification that is arbitrage-free. This modification once again opens up the application of ergodic diffusion models to problems in insurance and finance.

The outline of this paper is as follows. In Section 2 we briefly review the construction of local martingales based on ergodic diffusion processes. Section 3 sets out the financial market we consider and defines the economic notions of portfolios and arbitrage. The standard model specification considered in the existing literature is investigated, and associated arbitrage opportunities identified, in Section 4. This analysis is consequently used to construct an alternative, arbitrage-free specification in Section 5. An extension of the arbitrage-free specification to stock markets with a stochastic term structure of interest rates can be found in Section 6. Section 7 concludes.

2. Local Martingales Based on Ergodic Diffusions

Let $W(\cdot)$ be a Brownian motion. Consider a local martingale $X(\cdot)$ of the form

$$\begin{aligned} dX(t) &= \sigma(X(t))dW(t), \\ X(0) &= 0, \end{aligned} \tag{2.1}$$

where $\sigma(\cdot)$ is a continuous, strictly positive deterministic function. By Engelbert and Schmidt [20] it follows that a nonexploding solution to the stochastic differential equation (2.1) will always exist.

Consider an interval (l, r) with $-\infty < l < 0 < r < \infty$. Let $g(\cdot)$ be a continuous, bounded probability density function which is strictly positive on (l, r) and zero outside (l, r). Typically in applications to stock returns we are interested in the case when (l, r) is $(-\infty, \infty)$, and we will make this assumption for the analysis in Sections 4–7.

Using standard diffusion theory (cf. Karlin and Taylor [21]), it was noted by Bibby and Sørensen [14] and Rydberg [15] that $X(\cdot)$ can be constructed to be an ergodic diffusion with stationary density $g(\cdot)$ by selecting

$$\sigma(x) = \sqrt{\frac{k}{g(x)}}, \tag{2.2}$$

for some arbitrary constant $k > 0$. In particular they considered $g(\cdot)$ from the (generalized) hyperbolic class, which includes as special cases the Student-t and Normal Inverse Gaussian distributions. As an example, a local martingale $X(\cdot)$ with coefficient

$$\sigma^2(x) = k\frac{\sqrt{\pi\upsilon}\Gamma(\upsilon/2)}{\Gamma((\upsilon+1)/2)}\left(1+\frac{x^2}{\upsilon}\right)^{(\upsilon+1)/2} \tag{2.3}$$

will possess a Student-t (with υ degrees of freedom) stationary distribution, while a local martingale $X(\cdot)$ with

$$\sigma^2(x) = k\pi\left(1+x^2\right) \tag{2.4}$$

will possess a Cauchy stationary distribution. Other distributions can also be used, dependent on the characteristics of the process being modelled.

It is worth noting that the existence of a stationary distribution for (2.1) is closely related to the concept of volatility-induced stationarity in the interest rate modelling literature (Conley et al. [22], Nicolau [23]).

Note that the above is not the only method of constructing a Brownian-based model with a specified distribution. There are two alternative approaches. The first alternative (Bibby et al. [24], Borkovec and Klüppelberg [25]) constructs semimartingales with stationary density $g(\cdot)$ by considering diffusion processes with nonzero drift. We do not pursue this approach as our stock price construction considers local martingales, which necessarily have zero drift. The second alternative (Dupire [26], Madan and Yor [27]) uses the Fokker-Plank equation of a diffusion, with an additional time scaling assumption, to construct a diffusion with a specified marginal distribution. A significant drawback of this second alternative approach however is that the dynamics of the resulting local martingales will be time inhomogeneous in general. In contrast, the dynamics of the local martingale (2.1) is time homogeneous.

