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Research Article

Optimal Control with Partial Information for Stochastic Volterra Equations

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In the first part of the paper we obtain existence and characterizations of an optimal control for a linear quadratic control problem of linear stochastic Volterra equations. In the second part, using the Malliavin calculus approach, we deduce a general maximum principle for optimal control of general stochastic Volterra equations. The result is applied to solve some stochastic control problem for some stochastic delay equations.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space and $B(t), t \geq 0$ a $\mathcal{F}_t$-real valued Brownian motion. Let $R_0 = R \setminus \{0\}$ and $\nu(dz)$ a σ-finite measure on $(R_0, \mathcal{B}(R_0))$. Let $N(dt, dz)$ denote a stationary Poisson random measure on $R_+ \times R_0$ with intensity measure $dt\nu(dz)$. Denote by $\widetilde{N}(dt, dz) = N(dt, dz) - dt\nu(dz)$ the compensated Poisson measure. Suppose that we have a cash flow where the amount $X(t)$ at time t is modelled by a stochastic delay equation of the form:

$$dX(t) = \left\{ A_1(t)X(t) + A_2(t)X(t-h) + \int_{t-h}^{t} A_0(t,s)X(s)ds \right\} dt$$
$$+ C_1(t)dB(t) + \int_{R_0} C_2(t,z)\widetilde{N}(dt,dz); \quad t \geq 0,$$
$$X(t) = \eta(t); \quad t \in [-h, 0]. \tag{1.1}$$

Here $h > 0$ is a fixed delay and $A_1(t), A_2(t), A_0(t,s), C_1(t), C_2(t,z)$, and η are given bounded deterministic functions.

Suppose that we consume at the rate $u(t)$ at time t from this wealth $X(t)$, and that this consumption rate influences the growth rate of $X(t)$ both through its value $u(t)$ at time t and through its former value $u(t-h)$, because of some delay mechanisms in the system determining the dynamics of $X(t)$.

With such a consumption rate $u(t)$ the dynamics of the corresponding cash flow $X^u(t)$ is given by

$$\begin{aligned} dX^u(t) = \bigg\{ & A_1(t)X^u(t) + A_2(t)X^u(t-h) + \int_{t-h}^{t} A_0(t,s)X^u(s)ds \\ & + B_1(t)u(t) + B_2(t)u(t-h) \bigg\} dt + C_1(t)dB(t) \\ & + \int_{R_0} C_2(t,z)\widetilde{N}(dt,dz); \quad t \in [-h,0], \\ & X^u(t) = \eta(t); \quad t \le 0, \end{aligned} \tag{1.2}$$

where $B_1(t)$ and $B_2(t)$ are deterministic bounded functions.

Suppose that the consumer wants to maximize the combined utility of the consumption up to the terminal time T and the terminal wealth. Then the problem is to find $u(\cdot)$ such that

$$J(u) := E\left[\int_0^T U_1(t,u(t))dt + U_2(X^u(T))\right] \tag{1.3}$$

is maximal. Here $U(t,\cdot)$ and $U_2(\cdot)$ are given utility functions, possibly stochastic. See Section 4.

This is an example of a stochastic control problem with delay. Such problems have been studied by many authors. See, for example, [1–5] and the references therein. The methods used in these papers, however, do not apply to the cases studied here. Moreover, these papers do not consider partial information control (see below).

It was shown in [6] that the system (1.2) is equivalent to the following controlled stochastic Volterra equation:

$$\begin{aligned} X^u(t) = & \int_0^t K(t,s)u(s)ds + \int_0^t \Phi(t,s)C(s)dB(s) + \int_0^t \int_{R_0} \Phi(t,s)C_2(s,z)\widetilde{N}(ds,dz) \\ & + \Phi(t,0)\eta(0) + \int_{-h}^{0} \Phi(t,s+h)A_2(s+h)\eta(s)ds \\ & + \int_{-h}^{0} \left(\int_0^h \Phi(t,\tau)A_0(\tau,s)d\tau\right)\eta(s)ds, \end{aligned} \tag{1.4}$$

where

$$K(t,s) = \Phi(t,s)B_1(s) + \Phi(t,s+h)B_2(s+h), \tag{1.5}$$

and Φ is the transition function satisfying

$$\frac{\partial\Phi}{\partial t} = A_1(t)\Phi(t,s) + A_2(t)\Phi(t-h,s) + \int_{t-h}^{t} A_0(t,\tau)\Phi(\tau,s)d\tau,$$
$$\Phi(s,s) = I; \qquad \Phi(t,s) = 0 \quad \text{for } t < s. \tag{1.6}$$

So the control of the system (1.2) reduces to the control of the system (1.4). For more information about stochastic control of delay equations we refer to [6] and the references therein.

Stochastic Volterra equations are interesting on their own right, also for applications, for example, to economics or population dynamics. See, for example, Example 1.1 in [7] and the references therein.

In the first part of this paper, we study a linear quadratic control problem for the following controlled stochastic Volterra equation:

$$\begin{aligned} X^u(t) = \xi(t) &+ \int_0^t [K_1(t,s)X^u(s) + D_1(t,s)u(s) + K_2(t,s)]dB(s) \\ &+ \int_0^t\int_{R_0} K_4(t,s,z)X^u(s)\widetilde{N}(ds,dz) + \int_0^t D_2(t,s)X^u(s)ds \\ &+ \int_0^t\int_{R_0} D_3(t,s,z)u(s)\widetilde{N}(ds,dz) + \int_0^t\int_{R_0} K_5(t,s,z)\widetilde{N}(ds,dz) \\ &+ \int_0^t K_3(t,s)u(s)ds, \end{aligned} \tag{1.7}$$

where $u(t)$ is our control process and $\xi(t)$ is a given predictable process with $E[\xi^2(t)] < \infty$ for all $t \ge 0$, while K_i, D_i are bounded deterministic functions. In reality one often does not have the complete information when performing a control to a system. This means that the control processes is required to be predictable with respect to a subfiltration $\{\mathcal{G}_t\}$ with $\mathcal{G}_t \subset \mathcal{F}_t$. So the space of controls will be

$$U = \left\{u(s); u(s) \text{ is } \mathcal{G}_t\text{-predictable and such that } E\left[\int_0^T |u(s)|^2 ds\right] < \infty\right\}. \tag{1.8}$$

U is a Hilbert space equipped with the inner product

$$\langle u_1, u_2\rangle = E\left[\int_0^T u_1(s)u_2(s)ds\right]. \tag{1.9}$$

$\|\cdot\|$ will denote the norm in U. Let $\mathcal{A}_{\mathcal{G}}$ be a closed, convex subset of U, which will be the space of admissible controls. Consider the linear quadratic cost functional

$$J(u) = E\left[\int_0^T Q_1(s)u^2(s)ds + \int_0^T Q_2(s)X^u(s)^2 ds + \int_0^T Q_3(s)u(s)ds + \int_0^T Q_4(s)X^u(s)ds + a_1 X^u(T)^2 + a_2 X^u(T)\right] \tag{1.10}$$

and the value function

$$J = \inf_{u\in\mathcal{A}_{\mathcal{G}}} J(u). \tag{1.11}$$

In Section 2, we prove the existence of an optimal control and provide some characterizations for the control.

In the second part of the paper (from Section 3), we consider the following general controlled stochastic Volterra equation:

$$X^u(t) = \xi(t) + \int_0^t b(t,s,X^u(s),u(s),\omega)ds + \int_0^t \sigma(t,s,X^u(s),u(s),\omega)dB(s) + \int_0^t\int_{R_0} \theta(t,s,X^u(s),u(s),z,\omega)\widetilde{N}(ds,dz), \tag{1.12}$$

where $\xi(t)$ is a given predictable process with $E[\xi^2(t)] < \infty$ for all $t \geq 0$. The performance functional is of the following form:

$$J(u) = E\left[\int_0^T f(t,X^u(t),u(t),\omega)dt + g(X^u(T),\omega)\right], \tag{1.13}$$

where $b : [0,T]\times[0,T]\times R\times R\times\Omega \to R$, $\sigma : [0,T]\times[0,T]\times R\times R\times\Omega \to R$, $\theta : [0,T]\times[0,T]\times R\times R\times R_0\times\Omega \to R$ and $f : [0,T]\times R\times R\times\Omega \to R$ are $\mathcal{F}_t$-predictable and $g : R\times\Omega \to R$ is $\mathcal{F}_T$ measurable and such that

$$E\left[\int_0^T \left|f(t,X^u(t),u(t))\right|dt + \left|g(X^u(T))\right|\right] < \infty, \tag{1.14}$$

for any $u \in \mathcal{A}_{\mathcal{G}}$, the space of admissible controls. The problem is to find $\hat{u} \in \mathcal{A}_{\mathcal{G}}$ such that

$$\Phi := \sup_{u\in\mathcal{A}_{\mathcal{G}}} J(u) = J(\hat{u}). \tag{1.15}$$

Using the Malliavin calculus, inspired by the method in [8], we will deduce a general maximum principle for the above control problem.

