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*Research Article*

# Stochastic Integration in Abstract Spaces

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We establish the existence of a stochastic integral in a nuclear space setting as follows. Let $E$, $F$, and $G$ be nuclear spaces which satisfy the following conditions: the spaces are reflexive, complete, bornological spaces such that their strong duals also satisfy these conditions. Assume that there is a continuous bilinear mapping of $E \times F$ into $G$. If $H$ is an integrable, $E$-valued predictable process and $X$ is an $F$-valued square integrable martingale, then there exists a $G$-valued process $(\int H dX)_t$ called the stochastic integral. The Lebesgue space of these integrable processes is studied and convergence theorems are given. Extensions to general locally convex spaces are presented.

## 1. Introduction

In this note, we announce the existence of a stochastic integral in a nuclear space setting. The nuclear spaces are assumed to have special properties which are given in Section 3.1 below. Our main result will now be stated. All definitions and pertinent concepts will be given in Sections 2 and 3, as well as a presentation of the construction.

**Theorem 1.1.** *Let $E$, $F$, and $G$ be nuclear spaces which satisfy the special conditions listed in Section 3.1, and suppose that there is a continuous bilinear mapping of $E \times F$ into $G$. Assume that $X$ is an $F$-valued square integrable martingale.*

*If $H$ is a bounded $E$-valued predictable process, then there exists a $G$-valued process $(\int H\, dX)_t$, called the stochastic integral of $H$ with respect to $X$, which is a square integrable martingale.*

*If we further assume that $G$ has a countable basis of seminorms, then the above conclusion holds when $H$ is a predictable $E$-valued process, which is integrable with respect to $X$ (in this case, $H$ is, in general, unbounded).*

This result extends the theory of nuclear stochastic integration of Ustunel [1] in several directions. In [1] it is assumed that $F$ is the strong dual of $E$ and $G$ is the real number field, and furthermore $H$ is assumed to be bounded. To develop our theory, we modify the vector bilinear integral developed in [2] for Banach spaces. After defining the space $L^2_G$, $G$ locally convex, the above bilinear integration theory will be applied when we use the property that a complete nuclear space is a projective limit of a family of Hilbert spaces.

In Section 2 we will present the underlying integration theory, and apply this, in Section 3, to construct the stochastic integral.

We omit the proofs in some of the integration theorems since they follow along the usual lines, with appropriate modifications necessary in a general setting (see [2, 3]).

## 2. Bilinear Vector Integration Theory

### *2.1. The Banach Setting*

In this subsection, assume $E$, $F$, and $G$ are Banach spaces over the reals $\mathbb{R}$, with norms denoted by $|\cdot|$. Let $\Sigma$ be a $\sigma$-field of subsets of a set $T$, and assume $\mathbf{m} : \Sigma \to F$ is a $\sigma$-additive measure. We will assume that there is a continuous bilinear mapping $\Phi$ of $E \times F$ into $G$, which, in turn, yields a continuous linear map $\phi : E \to L(F, G)$, where $L(F, G)$ is the space of bounded linear operators from $F$ into $G$.

The *semivariation* of $\mathbf{m}$ relative to $\phi$, $E$, $F$, $G$, denoted by $\widehat{\mathbf{m}}$ is defined on $\Sigma$ as follows:

$$\widehat{\mathbf{m}}(A) = \sup|\Sigma e_i \mathbf{m}(A_i)|, \tag{2.1}$$

where the supremum is extended over all finite collections of elements $e_i$ in the unit ball $E_1$ of $E$ and over all finite disjoint collections of sets $A_i$ in $\Sigma$ which are contained in $A$. We are only interested in the case when $\widehat{\mathbf{m}}(T) < \infty$ in order to develop an integration theory of $E$-valued integrands. Sometimes we will write $\widehat{\mathbf{m}}$ as $\widehat{\mathbf{m}}_{E,G}$. Note that we write $e$ in place of $\phi(e)$.