3. Financial Market

For the financial market we consider a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and the time interval $[0, T]$, the filtration being generated by 1-dimensional Brownian Motion, augmented to satisfy the usual conditions. In cases where we consider two nonequivalent measures we will augment with respect to the null sets of both measures. Denote by $\mathcal{D}_P^*$ the equivalence class of all progressively measurable processes $\varphi(\cdot)$ satisfying

$$P\left(\int_0^T \varphi^2(t)dt < \infty\right) = 1. \tag{3.1}$$

For clarity of presentation, in Sections 3–5 we will assume that the only source of uncertainty in the financial market arises from the stock. Specifically, assume that there are two primary securities traded in the time interval $[0, T]$. The first is a savings account $B(\cdot)$ with

$$B(t) = e^{rt}, \tag{3.2}$$

for some constant short rate $r \in \mathbf{R}$. The second is a strictly positive stock price $S(\cdot)$, with $S(0) = 1$. We consider alternative specifications for $S(\cdot)$ in Sections 4 and 5.

An extension of our framework to include stochastic interest rates can be found in the Section 6.

Portfolios are formed by holding an amount of $\pi_0(\cdot)$ of the savings account and $\pi(\cdot)$ of the stock, with $\pi_0(\cdot)$, $\pi(\cdot)$ being progressively measurable. The value $V(\cdot)$ of a self-financing portfolio should satisfy for all $t \in [0, T]$,

$$\begin{aligned} dV(t) &= \frac{\pi_0(t)}{B(t)}dB(t) + \frac{\pi(t)}{S(t)}dS(t), \\ V(t) &= \pi_0(t) + \pi(t). \end{aligned} \tag{3.3}$$

To ensure that there are no pathological doubling strategies we also require that the discounted value of a portfolio $V(\cdot)/B(\cdot)$ is bounded below by -1. Portfolios satisfying this constraint are referred to as tame (Dybvig and Huang [28]). Finally, an arbitrage opportunity is defined as a tame portfolio with value process $V(\cdot)$ satisfying

$$\begin{aligned} &V(0) = 0, \\ &\frac{V(t)}{B(t)} \geq -1, \\ &V(T) > 0. \end{aligned} \tag{3.4}$$

4. Arbitrage Opportunities under the Standard Specification

Ergodic diffusion-based stock price models were first considered by Bibby and Sørensen [14] and Rydberg [15, 16]. Their models for the stock price process $S(\cdot)$ are of the form

$$S(t) = \exp\{\mu t + X(t)\}, \tag{4.1}$$

with $S(0) = 1$, constant $\mu \in \mathbf{R}$, and $X(\cdot)$ being a driftless ergodic diffusion with stationary density $g(\cdot)$ and associated state space $(-\infty, \infty)$. In the following we will call ergodic diffusion based models of the form (4.1) as being of the standard specification.

As $X(\cdot)$ is nonexplosive by construction, it follows that

$$P(S(T) > 0) = 1. \tag{4.2}$$

The model can equivalently be represented by the following stochastic differential equation by an application of Ito's formula:

$$dS(t) = \left(\mu + \frac{k}{2g(\ln S(t) - \mu t)}\right) S(t)dt + \sqrt{\frac{k}{g(\ln S(t) - \mu t)}} S(t)dW(t). \tag{4.3}$$

Representation (4.3) is sometimes more convenient for calculation purposes.

Bibby and Sørensen [14] and Rydberg [15, 16] report a good fit of the generalized hyperbolic ergodic diffusion to financial data. Unfortunately they also noted that no standard equivalent martingale measure can exist for the models they considered. By the fundamental theorem of asset pricing (Harrison and Kreps [17], Harrison and Pliska [18], Delbaen and Schachermayer [19]) it follows that arbitrage opportunities exist in these models. Furthermore, their discussion and proof suggests that the same problem may also apply to any ergodic diffusion-based model of the form (4.1). In the following we provide a detailed proof of the arbitrage opportunity for general ergodic diffusion models and perform further analysis on the technical features of this model specification which created these opportunities. Consequently in Section 5 we construct an alternative model specification that is arbitrage free.