Remark 1.1. Note that we are off the Markovian setting because the solution of the Volterra equation is not Markovian. Therefore the classical method of dynamic programming and the Hamilton-Jacobi-Bellman equation cannot be used here.

Remark 1.2. We emphasize that partial information is different from partial observation, where the control is based on noisy observations of the (current) state. For example, our discussion includes the case $\mathcal{G}_t = \mathcal{F}_{t-\delta}$ ($\delta > 0$ constant), which corresponds to delayed information flow. This case is not covered by partial observation models. For a comprehensive presentation of the linear quadratic control problem in the classical case with partial observation, see [9], with partial information see [10].

2. Linear Quadratic Control

Consider the controlled stochastic Volterra equation (1.7) and the control problem (1.10), (1.11). We have the following Theorem.

Theorem 2.1. *Suppose that $\int_{R_0} K_4^2(t,s,z)\nu(dz)$ is bounded and $Q_2(s) \geq 0$, $a_1 \geq 0$ and $Q_1(s) \geq \delta$ for some $\delta > 0$. Then there exists a unique element $u \in \mathcal{A}_{\mathcal{G}}$ such that*

$$J = J(u) = \inf_{v \in \mathcal{A}_{\mathcal{G}}} J(v). \tag{2.1}$$

Proof. For simplicity, we assume $D_3(t,s,z) = 0$ and $K_5(t,s,z) = 0$ in this proof because these terms can be similarly estimated as the corresponding terms for Brownian motion $B(\cdot)$. By (1.7) we have

$$\begin{aligned}
E\left[X^u(t)^2\right] &\leq 7E\left[\xi(t)^2\right] + 7E\left[\left(\int_0^t K_1(t,s)X^u(s)dB(s)\right)^2\right] + 7E\left[\left(\int_0^t D_1(t,s)u(s)dB(s)\right)^2\right] \\
&\quad + 7E\left[\left(\int_0^t K_2(t,s)dB(s)\right)^2\right] + 7E\left[\left(\int_0^t K_3(t,s)u(s)ds\right)^2\right] \\
&\quad + 7E\left[\left(\int_0^t D_2(t,s)X^u(s)ds\right)^2\right] + 7E\left[\left(\int_0^t \int_{R_0} K_4(t,s,z)X^u(s)\widetilde{N}(ds,dz)\right)^2\right] \\
&\leq 7E\left[\xi(t)^2\right] + 7E\left[\int_0^t K_1^2(t,s)X^u(s)^2 ds\right] + 7E\left[\int_0^t D_1^2(t,s)u(s)^2 ds\right] \\
&\quad + 7\int_0^t K_2^2(t,s)ds + 7\int_0^t K_3^2(t,s)ds E\left[\int_0^t u^2(s)ds\right] + 7tE\left[\int_0^t D_2^2(t,s)X^u(s)^2 ds\right] \\
&\quad + 7E\left[\int_0^t \left(\int_{R_0} K_4^2(t,s,z)\nu(dz)\right)X^u(s)^2 ds\right].
\end{aligned} \tag{2.2}$$

Applying Gronwall's inequality, there exists a constant C_1 such that

$$E\Big[X^u(t)^2\Big] \le \left(C_1 E\left[\int_0^t u^2(s)ds\right] + C_1\right)e^{C_1 T}. \tag{2.3}$$

Similar arguments also lead to

$$\begin{aligned} E\Big[(X^{u_1}(t) - X^{u_2}(t))^2\Big] \le C_2 e^{C_2 T}\Bigg(& E\left[\left(\int_0^t K_3(t,s)(u_2(s) - u_1(s))ds\right)^2\right] \\ & + E\left[\int_0^t D_1(t,s)^2 (u_2(s) - u_1(s))^2 ds\right]\Bigg) \end{aligned} \tag{2.4}$$

for some constant C_2. Now, let $u_n \in \mathcal{A}_{\mathcal{G}}$ be a minimizing sequence for the value function, that is, $\lim_{n\to\infty} J(u_n) = J$. From the estimate (2.3) we see that there exists a constant c such that

$$E\left[\int_0^T Q_3(s)u(s)ds + \int_0^T Q_4(s)X^u(s)ds + a_2 X^u(T)\right] \le c\|u\| + c. \tag{2.5}$$

Thus, by virtue of the assumption on Q_1, we have, for some constant M,

$$M \ge J(u_n) \ge \delta\|u_n\|^2 - c\|u_n\| - c. \tag{2.6}$$

This implies that $\{u_n\}$ is bounded in U, hence weakly compact. Let $u_{n_k}, k \ge 1$ be a subsequence that converges weakly to some element u_0 in U. Since $\mathcal{A}_{\mathcal{G}}$ is closed and convex, the Banach-Sack Theorem implies $u_0 \in \mathcal{A}_{\mathcal{G}}$. From (2.4) we see that $u_n \to u$ in U implies that $X^{u_n}(t) \to X^u(t)$ in $L^2(\Omega)$ for every $t \ge 0$ and $X^{u_n}(\cdot) \to X^u(\cdot)$ in U. The same conclusion holds also for $Z^u(t) := X^u(t) - X^0(t)$. Since Z^u is linear in u, we conclude that equipped with the weak topology both on U and $L^2(\Omega)$, $Z^u(t) : U \to L^2(\Omega)$ is continuous for every $t \ge 0$ and $Z^u(\cdot) : U \to U$ is continuous. Thus,

$$X^u(t) : U \longrightarrow L^2(\Omega), \qquad X^u(\cdot) : U \longrightarrow U \tag{2.7}$$

are continuous with respect to the weak topology of U and $L^2(\Omega)$. Since the functionals of X^u involved in the definition of $J(u)$ in (1.10) are lower semicontinuous with respect to the weak topology, it follows that

$$\begin{aligned}\lim_{k\to\infty} J(u_{n_k}) &= \lim_{k\to\infty} E\left[\int_0^T Q_1(s)u_{n_k}^2(s)ds + \int_0^T Q_2(s)X^{u_{n_k}}(s)^2 ds + \int_0^T Q_3(s)u_{n_k}(s)ds \right.\\ &\qquad \left. + \int_0^T Q_4(s)X^{u_{n_k}}(s)ds + a_1 X^{u_{n_k}}(T)^2 + a_2 X^{u_{n_k}}(T)\right] \\ &\geq E\left[\int_0^T Q_1(s)u_0^2(s)ds + \int_0^T Q_2(s)X^{u_0}(s)^2 ds + \int_0^T Q_3(s)u_0(s)ds \right.\\ &\qquad \left. + \int_0^T Q_4(s)X^{u_0}(s)ds + a_1 X^{u_0}(T)^2 + a_2 X^{u_0}(T)\right] \\ &= J(u_0),\end{aligned} \tag{2.8}$$

which implies that u_0 is an optimal control.

The uniqueness is a consequence of the fact that $J(u)$ is strictly convex in u which is due to the fact that X^u is affine in u and x^2 is a strictly convex function. The proof is complete. □

To characterize the optimal control, we assume $D_1(t,s) = 0$ and $D_3(t,s,z) = 0$; that is, consider the controlled system:

$$\begin{aligned}X^u(t) &= \xi(t) + \int_0^t [K_1(t,s)X^u(s) + K_2(t,s)]dB(s) + \int_0^t K_3(t,s)u(s)ds \\ &\quad + \int_0^t\int_{R_0} K_4(t,s,z)X^u(s)\widetilde{N}(ds,dz) + \int_0^t D_2(t,s)X^u(s)ds \\ &\quad + \int_0^t\int_{R_0} K_5(t,s,z)\widetilde{N}(ds,dz)\end{aligned} \tag{2.9}$$

Set

$$\begin{aligned}dF(t,s) &:= d_sF(t,s) \\ &= K_1(t,s)dB(s) + \int_{R_0} K_4(t,s,z)\widetilde{N}(ds,dz) + D_2(t,s)ds.\end{aligned} \tag{2.10}$$

For a predictable process $h(s)$, we have

$$\int_0^t h(s)dF(t,s) := \int_0^t K_1(t,s)h(s)dB(s) + \int_0^t\int_{R_0} K_4(t,s,z)h(s)\widetilde{N}(ds,dz) + \int_0^t D_2(t,s)h(s)ds. \tag{2.11}$$

Introduce

$$
\begin{aligned}
M_1(t) &= \xi(t) + \sum_{n=1}^{\infty} \int_0^t dF(t, s_1) \int_0^{s_1} dF(s_1, s_2) \\
&\quad \cdots \int_0^{s_{n-1}} \xi(s_n) dF(s_{n-1}, s_n), \\
M_2(t) &= \int_0^t K_2(t, s_1) dB(s_1) + \sum_{n=1}^{\infty} \int_0^t dF(t, s_1) \int_0^{s_1} dF(s_1, s_2) \\
&\quad \cdots \int_0^{s_{n-2}} dF(s_{n-2}, s_{n-1}) \int_0^{s_{n-1}} K_2(s_{n-1}, s_n) dB(s_n), \\
M_3(t) &= \int_0^t \int_{R_0} K_5(t, s_1, z) d\widetilde{N}(ds_1, dz) + \sum_{n=1}^{\infty} \int_0^t dF(t, s_1) \int_0^{s_1} dF(s_1, s_2) \\
&\quad \cdots \int_0^{s_{n-2}} dF(s_{n-2}, s_{n-1}) \int_0^{s_{n-1}} K_5(s_{n-1}, s_n, z) d\widetilde{N}(ds_n, dz), \\
L(t, s) &= K_3(t, s) + \sum_{n=1}^{\infty} \int_s^t dF(t, s_1) \int_s^{s_1} dF(s_1, s_2) \\
&\quad \cdots \int_s^{s_{n-1}} K_3(s_n, s) dF(s_{n-1}, s_n).
\end{aligned}
\tag{2.12}
$$