One can show that, for each $A \in \Sigma$, $\widehat{\mathbf{m}}(A) = \sup|\mathbf{m}_z|(A)$, where the supremum is taken over $z \in G_1'$, the unit ball of the dual $G'$ of $G$, and $\mathbf{m}_z : \Sigma \to E'$ is defined by $\mathbf{m}_z(A)e = \langle z, e\mathbf{m}(A)\rangle$, for $e \in E$. The total variation measure of $\mathbf{m}_z$ is denoted by $|\mathbf{m}_z|$. Let $\mathbf{m}_{E,G} = \{|\mathbf{m}_z| : z \in G_1'\}$. Thus, $\mathbf{m}_{E,G}$ is a bounded collection of positive $\sigma$-additive measures. If $c_o \not\subset G$ (e.g., if $G$ is a Hilbert space), then one can show that $\mathbf{m}_{E,G}$ is relatively weakly compact in the Banach space $ca(\Sigma)$ consisting of real-valued measures, with total variation norm. In this case, there exists a positive control measure $\lambda$ such that $\mathbf{m}_{E,G}$ is uniformly absolutely continuous with respect to $\lambda$. A set $Q \subset T$ is $\mathbf{m}$-*negligible* if it is contained in a set $A \in \Sigma$ such that $|\mathbf{m}|(A) = 0$.

The advantage of modifying the bilinear integration theory in [2] to the case where the integrand is operator-valued rather than the measure being operator-valued will become apparent when the nuclear stochastic integral is studied. This modification changes some of the results in the previous theory, but we are still able to construct the desired Lebesgue space of integrable functions and establish convergence theorems. We now sketch this theory.

Denote by $\mathcal{S} = \mathcal{S}_E$ the collection of $E$-valued simple functions. We say that $h : T \to E$ is measurable if there exists a sequence from $\mathcal{S}$ which converges pointwise to $h$. For such $h$, define

$$N(h) = \sup \int |h| \mathrm{d}|\mathbf{m}_z|, \tag{2.2}$$

where the supremum is taken over $z \in G_1'$. Let $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}(\mathbf{m}_{E,G})$ be the collection of all such $h$ with $N(h)$ finite. Then set $\mathcal{L} = \mathcal{L}(\mathbf{m}_{E,G})$ to be the closure of $\mathcal{S}$ in $\widetilde{\mathcal{F}}$. The space $\mathcal{L}$ with the seminorm $N$ is our Lebesgue space.

There are different, but equivalent ways to define $\int h\,d\mathbf{m}$ for $h \in \mathcal{L}$. We select one which yields more information (hence more usefulness) regarding the defining components. If $h \in \mathcal{L}$, one can show that there exists a *determining sequence* $\{h_n\}$ of elements in $\mathcal{S}$— that is, the sequence is Cauchy in $\mathcal{L}$, and $\{h_n\}$ converges in $\mathbf{m}$-measure, namely, $\widehat{\mathbf{m}}(|h - h_n| > \epsilon) \to 0$ for each $\epsilon > 0$. Define the integral of $h \in \mathcal{S}$ in the obvious manner. A determining sequence for $h$ has the property that $\{N(h_n 1_{(\cdot)})\}_n$ is uniformly absolutely continuous with respect to $\widehat{\mathbf{m}}$. Also $h_n \to h$ in $\mathcal{L}$. The setwise limit $\int_A h_n d\mathbf{m}$, $A \in \Sigma$, exists and defines a $\sigma$-additive measure on $\Sigma$. Denote this limit by $\int_A h\,d\mathbf{m}$. This limit is independent of the choice of the determining sequence for $h$. We refer to $\mathcal{L}$ as the space of integrable functions.