Theorem 4.1. *The financial market with stock price modelled by* (4.1) *admits arbitrage opportunities.*

Proof. By Ito's lemma the process $\widetilde{S}(\cdot) := S(\cdot)/B(\cdot)$ satisfies the stochastic differential equation

$$d\widetilde{S}(t) = \left(\mu - r + \frac{k}{2g\left(\ln \widetilde{S}(t) - (\mu - r)t\right)}\right)\widetilde{S}(t)dt + \sqrt{\frac{k}{g\left(\ln \widetilde{S}(t) - (\mu - r)t\right)}}\widetilde{S}(t)dW(t). \tag{4.4}$$

In a Brownian setting, the fundamental theorem of asset pricing states that there is no arbitrage if and only if there exists an equivalent measure such that $\widetilde{S}(\cdot)$ is a local martingale under the equivalent measure. As there is only one source of uncertainty, the market is complete. It follows that an equivalent local martingale measure exists if and only if there exists a strictly positive martingale $Z(\cdot)$ with $Z(0) = 1$, and

$$Z(t) = \exp\left\{-\int_0^t \gamma(u)dW(u) - \frac{1}{2}\int_0^t \gamma^2(u)du\right\}, \tag{4.5}$$

where $\gamma(\cdot)$ is the market price of risk process, with

$$\gamma(t) = \left(\frac{\mu - r}{\sqrt{k}}\right)\sqrt{g(X(t))} + \left(\frac{\sqrt{k}}{2}\right)\frac{1}{\sqrt{g(X(t))}}. \tag{4.6}$$

Notice that as $Z(\cdot)$ is a supermartingale as it is a local martingale that is bounded below. Consequently 0 will be absorbing if reached. As the processes $g(X(\cdot))$ and $1/g(X(\cdot))$ are continuous and non-explosive by construction, $\gamma(\cdot) \in \mathcal{D}_P^*$, and hence

$$P(Z(T) > 0) = 1 \tag{4.7}$$

(cf. Kazamaki [29], Liptser and Shiryaev [30, 6.1.1]).

The martingale property of $Z(\cdot)$ can be investigated by considering a candidate measure Q (cf. Kadota and Shepp [31], Delbaen and Shirakawa [32], Wong and Heyde [33], Rogers and Veraart [34] for an application of this technique in different settings), which is not assumed to be equivalent to P a priori, with Brownian motion $W_Q(\cdot)$, and a process $\widetilde{S}_Q(\cdot)$ satisfying

$$d\widetilde{S}_Q(t) = \sqrt{\frac{k}{g\left(\ln \widetilde{S}_Q(t) - (\mu - r)t\right)}}\widetilde{S}_Q(t)dW_Q(t), \tag{4.8}$$

with $\widetilde{S}_Q(0) = 1$. Under Q we assume that $\widetilde{S}_Q(\cdot)$ will be stopped if explosion (to 0 or ∞) occurs. Denote this stopping time as $\tau_{\widetilde{S}_Q}$. If P and Q are equivalent measures, then $\widetilde{S}(\cdot)$ and $\widetilde{S}_Q(\cdot)$ will be equivalent in law under Q.

Note that $\widetilde{S}_Q(\cdot)$ can also be represented in terms of a process $X_Q(\cdot)$, with

$$d\widetilde{S}_Q(t) = \sqrt{\frac{k}{g(X_Q(t))}}\widetilde{S}_Q(t)dW_Q(t), \tag{4.9}$$

$$X_Q(t) = \int_0^t \sqrt{\frac{k}{g(X_Q(u))}}dW_Q(u) - \int_0^t \left((\mu - r) + \left(\frac{1}{2}\right)\frac{k}{g(X_Q(u))}\right)du, \tag{4.10}$$

with $X_Q(\cdot)$ also being stopped at the explosion time $\tau_{\widetilde{S}_Q}$. Note also that if

$$Q\left(\tau_{\widetilde{S}_Q} \le T\right) > 0, \tag{4.11}$$

then the stochastic integral in (4.10) should be interpreted in the Liptser-Shiryaev [30, 4.2.9] sense.