Lemma 2.2. *Under our assumptions, the above series converges at least in $L^1(\Omega)$. Thus $M_i, i = 1, 2, 3$ and L are well-defined.*

Proof. We first note that

$$
\begin{aligned}
E\left[\left(\int_0^t h(s) dF(t, s)\right)^2\right] &= E\left[\int_0^t K_1^2(t, s) h^2(s) ds\right] + E\left[\int_0^t \int_{R_0} K_4^2(t, s, z) h^2(s) \nu(dz) ds\right] \\
&\quad + E\left[\left(\int_0^t D_2(t, s) h(s) ds\right)^2\right] \leq C_T E\left[\int_0^t g(t, s) h^2(s) ds\right]
\end{aligned}
\tag{2.13}
$$

for $t \leq T$, where

$$
g(t, s) = K_1^2(t, s) + \int_{R_0} K_4^2(t, s, z) \nu(dz) + D_2^2(t, s)
\tag{2.14}
$$

is a bounded deterministic function. Because of the similarity, let us prove only that M_1 is well-defined. Repeatedly using (2.13), we have

$$
\begin{aligned}
&E\left[\left(\int_0^t dF(t,s_1)\int_0^{s_1} dF(s_1,s_2)\cdots\int_0^{s_{n-1}}\xi(s_n)dF(s_{n-1},s_n)\right)^2\right]\\
&\quad\le C_T\int_0^t ds_1 g(t,s_1)E\left[\left(\int_0^{s_1} d\mathrm{F}(s_1,s_2)\cdots\int_0^{s_{n-1}}\xi(s_n)dF(s_{n-1},s_n)\right)^2\right]\\
&\quad\le\cdots\\
&\quad\le C_T^{n-1}\int_0^t ds_1 g(t,s_1)\int_0^{s_1} ds_2 g(s_1,s_2)\cdots\int_0^{s_{n-1}} ds_n g(s_{n-1},s_n)E\left[\xi^2(s_n)\right]\\
&\quad\le R_T^{n-1}E\left[\int_0^T\xi^2(s)ds\right]\frac{t^{n-1}}{(n-1)!}
\end{aligned}
\tag{2.15}
$$

for some constant R_T. This implies that

$$
\begin{aligned}
&E\left[\left|\int_0^t dF(t,s_1)\int_0^{s_1} dF(s_1,s_2)\cdots\int_0^{s_{n-1}}\xi(s_n)dF(s_{n-1},s_n)\right|\right]\\
&\quad\le R_T^{(n-1)/2}\left(E\left[\int_0^T\xi^2(s)ds\right]\right)^{1/2}\frac{t^{(n-1)/2}}{\sqrt{(n-1)!}}.
\end{aligned}
\tag{2.16}
$$

Thus, we have

$$
E[|M_1(t)|]\le E[|\xi(t)|]+\sum_{n=1}^{\infty}R_T^{(n-1)/2}\left(E\left[\int_0^T\xi^2(s)ds\right]\right)^{1/2}\frac{t^{(n-1)/2}}{\sqrt{(n-1)!}}<\infty. \tag{2.17}
$$

□

The following theorem is a characterization of the optimal control.

Theorem 2.3. *Assume that* $\int_{R_0}K_4^2(t,s,z)\nu(dz)$ *and* $\int_{R_0}K_5^2(t,s,z)\nu(dz)$ *are bounded and* $E[\int_0^T\xi^2(s)ds]<\infty$*. Suppose* $\mathcal{A}_{\mathcal{G}}=\mathcal{U}$*. Let* u *be the unique optimal control given in Theorem 2.1. Then* u *is determined by the following equation:*

$$
\begin{aligned}
&2Q_1(s)u(s)+2E\left[\int_0^T u(t)\left(\int_{s\vee t}^T Q_2(l)L(l,t)L(l,s)dl\right)dt\mid\mathcal{G}_s\right]\\
&\quad+2a_1E\left[\int_0^T u(t)L(T,t)L(T,s)dt\mid\mathcal{G}_s\right]+Q_3(s)+E\left[\int_s^T Q_4(l)L(l,s)dl\mid\mathcal{G}_s\right]\\
&\quad+2E\left[\int_s^T Q_2(l)(M_1(l)+M_2(l)+M_3(l))L(l,s)dl\mid\mathcal{G}_s\right]+a_2E[L(T,s)\mid\mathcal{G}_s]\\
&\quad+2a_1E[(M_1(T)+M_2(T)+M_3(T))L(T,s)\mid\mathcal{G}_s]=0,
\end{aligned}
\tag{2.18}
$$

almost everywhere with respect to $m(ds,d\omega):ds\times P(d\omega)$.

Proof. For any $w \in U$, since u is the optimal control, we have

$$J'(u)(w) = \frac{d}{d\varepsilon} J(u + \varepsilon w)\Big|_{\varepsilon=0} = 0. \tag{2.19}$$

This leads to

$$\begin{aligned} E\Bigg[& 2\int_0^T Q_1(s)u(s)w(s)ds + 2\int_0^T Q_2(s)X^u(s)\frac{d}{d\varepsilon}X^{u+\varepsilon w}(s)\Big|_{\varepsilon=0} ds \\ & + \int_0^T Q_3(s)w(s)ds + \int_0^T Q_4(s)\frac{d}{d\varepsilon}X^{u+\varepsilon w}(s)\Big|_{\varepsilon=0} ds \\ & + 2a_1 X^u(T)\frac{d}{d\varepsilon}X^{u+\varepsilon w}(T)\Big|_{\varepsilon=0} + a_2\frac{d}{d\varepsilon}X^{u+\varepsilon w}(T)\Big|_{\varepsilon=0}\Bigg] = 0 \end{aligned} \tag{2.20}$$

for all $w \in U$. By virtue of (2.9), it is easy to see that

$$Y^w(t) := \frac{d}{d\varepsilon}X^{u+\varepsilon w}(t)\Big|_{\varepsilon=0} \tag{2.21}$$

satisfies the following equation:

$$\begin{aligned} Y^w(t) = & \int_0^t K_1(t,s)Y^w(s)dB(s) + \int_0^t K_3(t,s)w(s)ds \\ & + \int_0^t \int_E K_4(t,s,z)Y^w(s)\widetilde{N}(ds,dz) + \int_0^t D_2(t,s)Y^w(s)ds. \end{aligned} \tag{2.22}$$

Remark that Y^w is independent of u. Next we will find an explicit expression for X^u. Let $dF(t,s)$ be defined as in (2.10). Repeatedly using (2.9) we have

$$\begin{aligned} X^u(t) = & \xi(t) + \int_0^t [K_1(t,s_1)X^u(s_1) + K_2(t,s_1)]dB(s_1) + \int_0^t K_3(t,s_1)u(s_1)ds_1 \\ & + \int_0^t \int_{R_0} K_4(t,s_1,z)X^u(s_1)\widetilde{N}(ds_1,dz) + \int_0^t D_2(t,s_1)X^u(s_1)ds \\ & + \int_0^t \int_{R_0} K_5(t,s_1,z)\widetilde{N}(ds_1,dz) \end{aligned}$$