**Theorem 2.1** (Vitali). *Let $\{h_n\}$ be a sequence of integrable functions. Let $h$ be an $E$-valued measurable function. Then $h \in \mathcal{L}$ and $h_n \to h$ in $\mathcal{L}$ if and only if*

(1) *$h_n \to h$ in $\mathbf{m}$-measure,*

(2) *$\{N(h_n 1_{(\cdot)})\}_n$ is uniformly absolutely continuous with respect to $\widehat{\mathbf{m}}$.*

**Theorem 2.2** (Lebesgue). *Let $g \in \mathcal{L}$, and let $\{h_n\}$ be a sequence of functions from $\mathcal{L}$. If $h_n \to h$ in $\mathbf{m}$-measure and $|h_n(\cdot)| \le |g(\cdot)|$ for each $n$, then $h \in \mathcal{L}$ and $h_n \to h$ in $\mathcal{L}$.*

**Theorem 2.3.** *If $\mathbf{m}_{E,G}$ is relatively weakly compact, then $\mathcal{L}$ contains the bounded measurable functions.*

### 2.2. *Application to the Stochastic Integral in Banach Spaces*

We retain the assumptions on $E$, $F$, $G$ as stated in Section 2.1. The stochastic setting is as follows (definitions and terminology are found in [4]). Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. $L^2_F(\mathbf{P})$ is a space of $\mathcal{F}$-measurable, $E$-valued functions such that $E(|f|^2) = \int |f|^2 d\mathbf{P} < \infty$, endowed with norm $|f| = E(|f|^2)^{1/2}$. Assume $(\mathcal{F}_t)_{t\ge 0}$ is a filtration which satisfies the usual conditions. Suppose $X : \mathbb{R}_+ \times \Omega \to F$ is a cadlag adapted process, with $X_t \in L^2_F$ for each $t$. Let $\mathcal{R}$ be the ring of subsets of $\mathbb{R}_+ \times \Omega$ generated by the predictable rectangles; thus $\sigma(\mathcal{R}) = \mathcal{P}$, the predictable $\sigma$-field. Let $\mathbf{m}$ $(= I_X)$ be the additive $L^2_F$-valued measure first defined on the predictable rectangles by $\mathbf{m}((s,t] \times A) = 1_A(X_t - X_s)$, $A \in \mathcal{F}_s$, and $\mathbf{m}(0_A) = 1_A X_0$, $A \in \mathcal{F}_0$. We regard $E$ as being continuously embedded into $L(L^2_F, L^2_G)$ in the obvious manner. The theory of [3] for Banach stochastic integration can be shown to apply in a parallel fashion to this setting, and we state a few pertinent results. If $c_o \not\subset F$, then $\mathbf{m}$ can be extended uniquely to a $\sigma$-additive $L^2_F$-valued measure if and only if $\mathbf{m}$ is bounded on $\mathcal{R}$. For our purposes in this paper, we will be interested only in the case when all the spaces are Hilbert spaces and $X$ is a square integrable martingale. In this case, $\widehat{\mathbf{m}}_{E,L^2_G}(\mathbb{R}_+ \times \Omega) < \infty$. As a result, we can construct the stochastic integral $(\int H\,dX)_t$, which is a process such that $\int_0^t H\,dX \in L^2_G$, and this process is a $G$-valued square integrable martingale. If we still denote the extension of $\mathbf{m}$ to $\mathcal{P}$ by $\mathbf{m}$, then $\int_0^t H\,dX$ is defined to be $\int H 1_{[0,t]} d\mathbf{m}$, where $H$ is integrable with respect to $\mathbf{m}$, that is, $H \in \mathcal{L}(\mathbf{m}_{E,L^2_G})$, and the Hilbert spaces involved in the bilinear theory are $E$, $L^2_F$, and $L^2_G$. This integral will be used to define the stochastic integral in nuclear spaces.

### 2.3. *The Definition of $L^2_G$, $G$ Locally Convex*

In this subsection, assume $(T, \Sigma, \mathbf{m})$ is a measure space, $\mathbf{m}$ is real-valued and $\sigma$-additive. Let $G$ be a complete locally convex space, and let $\mathcal{G}$ be a basis of seminorms defining the topology of $G$. A function $f : T \to G$ is *measurable* if it is the pointwise limit of simple $G$-valued measurable functions in $\mathcal{S}_G$. For $r \in \mathcal{G}$ and $h$ being measurable, let $N_r(h) = (\int r(h)^2 d|\mathbf{m}|)^{1/2}$. Let $\widetilde{\mathcal{F}}_G$ be the space of measurable functions $h$ such that $N_r(h) < \infty$ for each $r \in \mathcal{G}$. Then $\widetilde{\mathcal{F}}_G$ is a locally convex space with $\{N_r : r \in \mathcal{G}\}$ being a basis of seminorms. Define $L^2_G$, the space of integrable functions, to be the closure of $\mathcal{S}_G$ in $\widetilde{\mathcal{F}}_G$.