For calculation purposes it is convenient to consider, under the measure P, the related process $\widehat{S}(\cdot)$ defined by

$$\widehat{S}(t) = e^{-\mu t}S(t) = e^{X(t)}, \tag{4.12}$$

which, by the definition of $X(\cdot)$, satisfies

$$P\left(\widehat{S}(T) > 0\right) = 1. \tag{4.13}$$

Correspondingly, consider under the measure Q a process $\widehat{S}_Q(\cdot)$, defined by

$$\widehat{S}_Q(t) = e^{-(\mu-r)t}\widetilde{S}_Q(t) = e^{X_Q(t)}. \tag{4.14}$$

By Ito's lemma $\widehat{S}_Q(\cdot)$ satisfies

$$d\widehat{S}_Q(t) = (r-\mu)\widehat{S}_Q(t)dt + \sqrt{\frac{k}{g\left(\ln\left(\widehat{S}_Q(t)\right)\right)}}\widehat{S}_Q(t)dW_Q(t). \tag{4.15}$$

Define the Q local martingale $\eta(\cdot)$ by

$$\eta(t) = \exp\left\{\int_0^t \left(\frac{\mu-r}{\sqrt{k}}\right)\sqrt{g(X_Q(u))}dW_Q(u) - \frac{1}{2}\int_0^t \left(\frac{\mu-r}{\sqrt{k}}\right)^2 g(X_Q(u))du\right\}. \tag{4.16}$$

Observe that, by the boundedness of $g(\cdot)$ and Novikov's condition, $\eta(\cdot)$ is a strictly positive, true martingale. Hence we can define a measure Q_η equivalent to Q by

$$Q_\eta(A) = E_Q\left[\eta(T)1_A\right], \tag{4.17}$$

for some $\mathcal{F}_T$ measurable event A, and where 1_A represents the indicator function for A. Girsanov's theorem shows that under Q_η there exists a Brownian motion $W_{Q_\eta}(\cdot)$, and where $\widehat{S}_Q(\cdot)$ satisfies

$$d\widehat{S}_Q(t) = \sqrt{\frac{k}{g\left(\ln\left(\widehat{S}_Q(t)\right)\right)}}\widehat{S}_Q(t)dW_{Q_\eta}(t). \tag{4.18}$$

Consider the probability of explosion to 0 in finite time. This can be checked with a criterion of Delbaen and Shirakawa [32]. Denote $G(\cdot)$ as the cumulative distribution function corresponding to $g(\cdot)$. By the assumptions on $g(\cdot)$ in Section 2 we have, for all $\epsilon,\ M > 0$, with $\infty > M \geq \epsilon$,

$$\infty > \sup_{\epsilon\leq u\leq M}\left(\sqrt{\frac{k}{g(\ln(u))}}u\right) \geq \inf_{\epsilon\leq u\leq M}\left(\sqrt{\frac{k}{g(\ln(u))}}u\right) > 0 \tag{4.19}$$

as required by Delbaen and Shirakawa [32, Theorem 1.4]. Consequently the probability of hitting 0 is decided by the convergence or divergence of the integral

$$\int_0^1 \frac{g(\ln(y))}{ky}dy = \frac{G(0)-G(-\infty)}{k}, \tag{4.20}$$

with divergence being equivalent to a 0 probability of the process reaching 0. As $0 \leq G(\cdot) \leq 1$ it follows that (4.20) is finite, and hence $\widehat{S}_Q(\cdot)$ can hit 0 in finite time. As $\widehat{S}_Q(\cdot)$ is a Q_η local martingale bounded below by 0, it is also a supermartingale. Hence 0 is absorbing, and we have

$$Q_\eta\left(\widehat{S}_Q(T) = 0\right) > 0, \tag{4.21}$$

and by equivalence

$$Q\left(\widehat{S}_Q(T) = 0\right) > 0, \tag{4.22}$$

which by comparison to (4.13) implies that P and Q cannot be equivalent measures. □