$$
\begin{aligned}
&= \xi(t) + \int_0^t K_1(t,s_1)\Bigg[\xi(s_1) + \int_0^{s_1}[K_1(s_1,s_2)X^u(s_2) + K_2(s_1,s_2)]dB(s_2) \\
&\quad + \int_0^{s_1}\int_{R_0} K_4(s_1,s_2,z)X^u(s_2)\widetilde{N}(ds_2,dz) + \int_0^{s_1} K_3(s_1,s_2)u(s_2)ds_2 \\
&\quad + \int_0^{s_1} D_2(s_1,s_2)X^u(s_2)ds_2 + \int_0^{s_1}\int_{R_0} K_5(s_1,s_2,z)\widetilde{N}(ds_2,dz)\Bigg]dB(s_1) \\
&\quad + \int_0^t\int_{R_0} K_4(t,s_1,z)\Bigg[\xi(s_1) + \int_0^{s_1}[K_1(s_1,s_2)X^u(s_2) + K_2(s_1,s_2)]dB(s_2) \\
&\quad + \int_0^{s_1}\int_{R_0} K_4(s_1,s_2,z)X^u(s_2)\widetilde{N}(ds_2,dz) + \int_0^{s_1} K_3(s_1,s_2)u(s_2)ds_2 \\
&\quad + \int_0^{s_1} D_2(s_1,s_2)X^u(s_2)ds_2 + \int_0^{s_1}\int_{R_0} K_5(s_1,s_2,z)\widetilde{N}(ds_2,\mathrm{d}z)\Bigg]\widetilde{N}(ds_1,dz) \\
&\quad + \int_0^t\int_{R_0} D_2(t,s_1,z)\Bigg[\xi(s_1) + \int_0^{s_1}[K_1(s_1,s_2)X^u(s_2) + K_2(s_1,s_2)]dB(s_2) \\
&\quad + \int_0^{s_1}\int_{R_0} K_4(s_1,s_2,z)X^u(s_2)\widetilde{N}(ds_2,dz) + \int_0^{s_1} K_3(s_1,s_2)u(s_2)ds_2 \\
&\quad + \int_0^{s_1} D_2(s_1,s_2)X^u(s_2)ds_2 + \int_0^{s_1}\int_{R_0} K_5(s_1,s_2,z)\widetilde{N}(ds_2,dz)\Bigg]ds_1 \\
&\quad + \int_0^t K_2(t,s_1)dB(s_1) + \int_0^t K_3(t,s_1)u(s_1)ds_1 + \int_0^t\int_{R_0} K_5(t,s_1,z)\widetilde{N}(ds_1,dz) \\
&= \cdots \\
&= \xi(t) + \sum_{n=1}^{\infty}\int_0^t dF(t,s_1)\int_0^{s_1} dF(s_1,s_2)\cdots\int_0^{s_{n-1}}\xi(s_n)dF(s_{n-1},s_n) \\
&\quad + \sum_{n=1}^{\infty}\int_0^t dF(t,s_1)\int_0^{s_1} dF(s_1,s_2) \\
&\qquad \cdots\int_0^{s_{n-2}} dF(s_{n-2},s_{n-1})\int_0^{s_{n-1}} K_2(s_{n-1},s_n)dB(s_n) \\
&\quad + \sum_{n=1}^{\infty}\int_0^t dF(t,s_1)\int_0^{s_1} dF(s_1,s_2) \\
&\qquad \cdots\int_0^{s_{n-2}} dF(s_{n-2},s_{n-1})\int_0^{s_{n-1}} K_3(s_{n-1},s_n)u(s_n)ds_n \\
&\quad + \sum_{n=1}^{\infty}\int_0^t dF(t,s_1)\int_0^{s_1} dF(s_1,s_2) \\
&\qquad \cdots\int_0^{s_{n-2}} dF(s_{n-2},s_{n-1})\int_0^{s_{n-1}}\int_{R_0} K_5(s_{n-1},s_n,z)\widetilde{N}(ds_n,dz) \\
&\quad + \int_0^t K_2(t,s_1)dB(s_1) + \int_0^t K_3(t,s_1)u(s_1)ds_1 \\
&\quad + \int_0^t\int_{R_0} K_5(t,s_1,z)\widetilde{N}(ds_1,dz).
\end{aligned}
\tag{2.23}
$$

Similarly, we have the following expansion for Y^w:

$$Y^w(t) = \int_0^t K_3(t,s)w(s)ds + \sum_{n=1}^{\infty} \int_0^t dF(t,s_1) \int_0^{s_1} dF(s_1,s_2) \cdots \int_0^{s_{n-2}} dF(s_{n-2},s_{n-1}) \int_0^{s_{n-1}} K_3(s_{n-1},s_n)w(s_n)ds_n. \tag{2.24}$$

Interchanging the order of integration,

$$\begin{aligned} Y^w(t) &= \int_0^t w(s)\left[K_3(t,s) + \sum_{n=1}^{\infty} \int_s^t dF(t,s_1) \int_s^{s_1} dF(s_1,s_2) \cdots \int_s^{s_{n-1}} K_3(s_n,s)dF(s_{n-1},s_n)\right]ds \\ &= \int_0^t L(t,s)w(s)ds. \end{aligned} \tag{2.25}$$

Now substituting Y^w into (2.20) we obtain that

$$\begin{aligned} &E\left[2\int_0^T Q_1(s)u(s)w(s)ds + 2\int_0^T Q_2(s)X^u(s)\left(\int_0^s L(s,l)w(l)dl\right)ds\right] \\ &\quad + E\left[\int_0^T Q_3(s)w(s)ds + \int_0^T Q_4(s)\left(\int_0^s L(s,l)w(l)dl\right)ds\right] \\ &\quad + 2a_1E\left[\int_0^T X^u(T)L(T,s)w(s)ds + a_2\int_0^T L(T,s)w(s)ds\right] = 0 \end{aligned} \tag{2.26}$$

for all $w \in U$. Interchanging the order of integration and conditioning on $\mathcal{G}_s$ we see that (2.26) is equivalent to

$$\begin{aligned} &E\left[2\int_0^T Q_1(s)u(s)w(s)ds + 2\int_0^T w(s)E\left[\int_s^T Q_2(l)X^u(l)L(l,s)dl \mid \mathcal{G}_s\right]ds\right] \\ &\quad + E\left[\int_0^T Q_3(s)w(s)ds + \int_0^T w(s)E\left[\int_s^T Q_4(l)L(l,s)dl \mid \mathcal{G}_s\right]ds\right] \\ &\quad + 2a_1E\left[\int_0^T E[X^u(T)L(T,s) \mid \mathcal{G}_s]w(s)ds\right] \\ &\quad + a_2E\left[\int_0^T E[L(T,s) \mid \mathcal{G}_s]w(s)ds\right] = 0. \end{aligned} \tag{2.27}$$

Since this holds for all $w \in \mathcal{U}$, we conclude that

$$2Q_1(s)u(s) + 2E\left[\int_s^T Q_2(l)X^u(l)L(l,s)dl \mid \mathcal{G}_s\right] + Q_3(s) + E\left[\int_s^T Q_4(l)L(l,s)dl \mid \mathcal{G}_s\right] \\ + 2a_1E[X^u(T)L(T,s) \mid \mathcal{G}_s] + a_2E[L(T,s) \mid \mathcal{G}_s] = 0, \tag{2.28}$$

m-a.e. Note that $X^u(t)$ can be written as

$$X^u(t) = M_1(t) + M_2(t) + M_3(t) + \int_0^t u(s)L(t,s)ds. \tag{2.29}$$

Substituting $X^u(t)$ into (2.28), we get (2.18), completing the proof. □

Example 2.4. Consider the controlled system

$$X^u(t) = \xi(t) + \int_0^t K_2(t,s)dB(s) + \int_0^t K_3(t,s)u(s)ds \tag{2.30}$$

and the performance functional

$$J(u) = E\left[\int_0^T Q_1(s)u^2(s)ds + \int_0^T Q_3(s)u(s)ds + \int_0^T Q_4(s)X^u(s)ds + a_1X^u(T)^2 + a_2X^u(T)\right]. \tag{2.31}$$

Suppose $\mathcal{G}_t = \{\Omega, \emptyset\}$, meaning that the control is deterministic. In this case, we can find the unique optimal control explicitly. Noting that the conditional expectation reduces to expectation, the (2.18) for the optimal control u becomes

$$2Q_1(s)u(s) + 2a_1\left(\int_0^T u(t)K_3(T,t)dt\right)K_3(T,s) \\ + Q_3(s) + \int_s^T Q_4(l)K_3(l,s)dl + a_2K_3(T,s) + 2a_1g(T)K_3(T,s) = 0 \quad ds\text{-a.e.}, \tag{2.32}$$

where we have used the fact that $E[M_2(t)] = 0$, $M_1(t) = \xi(t), L(t,s) = K_3(t,s)$ in this special case. Put

$$b = \int_0^T u(t)K_3(T,t)dt. \tag{2.33}$$

Then (2.33) yields

$$u(s) = -a_1b\frac{K_3(T,s)}{Q_1(s)} + h(s), \quad ds\text{-a.e.}, \tag{2.34}$$

where

$$h(s) = -\frac{Q_3(s) + \int_s^T Q_4(l)K_3(l,s)dl}{2Q_1(s)} - \frac{a_2K_3(T,s) + 2a_1g(T)K_3(T,s)}{2Q_1(s)}. \tag{2.35}$$

Substitute the expression of u into (2.34) to get

$$-a_1b\int_0^T \frac{K_3(T,t)^2}{Q_1(t)}dt + \int_0^T h(t)K_3(T,t)dt = b. \tag{2.36}$$

Consequently,

$$b = \frac{1}{1 + a_1\int_0^T \left(K_3(T,t)^2/Q_1(t)\right)dt}\int_0^T h(t)K_3(T,t)dt. \tag{2.37}$$

Together with (2.35) we arrive at

$$u(s) = -a_1\left(\frac{1}{1 + a_1\int_0^T \left(K_3(T,t)^2/Q_1(t)\right)dt}\int_0^T h(t)K_3(T,t)dt\right)\frac{K_3(T,s)}{Q_1(s)} + h(s), \tag{2.38}$$

ds-a.e.