It can be shown that $L^2_G$ is the set of measurable functions $h$ which have a *determining sequence* $(h_n) \subset \mathcal{S}_G$, that is, the sequence satisfies for each $r \in \mathcal{G}$, $N_r(h_n - h_m) \to 0$ as $n, m \to \infty$, and for each $\epsilon > 0$ and $r \in \mathcal{G}$, we have $|\mathbf{m}|(r(h_n - h) > \epsilon) \to 0$ as $n \to \infty$. In this case, $\int_A h d\mathbf{m} = \lim \int_A h_n d\mathbf{m}$, $A \in \Sigma$, is unambiguously defined for each determining sequence (the definition of $\int h_n d\mathbf{m}$ is the obvious one).

The bounded measurable functions are in $L^2_G$, and the Vitali and the Lebesgue dominated convergence theorem hold. Moreover, we have the following theorem.

**Theorem 2.4.** *Let $G$ be a complete locally convex space with a countable basis of seminorms. Then $L^2_G$ is complete.*

### *2.4. A Remark on the Bilinear Mapping* $E \times F \to G$

Suppose $E$ and $G$ are locally convex spaces with $\mathcal{E}$ and $\mathcal{G}$ denoting their respective bases of defining seminorms. Assume $F$ is a Hilbert space and $\Phi : E \times F \to G$ is a continuous bilinear mapping that induces $\phi : E \to L(F, G)$. Using the continuity of $\Phi$, observe that for each $r \in \mathcal{G}$, there exists a $p \in \mathcal{E}$ such that $\Phi(U_p, F_1) \subset U_r$, where $U_p$ and $U_r$ are the closed balls induced by $p$ and $r$. If we define $p(r)$ to be the infimum over all $p$ for which the above inclusion holds, it turns out that $p(r)$ is a seminorm and $U_{p(r)}$ is the closed convex balanced hull of $\cup_p U_p$, where the union is taken over those $p$ in the above infimum. Also $p(r)(e) = \sup |ze|_{F'}$, where the supremum is taken over $z \in U_r^\circ$ $(ze : f \to \langle z, ef \rangle, f \in F)$. Call $p(r)$ the seminorm associated with $r$ and $\Phi$. Note that $E(U_{p(r)})$ is isometrically embedded in $L(F, G(U_r))$, where $E(U_{p(r)})$ is the Banach space consisting of equivalence classes modulo $\ker p(r)$, completed under the norm induced by $p(r)$; $G(U_r)$ is similarly defined.

## 3. The Nuclear Setting. The Construction of the Stochastic Integral

### *3.1. Square Integrable Martingales in Nuclear Spaces*

$(\Omega, \mathcal{F}, \mathbf{P})$ and $(\mathcal{F}_t)_{t\geq 0}$ are as in Section 2.2. Let $F$ denote a nuclear space which is reflexive, complete, bornological, and such that its strong dual $F'$ satisfies the same conditions. We say $F$ satisfies the *special conditions*. These special conditions are the hypotheses of Ustunel, who established fundamental results for square integrable martingales in this setting. Let $E$ be such a space. Then for $E$ and $E'$ there exist neighborhood bases of zero, $\mathcal{U}$, and $\mathcal{U}'$, respectively, such that for each $U \in \mathcal{U}$, the space $E(U)$ is a separable Hilbert space over the reals, and its separable dual is identified with the Hilbert space $E'[U^\circ]$ as defined in [5], where $U^\circ$ is the polar of $U$. Also, $\{U^\circ : U \in \mathcal{U}\}$ and $\{V^\circ : V \in \mathcal{U}'\}$ are bases of closed, convex, balanced bounded sets in $E'$, $E$, respectively. For $U \in \mathcal{U}$, we denote by $\mathcal{K}(U)$ the continuous canonical map from $E$ onto $E(U)$. If $U, V \in \mathcal{U}$ and $V \subset U$, then $\mathcal{K}(U, V)$ is the canonical mapping of $E(V)$ onto $E(U)$.