As this is a complete market we can identify a specific arbitrage portfolio and strategy using the techniques of Levental and Skorohod [35]. Firstly note that as $Z(\cdot)$ is not a true martingale, we have

$$E_P[Z(T)] = c, \tag{4.23}$$

for some constant $c < 1$. Define a P-martingale $\widetilde{Z}(\cdot)$ by

$$\widetilde{Z}(t) = E_P[Z(T) \mid \mathcal{F}_t], \tag{4.24}$$

which is a strictly positive martingale. It follows by Ito's lemma that there exist $\widetilde{\theta}(\cdot) \in \mathcal{P}_P^*$ such that we can represent $\widetilde{Z}(\cdot)$ in stochastic exponential form

$$\widetilde{Z}(t) = c\exp\left\{-\int_0^t \widetilde{\theta}(u)dW(u) - \frac{1}{2}\int_0^t \widetilde{\theta}^2(u)du\right\}. \tag{4.25}$$

Consider the process $V(\cdot)$, with

$$V(t) = B(t)\left(\frac{\widetilde{Z}(t) - Z(t) + 1 - c}{Z(t)}\right). \tag{4.26}$$

Applying Ito's lemma shows that $V(\cdot)$ is the value process of a self financing portfolio, with

$$\pi(t) = \left(\frac{(\mu - r)g(X(t))}{k} + \frac{1}{2}\right)V(t) + \left(\frac{(\mu - r)g(X(t))}{k} + \frac{1}{2} - \widetilde{\theta}(t)\frac{\sqrt{g(X(t))}}{\sqrt{k}}\frac{\widetilde{Z}(t)}{Z(t)}\right)B(t). \tag{4.27}$$

As $\widetilde{Z}_0(\cdot), Z_0(\cdot) > 0$, we have

$$V(0) = 0,$$
$$\frac{V(t)}{B(t)} = \left(\frac{\widetilde{Z}(t)}{Z(t)} + \frac{1-c}{Z(t)} - 1\right) \geq -1, \tag{4.28}$$
$$V(T) = (1-c)\frac{B(T)}{Z(T)} > 0,$$

implying that $V(\cdot)$ is an arbitrage opportunity.

5. Arbitrage-Free Specification

In the previous section we have shown that the model specification (4.1) will always admit arbitrage opportunities. In this section we further investigate the technical features of this specification that create the arbitrage and consequently construct a modification that is arbitrage free.

As Q is not an equivalent measure to P, the Q-local martingale $\widehat{Z}_Q(\cdot)$, where $\widehat{Z}_Q(0) = 1$, and

$$\widehat{Z}_Q(t) = \exp\left\{-\int_0^t \gamma_Q(u)dW_Q(u) - \int_0^t \frac{1}{2}\gamma_Q^2(u)du\right\}, \tag{5.1}$$

is not strictly positive, with

$$\gamma_Q(t) = -\left(\frac{\mu - r}{\sqrt{k}}\right)\sqrt{g(X_Q(t))} - \left(\frac{\sqrt{k}}{2}\right)\frac{1}{\sqrt{g(X_Q(t))}}, \tag{5.2}$$

and $X_Q(\cdot)$ defined by (4.10). By Kazamaki [29] and Liptser and Shiryaev [30, 6.1.1] we have

$$\left\{\widehat{Z}_Q(T) = 0\right\} \Longleftrightarrow \left\{\int_0^T \gamma_Q^2(t)du = \infty\right\}, \tag{5.3}$$

where all processes are stopped at the explosion time of $\int_0^{\cdot}\gamma_Q^2(t)du$, which we will denote as $\tau_{\langle\gamma_Q\rangle}$. Note that, by comparing (4.9) and (4.10) with (5.2), and as $g(\cdot)$ is bounded, we have $\tau_{\langle\gamma_Q\rangle} = \tau_{\tilde{S}_Q}$ by construction.

As $\widehat{Z}_Q(\cdot)$ is not strictly positive, we have

$$Q\left(\tau_{\langle\gamma_Q\rangle} \leq T\right) > 0, \tag{5.4}$$

and as $g(\cdot)$ is bounded it follows that the term in (5.2) causing (5.4) is of the form $1/g(X_Q(t))$. In comparison to the standard model specification (4.1) it is apparent that this term arose from a drift adjustment for the quadratic variation of $X(\cdot)$. This suggests that an alternative stock price model of the form

$$dS(t) = \mu S(t)dt + S(t)dX(t), \tag{5.5}$$

or, equivalently,

$$S(t) = \exp\left\{\mu t - \frac{1}{2}\langle X\rangle(t) + X(t)\right\}, \tag{5.6}$$

where $\langle X\rangle(\cdot)$ is the quadratic variation process of $X(\cdot)$, defined by

$$\langle X\rangle(t) = \int_0^t \frac{k}{g(X(u))}du, \tag{5.7}$$

will be arbitrage free.