3. A General Maximum Principle

In this section, we consider the following general controlled stochastic Volterra equation:

$$\begin{aligned} X^u(t) = \xi(t) &+ \int_0^t b(t,s,X^u(s),u(s),\omega)ds + \int_0^t \sigma(t,s,X^u(s),u(s),\omega)dB(s) \\ &+ \int_0^t\int_{R_0} \theta(t,s,X^u(s),u(s),z,\omega)\widetilde{N}(ds,dz), \end{aligned} \tag{3.1}$$

where $u(t)$ is our control process taking values in R and $\xi(t)$ is as in (1.7). More precisely, $u \in \mathcal{A}_{\mathcal{G}}$, where $\mathcal{A}_{\mathcal{G}}$ is a family of $\mathcal{G}_t$-predictable controls. Here $\mathcal{G}_t \subset \mathcal{F}_t$ is a given subfiltration and $b : [0,T]\times[0,T]\times R\times R\times\Omega \to R$, $\sigma : [0,T]\times[0,T]\times R\times R\times\Omega \to R$ and $\theta : [0,T]\times[0,T]\times R\times R\times R_0\times\Omega \to R$ are given measurable, $\mathcal{F}_t$-predictable functions. Consider a performance functional of the following form:

$$J(u) = E\left[\int_0^T f(t,X^u(t),u(t),\omega)dt + g(X^u(T),\omega)\right], \tag{3.2}$$

where $f : [0,T]\times R\times D\times\Omega \to R$ is $\mathcal{F}_t$ predictable and $g : R\times\Omega \to R$ is $\mathcal{F}_T$ measurable and such that

$$E\left[\int_0^T \left|f(t,X^u(t),u(t),\omega)\right|dt + \left|g(X^u(T),\omega)\right|\right] < \infty, \quad \forall u \in \mathcal{A}_{\mathcal{G}}. \tag{3.3}$$

The purpose of this section is to give a characterization for the critical point of $J(u)$. First, in the following two subsections we recall briefly some basic properties of Malliavin calculus for $B(\cdot)$ and $\widetilde{N}(\cdot,\cdot)$ which will be used in the sequel. For more information we refer to [11] and [12].

3.1. Integration by Parts Formula for $B(\cdot)$

In this subsection, $\mathcal{F}_T = \sigma(B(s), 0 \le s \le T)$. Recall that the Wiener-Ito chaos expansion theorem states that any $F \in L^2(\mathcal{F}_T, P)$ admits the representation

$$F = \sum_{n=0}^{\infty} I_n(f_n) \tag{3.4}$$

for a unique sequence of symmetric deterministic function $f_n \in L^2([0,T]^{\times n})$ and

$$I_n(f_n) = n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \dots, t_n) dB(t_1) dB(t_2) \cdots dB(t_n). \tag{3.5}$$

Moreover, the following isometry holds:

$$E\left[F^2\right] = \sum_{n=0}^{\infty} n! \|f_n\|^2_{L^2([0,T]^{\times n})}. \tag{3.6}$$

Let $D_{1,2}$ be the space of all $F \in L^2(\mathcal{F}_T, P)$ such that its chaos expansion (3.4) satisfies

$$\|F\|^2_{D_{1,2}} := \sum_{n=0}^{\infty} n n! \|f_n\|^2_{L^2([0,T]^{\times n})} < \infty. \tag{3.7}$$

For $F \in D_{1,2}$ and $t \in [0,T]$, the Malliavin derivative of F, $D_t F$, is defined by

$$D_t F = \sum_{n=0}^{\infty} n I_{n-1}(f_n(\cdot, t)), \tag{3.8}$$

where $I_{n-1}(f_n(\cdot,t))$ is the $n-1$ times iterated integral to the first $n-1$ variables of f_n keeping the last variable $t_n = t$ as a parameter. We need the following result.

Theorem A (Integration by parts formula (duality formula) for $B(\cdot)$). *Suppose that $h(t)$ is $\mathcal{F}_t$-adapted with $E[\int_0^T h^2(t)dt] < \infty$ and let $F \in D_{1,2}$. Then*

$$E\left[F \int_0^T h(t) dB(t)\right] = E\left[\int_0^T h(t) D_t F \, dt\right]. \tag{3.9}$$

3.2. Integration by Parts Formula for $\widetilde{N}$

In this section $\mathcal{F}_T = \sigma(\eta(s), 0 \le s \le T)$, where $\eta(s) = \int_0^s \int_{R_0} z\widetilde{N}(dr, dz)$. Recall that the Wiener-Ito chaos expansion theorem states that any $F \in L^2(\mathcal{F}_T, P)$ admits the representation

$$F = \sum_{n=0}^{\infty} I_n(f_n) \tag{3.10}$$

for a unique sequence of functions $f_n \in \widehat{L}^2((dt \times \nu)^n)$, where $\widehat{L}^2((dt \times \nu)^n)$ is the space of functions $f_n(t_1, z_1, \dots, t_n, z_n)$; $t_i \in [0, T]$, $z_i \in R_0$ such that $f_n \in L^2((dt \times \nu)^n)$ and f_n is symmetric with respect to the pairs of variables $(t_1, z_1), (t_2, z_2), \dots, (t_n, z_n)$. Here $I_n(f_n)$ is the iterated integral:

$$I_n(f_n) = n! \int_0^T \int_{R_0} \int_0^{t_n} \int_{R_0} \cdots \int_0^{t_2} \int_{R_0} f_n(t_1, z_1, \dots, t_n, z_n) \widetilde{N}(dt_1, dz_1) \cdots \widetilde{N}(dt_n, dz_n). \tag{3.11}$$

Moreover, the following isometry holds:

$$E\left[F^2\right] = \sum_{n=0}^{\infty} n! \|f_n\|^2_{L^2((dt \times \nu)^n)}. \tag{3.12}$$

Let $\widetilde{D}_{1,2}$ be the space of all $F \in L^2(\mathcal{F}_T, P)$ such that its chaos expansion (3.18) satisfies

$$\|F\|^2_{\widetilde{D}_{1,2}} := \sum_{n=0}^{\infty} n n! \|f_n\|^2_{L^2((dt \times \nu)^n)} < \infty. \tag{3.13}$$

For $F \in \widetilde{D}_{1,2}$ and $t \in [0, T]$, the Malliavin derivative of F, $D_{t,z}F$, is defined by

$$D_{t,z}F = \sum_{n=0}^{\infty} n I_{n-1}(f_n(\cdot, t, z)), \tag{3.14}$$

where $I_{n-1}(f_n(\cdot, t, z))$ is the $n-1$ times iterated integral with respect to the first $n-1$ pairs of variables of f_n keeping the last pair $(t_n, z_n) = (t, z)$ as a parameter. We need the following result

Theorem B (Integration by parts formula (duality formula) for $\widetilde{N}$). *Suppose $h(t, z)$ is $\mathcal{F}_t$-predictable with $E[\int_0^T \int_{R_0} h^2(t, z) dt \nu(dz)] < \infty$ and let $F \in \widetilde{D}_{1,2}$. Then*

$$E\left[F \int_0^T \int_{R_0} h(t, z) \widetilde{N}(dt, dz)\right] = E\left[\int_0^T \int_{R_0} h(t, z) D_{t,z}F \, dt \nu(dz)\right]. \tag{3.15}$$

3.3. Maximum Principles

Consider (3.1). We will make the following assumptions throughout this subsection.

(H.1) The functions $b : [0,T]\times[0,T]\times R\times R\times\Omega \to R$, $\sigma : [0,T]\times[0,T]\times R\times R\times\Omega \to R$, $\theta : [0,T]\times[0,T]\times R\times R\times R_0\times\Omega \to R$, $f : [0,T]\times R\times R\times\Omega \to R$, and $g : R\times\Omega \to R$ are continuously differentiable with respect to $x \in R$ and $u \in R$.

(H.2) For all $t \in (0,T)$ and all bounded $\mathcal{G}_t$-measurable random variables α the control

$$\beta_\alpha(s) = \alpha\chi_{[t,T]}(s) \tag{3.16}$$

belongs to $\mathcal{A}_{\mathcal{G}}$.

(H.3) For all $u, \beta \in \mathcal{A}_{\mathcal{G}}$ with β bounded, there exists $\delta > 0$ such that

$$u + y\beta \in \mathcal{A}_{\mathcal{G}} \quad \forall y \in (-\delta, \delta). \tag{3.17}$$

(H.4) For all $u, \beta \in \mathcal{A}_{\mathcal{G}}$ with β bounded, the process $Y^\beta(t) = (d/dy)X^{(u+y\beta)}(t)|_{y=0}$ exists and satisfies the following equation:

$$\begin{aligned} Y^\beta(t) = &\int_0^t \frac{\partial b}{\partial x}(t,s,X^u(s),u(s))Y^\beta(s)ds + \int_0^t \frac{\partial b}{\partial u}(t,s,X^u(s),u(s))\beta(s)ds \\ &+ \int_0^t \frac{\partial \sigma}{\partial x}(t,s,X^u(s),u(s))Y^\beta(s)dB(s) + \int_0^t \frac{\partial \sigma}{\partial u}(t,s,X^u(s),u(s))\beta(s)dB(s) \\ &+ \int_0^t\int_{R_0} \frac{\partial \theta}{\partial x}(t,s,X^u(s),u(s),z)Y^\beta(s)\widetilde{N}(ds,dz) \\ &+ \int_0^t\int_{R_0} \frac{\partial \theta}{\partial u}(t,s,X^u(s),u(s),z)\beta(s)\widetilde{N}(ds,dz). \end{aligned} \tag{3.18}$$

(H.5) For all $u \in \mathcal{A}_{\mathcal{G}}$, the Malliavin derivatives $D_t(g'(X^u(T)))$ and $D_{t,z}(g'(X^u(T)))$ exist.