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space with $(\mathcal{F}_t)_{t\geq 0}$ being a filtration satisfying the usual conditions. The set $X = \{X^U : U \in \mathcal{U}\}$ is called a projective system of square integrable martingales if for each $U$, we have that $X^U$ is an $E(U)$-valued square integrable martingale, and if whenever $U, V \in \mathcal{U}$ and $V \subset U$, then $\mathcal{K}(U,V)X^V$ and $X^U$ are indistinguishable. We also assume $X^U$ is cadlag for each $U$. One says that $X$ has a limit in $E$ if there exists a weakly adapted mapping $\widetilde{X}$ on $\mathbb{R}_+ \times \Omega$ into $E$ such that $\mathcal{K}(U)\widetilde{X}$ is a modification of $X^U$ for each $U \in \mathcal{U}$.

The next theorem is crucial for defining the stochastic integral. Ustunel [1, Section II.4] assumed the existence of a limit in $E$ for $X$. This hypothesis was removed in [6]. We now state the theorem and provide a brief sketch of the proof, which uses a technique of Ustunel.

**Theorem 3.1.** *Let $X$ be a projective system of square integrable martingales. Then there exists a limit $\widetilde{X}$ in $E$ of $X$ which is strongly cadlag in $E$, and for which $\mathcal{K}(U)\widetilde{X}$ is a modification of $X^U$ for each $U \in \mathcal{U}$. Moreover, there exists a $V \in \mathcal{U}'$ such that $\widetilde{X}$ takes its values in $E[V^\circ]$.*

Let $\mathfrak{M}^2$ denote the space of real-valued square integrable martingales. Define a mapping $\mathbf{T} : E' \to \mathfrak{M}^2$ by $\mathbf{T}(e') = \langle e', X^U \rangle$, where $U$ is chosen in $\mathcal{U}$ so that $e' \in E'[U^\circ]$. Argue that $\mathbf{T}$ is well defined and linear. If $e'_n \to e'$ in $E'[U^\circ]$ for some $U \in \mathcal{U}$, then

$$\left|\mathbf{T}(e'_n)_\infty - \mathbf{T}(e')_\infty\right| \leq \|X^U_\infty\|_{E(U)} \left\|e'_n - e'\right\|_{E'[U^\circ]}; \tag{3.1}$$

hence $\{\mathbf{T}(e'_n)_\infty\}$ converges to $(\mathbf{T}(e')_\infty)$ in $L^2(\mathcal{P}) = L^2$, and thus $\mathbf{T}(e'_n) \to T(e')$ in $\mathfrak{M}^2$. Consequently, $\mathbf{T}$ is continuous on $E'[U^\circ]$. Since $E'$ is bornological, $\mathbf{T}$ is continuous on $E'$. As a result, $\mathbf{T} : E' \to \mathfrak{M}^2$ is a nuclear map of the form

$$\mathbf{T}(e') = \sum \lambda_i < e_i, \quad e' > M^i, \tag{3.2}$$

where $\{\lambda_i\} \in l^1$, $\{e_i\}$ is equicontinuous in $E$, and $(M^i)$ is bounded in $\mathfrak{M}^2$. Choose $V \in \mathcal{U}'$ such that all $e_i \in V^\circ$. Define the process $\widetilde{X}$ by $\widetilde{X_t} = \sum \lambda_i e_i M^i_t$, where we choose $(\widetilde{X_t})$ to be a cadlag version. Then $\widetilde{X}$ is the desired process.

From now on, we identify $X$ and $\widetilde{X}$, and we assume that $X$ takes its values in the Hilbert space $E[V^\circ]$.