Theorem 5.1. *The financial market with stock model* (5.6)-(5.7) *does not admit arbitrage opportunities.*

Proof. Note firstly that we have, by the assumptions on $g(\cdot)$ in Section 2,

$$P(\langle X\rangle(T) < \infty) = 1. \tag{5.8}$$

Hence by Kazamaki [29] and Liptser and Shiryaev [30, 6.1.1] it follows that $S(\cdot)$ is strictly positive over $[0, T]$.

From Ito's lemma the discounted stock price process $\widetilde{S}(\cdot) = S(\cdot)/B(\cdot)$ satisfies

$$d\widetilde{S}(t) = (\mu - r)\widetilde{S}(t)dt + \sqrt{\frac{k}{g(X(t))}}\widetilde{S}(t)dW(t). \tag{5.9}$$

An equivalent local martingale measure exists if and only if there exists a strictly positive martingale $Z(\cdot)$, with $Z(0) = 1$, and

$$Z(t) = \exp\left\{-\int_0^t \theta(u)dW(u) - \frac{1}{2}\int_0^t \theta^2(u)du\right\}, \tag{5.10}$$

where $\theta(\cdot)$ is the market price of risk process, with

$$\theta(t) = \left(\frac{\mu - r}{\sqrt{k}}\right)\sqrt{g(X(t))}. \tag{5.11}$$

By construction the process $\sqrt{g(X(\cdot))}$ is continuous and bounded. Hence by Novikov's condition it follows that $Z(\cdot)$ is a strictly positive martingale and an (unique) equivalent local martingale measure Q exists. By the fundamental theorem of asset pricing it follows that the model is arbitrage free. $\square$

Under the unique equivalent local martingale measure Q the discounted stock price $\widetilde{S}(\cdot)$ satisfies

$$d\widetilde{S}(t) = \sqrt{\frac{k}{g(X(t))}}\widetilde{S}(t)dW^Q(t), \tag{5.12}$$

with

$$X(t) = \int_0^t \sqrt{\frac{k}{g(X(u))}}dW^Q(u) - \int_0^t (\mu - r)du. \tag{5.13}$$

6. Stock Markets with Stochastic Term Structure of Interest Rates

In Sections 3 to 5 we considered a financial market with a constant interest rate for clarity of presentation. In practical applications however the long term nature of many problems (e.g., in insurance and pensions) imply that such an assumption may be inappropriate. In this section we extend Theorem 5.1 to stock markets with a stochastic term structure of interest rates.

To allow for imperfect correlation between stock and interest rates we now consider a probability space $(\Omega, \mathcal{G}, \mathbf{P})$ and the time interval $[0, T]$, the filtration being generated by 2 dimensional Brownian motion $(W(\cdot), W_r(\cdot))$, augmented to satisfy the usual conditions. Following Section 5, we consider a stock price model of the form

$$dS(t) = \mu S(t)dt + S(t)dX(t), \tag{6.1}$$

where the local martingale $X(\cdot)$ is assumed to be

$$\begin{gathered} dX(t) = \sqrt{\frac{k}{g(x)}}\left(\rho\, dW(t) + \sqrt{1-\rho^2}\, dW_r(t)\right), \\ X(0) = 0, \end{gathered} \tag{6.2}$$

for constants $k > 0$, $\rho \in (-1, 1)$, and stationary density $g(\cdot)$.