In the sequel, we omit the random parameter ω for simplicity. Let $J(u)$ be defined as in (3.2).

(H.6) The functions $(\partial b/\partial u)(t,s,x,u)^2$, $(\partial b/\partial x)(t,s,x,u)^2$, $(\partial \sigma/\partial u)(t,s,x,u)^2$, $(\partial \sigma/\partial x)(t,s,x,u)^2$, and $\int_{R_0}(\partial\theta/\partial u)(t,s,x,u,z)^2\nu(dz)$, $\int_{R_0}(\partial\theta/\partial x)(t,s,x,u,z)^2\nu(dz)$ are bounded on $[0,T]\times[0,T]\times R\times R\times\Omega$.

Theorem 3.1 (Maximum principle I for optimal control of stochastic Volterra equations). *(1) Suppose that $\hat{u}$ is a critical point for $J(u)$ in the sense that $(d/dy)J(\hat{u}+y\beta)|_{y=0}=0$ for all bounded $\beta\in\mathcal{A}_{\mathcal{G}}$. Then*

$$\begin{aligned}E\Bigg[\Bigg\{&\int_t^T \frac{\partial f}{\partial x}\left(s,\widehat{X}(s),\hat{u}(s)\right)\Lambda(s,t)ds+\int_t^T \frac{\partial f}{\partial u}\left(s,\widehat{X}(s),\hat{u}(s)\right)ds\\&+\int_t^T \frac{\partial b}{\partial x}\left(T,s,\widehat{X}(s),\hat{u}(s)\right)\Lambda(s,t)g'\left(\widehat{X}(T)\right)ds\\&+\int_t^T \frac{\partial b}{\partial u}\left(T,s,\widehat{X}(s),\hat{u}(s)\right)g'\left(\widehat{X}(T)\right)ds\\&+\int_t^T \frac{\partial \sigma}{\partial x}\left(T,s,\widehat{X}(s),\hat{u}(s)\right)\Lambda(s,t)D_s\left(g'\left(\widehat{X}(T)\right)\right)ds\\&+\int_t^T \frac{\partial \sigma}{\partial u}\left(T,s,\widehat{X}(s),\hat{u}(s)\right)D_s\left(g'\left(\widehat{X}(T)\right)\right)ds\\&+\int_t^T \left(\int_{R_0}\frac{\partial \theta}{\partial x}\left(T,s,\widehat{X}(s),\hat{u}(s),z\right)\Lambda(s,t)D_{s,z}\left(g'\left(\widehat{X}(T)\right)\right)\nu(dz)\right)ds\\&+\int_t^T \left(\int_{R_0}\frac{\partial \theta}{\partial u}\left(T,s,\widehat{X}(s),\hat{u}(s),z\right)D_{s,z}\left(g'\left(\widehat{X}(T)\right)\right)\nu(dz)\right)ds\Bigg\}\mid\mathcal{G}_t\Bigg]=0,\end{aligned}\tag{3.19}$$

where $\Lambda(s,t)$ is defined in (3.29) *below and $\widehat{X}=X^{\hat{u}}$.*

(2) Conversely, suppose $\hat{u}\in\mathcal{A}_{\mathcal{G}}$ such that (3.19) *holds. Then $\hat{u}$ is a critical point for $J(\cdot)$.*

Proof. (1) Suppose that $\hat{u}$ is a critical point for $J(u)$. Let $\beta\in\mathcal{A}_{\mathcal{G}}$ be bounded. Write $\widehat{X}=X^{\hat{u}}$. Then

$$0=\frac{d}{dy}J(\hat{u}+y\beta)\Big|_{y=0}=E\left[\int_0^T\left\{\frac{\partial f}{\partial x}\left(t,\widehat{X}(t),\hat{u}(t)\right)Y^\beta(t)+\frac{\partial f}{\partial u}\left(t,\widehat{X}(t),\hat{u}(t)\right)\beta(t)\right\}dt+g'\left(\widehat{X}(T)\right)Y^\beta(T)\right],\tag{3.20}$$

where

$$\begin{aligned}Y^\beta(t)&=\frac{d}{dy}X^{(\hat{u}+y\beta)}(t)\Big|_{y=0}\\&=\int_0^t\frac{\partial b}{\partial x}\left(t,s,\widehat{X}(s),\hat{u}(s)\right)Y^\beta(s)ds+\int_0^t\frac{\partial b}{\partial u}\left(t,s,\widehat{X}(s),\hat{u}(s)\right)\beta(s)ds\\&\quad+\int_0^t\frac{\partial \sigma}{\partial x}\left(t,s,\widehat{X}(s),\hat{u}(s)\right)Y^\beta(s)dB(s)+\int_0^t\frac{\partial \sigma}{\partial u}\left(t,s,\widehat{X}(s),\hat{u}(s)\right)\beta(s)dB(s)\\&\quad+\int_0^t\int_{R_0}\frac{\partial \theta}{\partial x}\left(t,s,\widehat{X}(s),\hat{u}(s),z\right)Y^\beta(s)\widetilde{N}(ds,dz)\\&\quad+\int_0^t\int_{R_0}\frac{\partial \theta}{\partial u}\left(t,s,\widehat{X}(s),\hat{u}(s),z\right)\beta(s)\widetilde{N}(ds,dz).\end{aligned}\tag{3.21}$$

By the duality formulae (3.9) and (3.15), we have

$$
\begin{aligned}
E\Big[g'\Big(\widehat{X}(T)\Big)Y^{\beta}(T)\Big] &= E\left[\int_0^T \frac{\partial b}{\partial x}\Big(T,s,\widehat{X}(s),\widehat{u}(s)\Big)Y^{\beta}(s)g'\Big(\widehat{X}(T)\Big)ds\right] \\
&\quad + E\left[\int_0^T \frac{\partial b}{\partial u}\Big(T,s,\widehat{X}(s),\widehat{u}(s)\Big)\beta(s)g'\Big(\widehat{X}(T)\Big)ds\right] \\
&\quad + E\left[\left(\int_0^T \frac{\partial \sigma}{\partial x}\Big(T,s,\widehat{X}(s),\widehat{u}(s)\Big)Y^{\beta}(s)dB(s)\right)g'\Big(\widehat{X}(T)\Big)\right] \\
&\quad + E\left[\left(\int_0^T \frac{\partial \sigma}{\partial u}\Big(T,s,\widehat{X}(s),\widehat{u}(s)\Big)\beta(s)dB(s)\right)g'\Big(\widehat{X}(T)\Big)\right] \\
&\quad + E\left[\left(\int_0^T\int_{R_0} \frac{\partial \theta}{\partial x}\Big(T,s,\widehat{X}(s),\widehat{u}(s),z\Big)Y^{\beta}(s)\widetilde{N}(ds,dz)\right)g'\Big(\widehat{X}(T)\Big)\right] \\
&\quad + E\left[\left(\int_0^T\int_{R_0} \frac{\partial \theta}{\partial u}\Big(T,s,\widehat{X}(s),\widehat{u}(s),z\Big)\beta(s)\widetilde{N}(ds,dz)\right)g'\Big(\widehat{X}(T)\Big)\right] \\
&= E\left[\int_0^T \frac{\partial b}{\partial x}\Big(T,s,\widehat{X}(s),\widehat{u}(s)\Big)Y^{\beta}(s)g'\Big(\widehat{X}(T)\Big)ds\right] \\
&\quad + E\left[\int_0^T \frac{\partial b}{\partial u}\Big(T,s,\widehat{X}(s),\widehat{u}(s)\Big)\beta(s)g'\Big(\widehat{X}(T)\Big)ds\right] \\
&\quad + E\left[\int_0^T \frac{\partial \sigma}{\partial x}\Big(T,s,\widehat{X}(s),\widehat{u}(s)\Big)Y^{\beta}(s)D_s\Big(g'\Big(\widehat{X}(T)\Big)\Big)ds\right] \\
&\quad + E\left[\int_0^T \frac{\partial \sigma}{\partial u}\Big(T,s,\widehat{X}(s),\widehat{u}(s)\Big)\beta(s)D_s\Big(g'\Big(\widehat{X}(T)\Big)\Big)ds\right] \\
&\quad + E\left[\left(\int_0^T\int_{R_0} \frac{\partial \theta}{\partial x}\Big(T,s,\widehat{X}(s),\widehat{u}(s),z\Big)Y^{\beta}(s)D_{s,z}\Big(g'\Big(\widehat{X}(T)\Big)\Big)\nu(dz)ds\right)\right] \\
&\quad + E\left[\left(\int_0^T\int_{R_0} \frac{\partial \theta}{\partial u}\Big(T,s,\widehat{X}(s),\widehat{u}(s),z\Big)\beta(s)D_{s,z}\Big(g'\Big(\widehat{X}(T)\Big)\Big)\nu(dz)ds\right)\right].
\end{aligned}
\tag{3.22}
$$