### 3.2. Construction of the Stochastic Integral

Assume that $E$, $F$, and $G$ are nuclear spaces over the reals satisfying the special conditions set forth in Section 3.1. Also assume that $\Phi : E \times F \to G$ is a continuous bilinear mapping. The neighborhood bases of zero in $E$ and $G$ are denoted by $\mathcal{U}_E$ and $\mathcal{U}_G$. Let $X : \mathbb{R}_+ \times \Omega \to F$ be a square integrable martingale. By Theorem 2.4, we may assume $X$ is Hilbert space valued. As a result, we may now assume $F$ is a real Hilbert space. The bilinear map $\Phi$ induces a continuous linear map $\phi : E \to L(F,G)$, which in turn induces the continuous linear map $\overline{\phi} : E \to L(L^2_F, L^2_G)$, where $L^2_G$ is the space constructed in Section 2.3.

Since $c_o \not\subset F$, the stochastic measure $\mathbf{m}$ $(= I_X)$ first defined on the predictable rectangles can be extended to a $\sigma$-additive measure, still denoted by $\mathbf{m}$, $\mathbf{m} : \mathcal{P} \to L^2_F$. Note that if $K_1$ and $K_2$ are Hilbert spaces, then $\mathbf{m}$ has finite semivariation with respect to every continuous linear embedding of $K_1$ into $L(L^2_F, K_2)$.

If $z \in (L^2_G)'$, we define $\mathbf{m}_z : \mathcal{P} \to E'$ by $\mathbf{m}_z(A)e = \langle z, e\mathbf{m}(A)\rangle$, for $e \in E$. Given any $r \in \mathcal{G}$, if $z \in U^\circ_{N_r}$, then $\mathbf{m}_z : \mathcal{P} \to E'[U^\circ_{p(N_r)}] = E(U_{p(N_r)})'$ (where $p(N_r)$ is the seminorm associated with $N_r$ relative to the mapping $E \to L(L^2_F, L^2_G)$) by

$$\mathbf{m}_z(A)[e]_{p(N_r)} = \left\langle z, [e]_{p(N_r)}\mathbf{m}(A)\right\rangle = \langle z, e\mathbf{m}(A)\rangle. \tag{3.3}$$

In fact, $p(N_r) = p(r)$, relative to the mapping $E \to L(F, G)$. Let $\mathbf{m}_r = \{|\mathbf{m}_z| : z \in U^\circ_{N_r}\}$. Then $\widehat{\mathbf{m}}_r(A) = \sup|\mathbf{m}_z|(A)$, where the supremum is extended over $z \in U^\circ_{N_r}$. Observe that $\widehat{\mathbf{m}}_r$ is the semivariation of $\mathbf{m}$ relative to $E(U_{p(r)})$, $L^2_G(U_{N_r})$ which arises from the isometric mapping of $E(U_{p(r)})$ into $L(L^2_F, L^2_G(U_{N_r}))$. One can show that $L^2_G(U_{N_r})$ is isometrically embedded in the Hilbert space $L^2_{G(U_r)}$ and, as a result, $\mathbf{m}$ has a finite semivariation relative to each of these embeddings; thus $\widehat{\mathbf{m}}_r$ is finite for each $r \in \mathcal{G}$, and $\mathbf{m}_r$ is relatively weakly compact in $ca(\mathcal{P})$.

A process $H : \mathbb{R}_+ \times \Omega \to E$ is a predictable process, or simply measurable, if it is the pointwise limit of processes from $\mathcal{S}_E$, the simple predictable $E$-valued processes. For such a measurable process $H$, define, for $r \in \mathcal{G}$,