Interest rate variability will be introduced via the second Brownian motion $W_r(\cdot)$ and its augmented filtration $\mathcal{F}^r_{(\cdot)}$, with $\mathcal{F}^r \subset \mathcal{G}$. Following Heath et al. [36] (cf. Musiela and Rutkowski [37, Chapter 11]), assume that for every $U \le T$, the instantaneous forward rate $f(t, U)$ follows

$$f(t, U) = f(0, U) + \int_0^t \alpha_f(u, U)du + \int_0^t \sigma_f(u, U)dW_r(u), \tag{6.3}$$

for a Borel measurable function $f(0,\cdot)$, and $\mathcal{F}^r$ progressively measurable processes $\alpha_f(\cdot,U)$ and $\sigma_f(\cdot,U)$ satisfying

$$\int_0^U \left|\alpha_f(u,U)\right|du + \int_0^U \left|\sigma_f(u,U)du\right|^2 < \infty. \tag{6.4}$$

Assume $\alpha_f(t,U)$, $\sigma_f(t,U)$, and $f(0,U)$ are differentiable with respect to U, with bounded first derivatives $\alpha'_f(t,U)$, $\sigma'_f(t,U)$, and $f'(0,U)$. It is known (cf. Musiela and Rutkowski [37, Proposition 11.1.1]) that the short rate process $r(\cdot)$ is a continuous semimartingale, with

$$r(t) = r(0) + \int_0^t \zeta(u)du + \int_0^t \sigma_f(u,u)dW_r(u), \tag{6.5}$$

where

$$\zeta(t) = \alpha_f(t,t) + f'(0,t) + \int_0^t \alpha'_f(u,t)du + \int_0^t \sigma'_f(u,t)dW_r(u). \tag{6.6}$$

Finally, assume that there exists a $\mathcal{F}^r$ progressively measurable process $\lambda(\cdot) \in \mathcal{D}_P^*$ such that, for any $U \leq T$, we have

$$\int_t^U \alpha_f(t,v)dv = \frac{1}{2}\left(\int_t^U \sigma_f(t,v)dv\right)^2 - \left(\int_t^U \sigma_f(t,v)dv\right)\lambda(t), \tag{6.7}$$

and such that the process $Z_r(\cdot)$, with

$$Z_r(t) = \exp\left\{\int_0^t \lambda(u)dW_r(u) - \frac{1}{2}\int_0^t \lambda^2(u)du\right\}, \tag{6.8}$$

is a strictly positive martingale. This assumption is standard in literature and can intuitively be interpreted as assuming that the interest rate market is internally arbitrage free.

Under the above setup the savings account $B(\cdot)$ satisfies

$$B(t) = e^{\int_0^t r(u)du}, \tag{6.9}$$

and the price $B(\cdot,U)$ of a U-maturity Zero Coupon Bond process, with initial value $B(0,U)$, satisfies

$$dB(t,U) = a(t,U)B(t,U)dt + b(t,U)B(t,U)dW_r(t), \tag{6.10}$$

where

$$a(t,U) = f(t,t) - \int_t^U \alpha_f(t,v)dv + \frac{1}{2}\left(\int_t^U \sigma_f(t,v)dv\right)^2,$$
$$b(t,U) = -\int_t^U \sigma_f(t,v)dv. \tag{6.11}$$

Theorem 6.1. *The financial market with stock model* (6.1)-(6.2) *and interest rate term structure modelled by* (6.3)–(6.8) *does not admit arbitrage opportunities.*

Proof. The discounted stock price process $\tilde{S}(\cdot) = S(\cdot)/B(\cdot)$ satisfies

$$d\tilde{S}(t) = (\mu - r(t))\tilde{S}(t)dt + \sqrt{\frac{k}{g(X(t))}}\tilde{S}(t)\left(\rho dW(t) + \sqrt{1-\rho^2}dW_r(t)\right) \tag{6.12}$$

under the measure P. As $Z_r(\cdot)$ is a strictly positive P martingale by assumption, we can define a measure P_{Z_r} equivalent to P by

$$P_{Z_r}(A) = E_P[Z_r(T)1_A], \tag{6.13}$$

for some $\mathcal{G}_T$ measurable event A, and where 1_A represents the indicator function for A. By Girsanov's theorem, under P_{Z_r} we have a 2-dimensional Brownian motion $(W(\cdot), W_{r,P_{Z_r}}(\cdot))$, with

$$W_{r,P_{Z_r}}(t) = W_r(t) - \int_0^t \lambda(u)du, \tag{6.14}$$

with $\lambda(\cdot)$ defined by (6.7). Under P_{Z_r} the short rate process $r(\cdot)$ satisfies

$$r(t) = r(0) + \int_0^t (\zeta(u) + \sigma_f(u,u)\lambda(u))du + \int_0^t \sigma_f(u,u)dW_{r,P_{Z_r}}(u), \tag{6.15}$$

which is continuous and is independent of $W(\cdot)$ by the assumptions on the coefficients of (6.3).