Let α be bounded, $\mathcal{G}_t$ measurable. Choose $\beta_\alpha(s) = \alpha\chi_{[t,T]}(s)$ and substitute (3.22) into (3.20) to obtain

$$\begin{aligned}
&E\left[\int_t^T\left\{\frac{\partial f}{\partial x}\left(s,\widehat{X}(s),\widehat{u}(s)\right)Y^{\beta_\alpha}(s)ds+\alpha\int_t^T\frac{\partial f}{\partial u}\left(s,\widehat{X}(s),\widehat{u}(s)\right)ds\right\}\right]\\
&\quad+E\left[\int_t^T\frac{\partial b}{\partial x}\left(T,s,\widehat{X}(s),\widehat{u}(s)\right)Y^{\beta_\alpha}(s)g'\left(\widehat{X}(T)\right)ds\right]\\
&\quad+E\left[\alpha\int_t^T\frac{\partial b}{\partial u}\left(T,s,\widehat{X}(s),\widehat{u}(s)\right)g'\left(\widehat{X}(T)\right)ds\right]\\
&\quad+E\left[\int_t^T\frac{\partial \sigma}{\partial x}\left(T,s,\widehat{X}(s),\widehat{u}(s)\right)Y^{\beta_\alpha}(s)D_s\left(g'\left(\widehat{X}(T)\right)\right)ds\right]\\
&\quad+E\left[\alpha\int_t^T\frac{\partial \sigma}{\partial u}\left(T,s,\widehat{X}(s),\widehat{u}(s)\right)D_s\left(g'\left(\widehat{X}(T)\right)\right)ds\right]\\
&\quad+E\left[\left(\int_t^T\int_{R_0}\frac{\partial \theta}{\partial x}\left(T,s,\widehat{X}(s),\widehat{u}(s),z\right)Y^{\beta}(s)D_{s,z}\left(g'\left(\widehat{X}(T)\right)\right)\nu(dz)ds\right)\right]\\
&\quad+E\left[\alpha\int_t^T\int_{R_0}\frac{\partial \theta}{\partial u}\left(T,s,\widehat{X}(s),\widehat{u}(s),z\right)D_{s,z}\left(g'\left(\widehat{X}(T)\right)\right)\nu(dz)ds\right]=0,
\end{aligned} \tag{3.23}$$

where $Y^{\beta_\alpha}(l) = 0$ for $l \leq t$, and for $l \geq t$,

$$\begin{aligned}
Y^{\beta_\alpha}(l) &= \int_t^l \frac{\partial b}{\partial x}\left(l,s,\widehat{X}(s),\widehat{u}(s)\right)Y^{\beta_\alpha}(s)ds\\
&\quad+\alpha\int_t^l\frac{\partial b}{\partial u}\left(l,s,\widehat{X}(s),\widehat{u}(s)\right)ds\\
&\quad+\int_t^l\frac{\partial \sigma}{\partial x}\left(l,s,\widehat{X}(s),\widehat{u}(s)\right)Y^{\beta_\alpha}(s)dB(s)\\
&\quad+\alpha\int_t^l\frac{\partial \sigma}{\partial u}\left(l,s,\widehat{X}(s),\widehat{u}(s)\right)dB(s)\\
&\quad+\int_t^l\int_{R_0}\frac{\partial \theta}{\partial x}\left(l,s,\widehat{X}(s),\widehat{u}(s),z\right)Y^{\beta}(s)\widetilde{N}(ds,dz)\\
&\quad+\alpha\int_t^l\int_{R_0}\frac{\partial \theta}{\partial u}\left(l,s,\widehat{X}(s),\widehat{u}(s),z\right)\widetilde{N}(ds,dz).
\end{aligned} \tag{3.24}$$

For $l \geq s$, put

$$\begin{aligned} d\Gamma(l,s) &:= d_s\Gamma(l,s) \\ &= \frac{\partial b}{\partial x}\Big(l,s,\widehat{X}(s),\widehat{u}(s)\Big)ds + \frac{\partial \sigma}{\partial x}\Big(l,s,\widehat{X}(s),\widehat{u}(s)\Big)dB(s) \\ &\quad + \int_{R_0} \frac{\partial \theta}{\partial x}\Big(l,s,\widehat{X}(s),\widehat{u}(s),z\Big)\widetilde{N}(ds,dz). \end{aligned} \tag{3.25}$$

This means that for a predictable process $h(s)$, we have

$$\begin{aligned} \int_t^l h(s)d\Gamma(l,s) &= \int_t^l \frac{\partial b}{\partial x}\Big(l,s,\widehat{X}(s),\widehat{u}(s)\Big)h(s)ds + \int_t^l \frac{\partial \sigma}{\partial x}\Big(l,s,\widehat{X}(s),\widehat{u}(s)\Big)h(s)dB(s) \\ &\quad + \int_t^l \int_{R_0} \frac{\partial \theta}{\partial x}\Big(l,s,\widehat{X}(s),\widehat{u}(s),z\Big)h(s)\widetilde{N}(ds,dz). \end{aligned} \tag{3.26}$$

Set

$$\begin{aligned} D(l,t) &= \int_t^l \frac{\partial b}{\partial u}\Big(l,s,\widehat{X}(s),\widehat{u}(s)\Big)ds \\ &\quad + \int_t^l \frac{\partial \sigma}{\partial u}\Big(l,s,\widehat{X}(s),\widehat{u}(s)\Big)dB(s) \\ &\quad + \int_t^l \int_{R_0} \frac{\partial \theta}{\partial u}\Big(l,s,\widehat{X}(s),\widehat{u}(s),z\Big)\widetilde{N}(ds,dz). \end{aligned} \tag{3.27}$$

Repeatedly using the linear equation (3.24), as in the proof of (2.23), we obtain

$$Y^{\beta_\alpha}(l) = \alpha\Lambda(l,t), \tag{3.28}$$

where

$$\begin{aligned} \Lambda(l,t) = D(l,t) + \sum_{k=1}^{\infty} \int_t^l d\Gamma(l,s_1) \int_t^{s_1} d\Gamma(s_1,s_2) \\ \cdots \int_t^{s_{k-1}} D(s_k,t)d\Gamma(s_{k-1},s_k). \end{aligned} \tag{3.29}$$

As in the proof of Lemma 2.2, we can check that the above series converges in $L^1(\Omega)$ under the assumption (H.6). We substitute (3.28) into (3.23) to get

$$
\begin{aligned}
E\Bigg[\alpha\Bigg\{&\int_t^T \frac{\partial f}{\partial x}\Big(s,\widehat{X}(s),\widehat{u}(s)\Big)\Lambda(s,t)ds+\int_t^T \frac{\partial f}{\partial u}\Big(s,\widehat{X}(s),\widehat{u}(s)\Big)ds\\
&+\int_t^T \frac{\partial b}{\partial x}\Big(T,s,\widehat{X}(s),\widehat{u}(s)\Big)\Lambda(s,t)g'\Big(\widehat{X}(T)\Big)ds\\
&+\int_t^T \frac{\partial b}{\partial u}\Big(T,s,\widehat{X}(s),\widehat{u}(s)\Big)g'\Big(\widehat{X}(T)\Big)ds\\
&+\int_t^T \frac{\partial \sigma}{\partial x}\Big(T,s,\widehat{X}(s),\widehat{u}(s)\Big)\Lambda(s,t)D_s\Big(g'\Big(\widehat{X}(T)\Big)\Big)ds\\
&+\int_t^T \frac{\partial \sigma}{\partial u}\Big(T,s,\widehat{X}(s),\widehat{u}(s)\Big)D_s\Big(g'\Big(\widehat{X}(T)\Big)\Big)ds\\
&+\int_t^T\int_{R_0} \frac{\partial \theta}{\partial x}\Big(T,s,\widehat{X}(s),\widehat{u}(s),z\Big)\Lambda(s,t)D_{s,z}\Big(g'\Big(\widehat{X}(T)\Big)\Big)\nu(dz)ds\\
&+\int_t^T\int_{R_0} \frac{\partial \theta}{\partial u}\Big(T,s,\widehat{X}(s),\widehat{u}(s),z\Big)D_{s,z}\Big(g'\Big(\widehat{X}(T)\Big)\Big)\nu(dz)ds\Bigg\}\Bigg]=0.
\end{aligned}
\tag{3.30}
$$