$$\mathfrak{N}_r(H) = \sup \int p(r)(H)\, d|\mathbf{m}_z|, \tag{3.4}$$

where the supremum is extended over $z \in U^\circ_{N_r}$. Let $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}(\mathbf{m}_{E,L^2_G})$ be the space of measurable functions $H$ such that $\mathfrak{N}_r(H) < \infty$ for each $r \in \mathcal{G}$. Then $\widetilde{\mathcal{F}}$ is a locally convex space containing $\mathcal{S}_E$. Let $\mathcal{L} = \mathcal{L}(\mathbf{m}_{E,L^2_G})$ denote the closure of $\mathcal{S}_E$ in $\widetilde{\mathcal{F}}$. One can show that for each $H \in \mathcal{L}$ there exists a determining sequence $(H_n)$ from $\mathcal{S}_E$ such that $(H_n)$ is mean Cauchy in $\mathcal{L}$ ($\mathfrak{N}_r(H_n - H_m) \to_{n,m} 0$), for each $r \in \mathcal{G}$, and $\widehat{\mathbf{m}}_r(p(r)(H_n - H) > \epsilon) \to_n 0$ for each $\epsilon > 0$ and $r \in \mathcal{G}$.

Now assume $G$ has a countable basis of seminorms, that is, $G$ is now a nuclear Fréchet space. Thus there exists a positive measure $\lambda$ such that $\widehat{\mathbf{m}}_r \ll \lambda$ for each $r \in \mathcal{G}$. Since $L^2_G$ is complete and, for $H \in \mathcal{S}_E$, we have $N_r(\int H\, d\mathbf{m}) \le \mathfrak{N}_r(H)$, where the integral is defined in the obvious way, then for general $H \in \mathcal{L}$ with determining sequence $(H_n)$, we can define

$$\int_{(\cdot)} H\, d\mathbf{m} = \lim \int_{(\cdot)} H_n d\mathbf{m} \in L^2_G. \tag{3.5}$$

The completeness of $L^2_G$ ensures that $\int_A H\, d\mathbf{m}$ is a function in $L^2_G$. Define the process $(\int H\, dX)_t = \int_0^t H\, dX$ by $\int_0^t H\, dX = \int H 1_{[0,t]} d\mathbf{m}$, called the *stochastic integral of $H$ with respect to $X$*. We say $H$ is integrable with respect to $X$ if $H \in \mathcal{L}$. If $H \in \mathcal{S}_E$, one can show that $(\int H\, dX)_t$ is a $G$-valued square integrable martingale. By means of using determining sequences, the general stochastic integral enjoys this property.

Next, assume that $G$ just satisfies the special conditions (no longer nuclear Fréchet). Let $H$ be a bounded measurable $E$-valued process; hence the range of $H$ is contained in a closed, bounded, convex, balanced set $B_1$, where $E[B_1]$ is a Hilbert space. By the continuity of $\Phi$, it follows that $\Phi(B_1, F_1)$ is contained in a bounded set $B$ having the same properties as $B_1$, and $G[B]$ is a Hilbert space.

Algebraically, $\Phi$ induces $\Phi_0 : E[B_1] \times F \to G[B]$ which is bilinear, and since $\Phi_0^{-1}(\alpha B) \supset (\alpha B_1) \times F$ for every $\alpha \in \mathbb{R}$, $\Phi_0$ is continuous. As a result, this induces a continuous linear map

$\phi_0 : E[B_1] \to L(F, G[B])$, which in turn induces the continuous linear map $\widetilde{\phi}_0 : E[B_1] \to L(L_F^2, L_{G[B]}^2)$. Hence we can define $\mathbf{m} = I_X : \mathcal{P} \to L_F^2$ as before, which is $\sigma$-additive and has finite semivariation relative to $\widetilde{\phi}_0$.

Since $H$ is measurable, it is the pointwise limit of functions from $\mathcal{S}_E$, and thus if $x' \in E'$, $x'H_n \to x'H$. This implies that $(x'H)^{-1}(\mathcal{O}) \in \mathcal{P}$ for any open subset $\mathcal{O}$ of the reals. By the reflexivity of $E$,

$$E[B_1]' = E[B_1] = \left[E'(B_1^\circ)\right]' = E'(B_1^\circ), \tag{3.6}$$

since we have chosen $B_1 = V^\circ \in \mathcal{U}'$. Let $e' \in E[B_1]'$; then $e' = [x']_{B_1^\circ}$, and for $e \in E[B_1]$, it follows that $\langle e', e\rangle = \langle [x']_{B_1^\circ}, e\rangle = \langle x', e\rangle$, that is, $x'H = e'H$. As a consequence, $H : \mathbb{R}_+ \times \Omega \to E[B_1]$ is weakly measurable, and since $E[B_1]$ is separable, by the Pettis theorem we conclude that $H$ is bounded and measurable as an $E[B_1]$-valued function.