Consider now the nonnegative P_{Z_r} local martingale $Z(\cdot)$, with

$$Z(t) = \exp\left\{-\int_0^t \theta(u)dW(u) - \frac{1}{2}\int_0^t \theta^2(u)du\right\}, \tag{6.16}$$

and where $\theta(\cdot)$ is the market price of risk process corresponding to $W(\cdot)$, with

$$\theta(t) = \left(\frac{\mu - r(t)}{\rho\sqrt{k}}\right)\sqrt{g(X(t))} + \frac{\sqrt{1-\rho^2}}{\rho}\lambda(t). \tag{6.17}$$

Notice that $r(\cdot) \in \mathcal{D}_P^*$ by the continuity of $r(\cdot)$. As $\lambda(\cdot) \in \mathcal{D}_P^*$ we also have $(r(\cdot) + \lambda(\cdot)) \in \mathcal{D}_P^*$, and hence by equivalence

$$P_{Z_r}\left(\int_0^T (r(u) + \lambda(u))^2 du < \infty\right) = 1. \tag{6.18}$$

Consequently, by the boundedness of $g(\cdot)$, we have

$$P_{Z_r}\left(\int_0^T \theta^2(u) du < \infty\right) = 1, \tag{6.19}$$

and, by noting that $r(\cdot)$ and $\lambda(\cdot)$ are independent of $W(\cdot)$, we have (cf. Liptser and Shiryaev, [30, Example 6.2.4])

$$E_{P_{Z_r}}\left[e^{(1/2)\int_0^T \theta^2(t)dt} \mid \mathcal{F}_T^r\right] < \infty. \tag{6.20}$$

It follows by Novikov's condition that

$$E_{P_{Z_r}}\left[Z(T) \mid \mathcal{F}_T^r\right] = 1 \tag{6.21}$$

and in particular,

$$E_{P_{Z_r}}[Z(T)] = E_{P_{Z_r}}\left[E_{P_{Z_r}}\left[Z(T) \mid \mathcal{F}_T^r\right]\right] = 1. \tag{6.22}$$

Equations (6.19) and (6.22) imply that $Z(\cdot)$ is a strictly positive P_{Z_r} martingale, and hence we can define a measure Q equivalent to P_{Z_r} by

$$Q(A) = E_Q[Z(T)1_A], \tag{6.23}$$

for some $\mathcal{G}_T$ measurable event A, and where 1_A represents the indicator function for A. By Girsanov's Theorem it follows that discounted asset prices are local martingales under Q. As Q is equivalent to P_{Z_r} which is in turn equivalent to P, Q is an equivalent local martingale measure. By the fundamental theorem of asset pricing it follows that the model is arbitrage free. □

7. Conclusions

In this paper we investigated the arbitrage-free property of the class of stock price models where the local martingale component is based on an ergodic diffusion with a specified stationary distribution. The dynamics of these models are time homogeneous and, as it is based on Brownian motion, tractable. The financial market under these models will be complete, and hence the valuation of options and guarantees can be performed without requiring extra assumptions regarding the market price of risk. In this paper we provided a detailed proof that any ergodic diffusion process used as a stock return model, and as specified in the existing literature, will admit arbitrage in general. We further analyzed the technical cause for these arbitrage opportunities and consequently constructed a modification that is arbitrage-free. This arbitrage free property is shown to be true in financial markets both with and without stochastic interest rates. Our modification once again opens up the application of ergodic diffusion models to problems in insurance and finance.

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