Since α is arbitrary, it follows that

$$
\begin{aligned}
E\Bigg[&\int_t^T \frac{\partial f}{\partial x}\Big(s,\widehat{X}(s),\widehat{u}(s)\Big)\Lambda(s,t)ds+\int_t^T \frac{\partial f}{\partial u}\Big(s,\widehat{X}(s),\widehat{u}(s)\Big)ds\\
&+\int_t^T \frac{\partial b}{\partial x}\Big(T,s,\widehat{X}(s),\widehat{u}(s)\Big)\Lambda(s,t)g'\Big(\widehat{X}(T)\Big)ds\\
&+\int_t^T \frac{\partial b}{\partial u}\Big(T,s,\widehat{X}(s),\widehat{u}(s)\Big)g'\Big(\widehat{X}(T)\Big)ds\\
&+\int_t^T \frac{\partial \sigma}{\partial x}\Big(T,s,\widehat{X}(s),\widehat{u}(s)\Big)\Lambda(s,t)D_s\Big(g'\Big(\widehat{X}(T)\Big)\Big)ds\\
&+\int_t^T \frac{\partial \sigma}{\partial u}\Big(T,s,\widehat{X}(s),\widehat{u}(s)\Big)D_s\Big(g'\Big(\widehat{X}(T)\Big)\Big)ds\\
&+\int_t^T\int_{R_0} \frac{\partial \theta}{\partial x}\Big(T,s,\widehat{X}(s),\widehat{u}(s),z\Big)\Lambda(s,t)D_{s,z}\Big(g'\Big(\widehat{X}(T)\Big)\Big)\nu(dz)ds\\
&+\int_t^T\int_{R_0} \frac{\partial \theta}{\partial u}\Big(T,s,\widehat{X}(s),\widehat{u}(s),z\Big)D_{s,z}\Big(g'\Big(\widehat{X}(T)\Big)\Big)\nu(dz)ds\mid \mathcal{G}_t\Bigg]=0,
\end{aligned}
\tag{3.31}
$$

completing the proof of (1).

(2) Suppose that (3.19) holds for some $\hat{u} \in \mathcal{A}_{\mathcal{G}}$. Running the arguments in the proof of (1) backwards, we see that (3.20) holds for all bounded $\beta \in \mathcal{A}_{\mathcal{G}}$ of the form $\beta(s) = \alpha\chi_{[t,T]}(s)$. This is sufficient because the set of linear combinations of such β is dense in $\mathcal{A}_{\mathcal{G}}$. □

Next we consider the case where the coefficients are independent of x. The maximum principle will be simplified significantly. Fix a control $\hat{u} \in \mathcal{A}_{\mathcal{G}}$ with corresponding state process $\widehat{X}(t)$. Define the associated Hamiltonian process $H(t,u)$ by

$$H(t,u) = f(t,u) + b(T,t,u)g'\left(\widehat{X}(T)\right) + \sigma(T,t,u)D_t\left(g'\left(\widehat{X}(T)\right)\right) + \int_{R_0} \theta(T,t,u,z)D_{t,z}\left(g'\left(\widehat{X}(T)\right)\right)\nu(dz); \quad t \in [0,T],\ u \in R. \tag{3.32}$$

Theorem 3.2 (Maximum principle II for optimal control of stochastic Volterra equations). *Suppose that f,b,σ,θ are all independent of x. Then the followings are equivalent.*

(i) *$\hat{u}$ is a critical point for $J(u)$.*

(ii) *For each $t \in [0,T]$, $u = \hat{u}(t)$ is a critical point for $u \to E[H(t,u) \mid \mathcal{G}_t]$, in the sense that*

$$\frac{\partial}{\partial u}E[H(t,u) \mid \mathcal{G}_t]_{u=\hat{u}(t)} = 0. \tag{3.33}$$

Proof. Suppose that f,b,σ,θ are all independent of x. Then (3.19) reduces to

$$\begin{aligned} E\Bigg[&\int_v^T \frac{\partial f}{\partial u}(s,\hat{u}(s))ds \\ &+ \int_v^T \frac{\partial b}{\partial u}(T,s,\hat{u}(s))g'\left(\widehat{X}(T)\right)ds \\ &+ \int_v^T \frac{\partial \sigma}{\partial u}(T,s,\hat{u}(s))D_s\left(g'\left(\widehat{X}(T)\right)\right)ds \\ &+ \int_v^T\int_{R_0} \frac{\partial \theta}{\partial u}(T,s,\hat{u}(s),z)D_{s,z}\left(g'\left(\widehat{X}(T)\right)\right)\nu(dz)ds \mid \mathcal{G}_v\Bigg] = 0 \quad \forall v \in [0,T]. \end{aligned} \tag{3.34}$$

By inserting $\mathcal{G}_t$ we deduce that for all $v \geq t$,

$$\begin{aligned} E\Bigg[&\int_v^T \frac{\partial f}{\partial u}(s,\hat{u}(s))ds \\ &+ \int_v^T \frac{\partial b}{\partial u}(T,s,\hat{u}(s))g'\left(\widehat{X}(T)\right)ds \\ &+ \int_v^T \frac{\partial \sigma}{\partial u}(T,s,\hat{u}(s))D_s\left(g'\left(\widehat{X}(T)\right)\right)ds \\ &+ \int_v^T\int_{R_0} \frac{\partial \theta}{\partial u}(T,s,\hat{u}(s),z)D_{s,z}\left(g'\left(\widehat{X}(T)\right)\right)\nu(dz)ds \mid \mathcal{G}_t\Bigg] = 0. \end{aligned} \tag{3.35}$$

Taking the right derivative with respect to v at the point t we obtain (3.33). □

4. Applications to Stochastic Delay Control

We now apply the general maximum principle for optimal control of Volterra equations to the stochastic delay problem (1.2)-(1.3) in the Introduction, by using the equivalence between (1.2) and (1.4). Note that in this case we have (see (3.1), (3.2) and compare with (1.2), (1.3))

$$f(t,x,u) = U_1(t,u), g(x) = U_2(x), b(t,s,x,u) = K(t,s)u,$$
$$\sigma(t,s,x,u) = \Phi(t,s)C(s), \Theta(t,s,x,u,z) = \Phi(t,s)C_2(s,z),$$
$$\xi(t) = \Phi(t,0)\eta(0) + \int_{-h}^{0} \Phi(t,s+h)A_2(s+h)\eta(s)ds + \int_{-h}^{0}\left(\int_{0}^{h} \Phi(t,\tau)A_0(\tau,s)d\tau\right)\eta(s)ds. \tag{4.1}$$

Hence the system (1.4) satisfies the conditions of Theorem 3.2. By (3.32) we get the Hamiltonian

$$H(t,u) = U_1(t,u) + K(T,t)uU_2'\left(\widehat{X}(T)\right) + \Phi(T,t)C(t)D_t\left(g'\left(\widehat{X}(T)\right)\right) + \int_{R_0} \Phi(T,t)C_2(t,z)D_{t,z}\left(U_2'\left(\widehat{X}(T)\right)\right)\nu(dz). \tag{4.2}$$

Therefore by Theorem 3.2 we get the following condition for an optimal harvesting rate $\widehat{u}(t)$:

$$E\left[U_1'(t,\widehat{u}(t),\omega) + K(T,t)U_2'\left(\widehat{X}(T),\omega\right) \mid \mathcal{G}_t\right] = 0, \tag{4.3}$$

where $\widehat{X}(T) = X^{\widehat{u}}(T)$ and $U_i' = (\partial/\partial x)U_i;\ i = 1,2$.

Now suppose that U_1 and U_2 are stochastic utilities of the form

$$U_1(t,u,\omega) = \gamma_t(\omega)\widetilde{U}_1(t,u), \quad \omega \in \Omega, \tag{4.4}$$
$$U_2(x,\omega) = \zeta(\omega)\widetilde{U}_2(x), \quad \omega \in \Omega, \tag{4.5}$$

where $\gamma_t(\omega) > 0$ is $\mathcal{F}_t$-adapted, $\zeta(\omega)$ is $\mathcal{F}_T$-measurable, and $\widetilde{U}_1, \widetilde{U}_2$ are concave, C^1-functions on $(0,\infty)$ and R, respectively. The (4.3) simplifies to

$$\widetilde{U}_1'(t,\widehat{u}(t))E\left[\gamma_t\mathcal{G}_t\right] = -K(T,t)E\left[\zeta\widetilde{U}_2'\left(\widehat{X}(T)\right)\mathcal{G}_t\right]. \tag{4.6}$$

This gives a relation between the optimal control $\widehat{u}(t)$ and the corresponding optimal terminal wealth $\widehat{X}(T)$. In particular, if

$$\widetilde{U}_2(x) = x, \tag{4.7}$$

we get

$$\widetilde{U}_1'(t,\widehat{u}(t)) = -\frac{K(T,t)E[\zeta \mid \mathcal{G}_t]}{E[\gamma_t \mid \mathcal{G}_t]}. \tag{4.8}$$

We have proved the following.

Corollary 4.1. *The optimal consumption rate* $\widehat{u}(t)$ *for the stochastic delay system* (1.2), (4.4), (4.5), (4.7) *and the performance functional*

$$J(u) = E\left[\int_0^T \gamma_t(\omega)\widetilde{U}_1(t,u(t))dt + \zeta(\omega)X^u(T)\right] \tag{4.9}$$

with partial information $\mathcal{G}_t$ *is given by* (4.8).

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