We now use the integration theory in Section 2.1. There exists a control measure $\lambda$ in this setting, since $c_o \not\subset G[B]$; hence it follows that the space of integrable functions, relative to the map $\widetilde{\phi}_0$, contains the bounded measurable functions. Thus $\int H\,dX = \int H\,d\mathbf{m} \in L_{G[B]}^2$, and the process $\left(\int H\,dX\right)_t = \int H 1_{[0,t]} d\mathbf{m}$ defines the stochastic integral; note that this process is a square integrable martingale. Since the norm on $G[B]$ is stronger than any $r \in \mathcal{G}$, one can show that $L_{G[B]}^2$ is continuously injected in $L_G^2$.

*Remarks 3.2.* (1) When we assumed $G$ was a nuclear Fréchet space, we constructed the stochastic integral for every $H$ integrable with respect to $X$. In particular, if $H$ is bounded, the stochastic integral agrees with the one constructed by means of using $L_{G[B]}^2$.

(2) Suppose $G$ is nuclear Fréchet and $H$ is integrable relative to $\phi : E \to L(L_F^2, L_G^2)$. For each seminorm $N_r$ on $L_G^2$, there is a seminorm $p(r) \in \mathcal{E}$ which induces the isometric embedding $\widetilde{\phi}$ of $E(U_{p(r)})$ into $L(L_F^2, L_G^2(U_{N_r}))$, where $L_G^2(U_{N_r})$ is a Hilbert space since it is isometrically embedded in $L_{G(U_r)}^2$. Thus each $[H]_{p(r)} : \mathbb{R}_+ \times \Omega \to E(U_{p(r)})$ is integrable relative to $\widetilde{\phi}$ and gives rise to the stochastic integral defined by $\left(\int [H]_{p(r)} 1_{[0,t]} dX\right)_{t \geq 0}$, which is a square integrable martingale. The projective system of square integrable martingales $\{[H]_{p(r)}\}_{r \in \mathcal{G}}$ has a limit in $G$, and this limit is $\left(\int H\,dX\right)_t := (M)_t$, $M_\infty = \int H\,dX$.

Since there is a control measure for $\mathbf{m}_{E, L_G^2}$, one can show that $E(M_\infty \mid \mathcal{F}_t) = M_t$.

## References


[1] S. Ustunel, "Stochastic integration on nuclear spaces and its applications," *Annales de l'Institut Henri Poincaré*, vol. 18, no. 2, pp. 165–200, 1982.

[2] J. K. Brooks and N. Dinculeanu, "Lebesgue-type spaces for vector integration, linear operators, weak completeness and weak compactness," *Journal of Mathematical Analysis and Applications*, vol. 54, no. 2, pp. 348–389, 1976.

[3] J. K. Brooks and N. Dinculeanu, "Stochastic integration in Banach spaces," in *Seminar on Stochastic Processes, 1990 (Vancouver, BC, 1990)*, vol. 24, pp. 27–115, Birkhäuser, Boston, Mass, USA, 1991.

[4] C. Dellacherie and P.-A. Meyer, *Probabilities and Potential*, vol. 29 of *North-Holland Mathematics Studies*, North-Holland, Amsterdam, The Netherlands, 1978.

[5] F. Trèves, *Topological Vector Spaces, Distributions and Kernels*, Academic Press, New York, NY, USA, 1967.

[6] J. K. Brooks and D. K. Neal, "Generalized stochastic integration on nuclear spaces," *Atti del Seminario Matematico e Fisico dell'Università di Modena*, vol. 46, pp. 83–98, 1998.