ON A VOLTERRA STIELTJES INTEGRAL EQUATION*

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ABSTRACT

The paper deals with a study of linear Volterra integral equations involving Lebesgue-Stieltjes integrals in two independent variables. The authors prove an existence theorem using the Banach fixed-point principle. An *explicit example* is also considered.

Key words: linear Volterra integral equations, Lebesgue-Stieltjes integrals, Banach fixed-point principle, Bessel functions.

AMS subject classification: 45D05, 35L99

1. INTRODUCTION

While studying hyperbolic p.d.e. $u_{xy} + cu = g(x, y)$, one constructs a Riemann function $R(x, y, \xi, \eta)$ which in this case turns out to be $J_0(\sqrt{4c(x-\xi)(y-\eta)})$ [3, Page 123, Ex. 4] where $J_0(z)$ denotes Bessel's function of the first kind of order zero. This fact leads one to hope that there may be situations wherein solutions of hyperbolic equations may involve, in addition to $J_0(z)$ the other Bessel functions $J_1(z)$, $J_2(z)$, ... This paper aims to achieve this conclusion.

Below we consider a linear integral equation of order two of Volterra type involving Lebesgue-Stieltjes integrals. Initial value problems for hyperbolic p.d.e. are

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particular situations of this type of integral equations.

It is well known that the elements of the iterative method are abstracted into a Banach fixed point theorem, and as such, its application not only establishes the existence of a unique solution, but also suggests a constructive approach to solutions of IVPs. In this paper, we prove the existence of solutions for a class of Volterra-Lebesgue-Stieltjes integral equations. The technique involves Banach fixed point principle.

To illustrate our results, we have constructed a Volterra equation of order two involving integrals w.r.t. $\alpha(x)$ and $\beta(x)$ having only one discontinuity. (The problem becomes complicated if more than one discontinuities exist). The interesting part is that the given equation yields a solution $u(x, y)$ which takes different representation in different domains and that these representations involve the infinite set of Bessel functions $J_0, J_1, J_2, \ldots$ and that the series representing solutions are convergent.

2. NOTATION AND PRELIMINARIES

(i) Let $K_1 = [a_1, b_1]$ and $K_2 = [a_2, b_2]$ belong to R and $K = K_1 \times K_2 = [a, b]$ in R_2 with $a = (a_1, a_2)$ and $b = (b_1, b_2)$. A function $g: K \to R$ is said to be of bounded variation if the corresponding interval function

$G(I) = g(d) - g(d_1, c_2) - g(c_1, d_2) + g(c)$ for $I = [c, d]$ is of bounded variation (in the sense of Vitali), i.e. there exists $M > 0$ such that $\Sigma\, [G(I_i)] \leq M$ for every finite collection of intervals I_i from K; see e.g. Chap. III.4 in [1] or Chap. VII in [2].

(ii) Let X be the space of $g: K \to R$ such that g is of bounded variation,

$g(.,a_2)$ and $g(a_1,.)$ are of bounded variation and g is right-continuous at a and every interior point of K. Then X is a Banach space under the norm

(1) $$\| g \| = | g(a) | + V(g(.,a_2)) + V(g(a_1,.)) + V(g),$$

where V denotes total variation. We write V_s, V_t and and V_x for the total variations on $[a_1, s]$, $[a_2, t]$ and $[a, x]$, respectively.

(iii) Let $\alpha : K_1 \to R$ and $\beta : K_2 \to R$ be two functions satisfying hypothesis

(H) α and β are of bounded variation, right-continuous and have only isolated discontinuities.

If $u \in X$ and $k : K \to R$ is continuous then the Lebesgue-Stieltjes integral

(2) $$(Tu)(x) = \int_{[a,x]} k(s,t)\, u(s,t)\, d\alpha(s)\, d\beta(t)$$

is defined, the operator T maps X into itself, and we have

(3) $$V_x(Tu) \le \int_{[a,x]} |k(s,t)|\, |u(s,t)|\, dV_s(\alpha)\, dV_t(\beta) \text{ on } K.$$

3. MAIN RESULT

We consider integral equation

(4) $$u(x) = g(x) + \int_{[a,x]} k(s,t)\, u(s,t)\, d\alpha(s)\, d\beta(t).$$

A solution is understood to be a function $u \in X$ satisfying (4) on K.

Theorem. Let $g \in X$, $k : K \to R$ be continuous and α, β satisfy hypothesis (H). Assume also that

$$(5) \quad \int_K |k(s,t)| \{dV_s(\alpha_1)\, dV_t(\beta_2) + dV_s(\alpha_2)\, dV_t(\beta_1) + dV_s(\alpha_2)\, dV_t(\beta_2)\} < 1 .$$

Then (4) has a unique solution u, and u can be obtained by successive approximation starting with g.

Proof. Consider the closed D from X, defined by

$$D = \{u \in X : u(a_1, .) = g(a_1, .) \text{ and } u(., a_2) = g(., a_2)\} .$$

Instead of the metric given by $\| . \|$ we consider D with the equivalent metric given by

$$\|u\|_\mu = \sup_K V_x(u)\, e^{-z(x)} \quad \text{with} \quad z(x) = \mu \int_{[a,x]} |k(s,t)|\, dV_s(\alpha)\, dV_t(\beta) ,$$

where $\mu > 0$ will be chosen appropriately.

Let $T_1 = T + g$ with T from (2). Evidently T_1 maps D into itself, and (3) implies

$$(6) \quad V_x(T_1u - T_1v) \le \|u - v\|_\mu \int_{[a,x]} k(s,t)\, e^{z(s,t)}\, dV_s(\alpha)\, dV_t(\beta) .$$

Now, using the decompositions of α and β and noticing that

$dV_s(\alpha_1) = |\alpha'_1(s)|\, ds$ and $dV_t(\beta'_1) = |\beta'_1(t)|\, dt$, we have the integral in

(6) $\le \int_{[a,x]} h(s,t) \exp(\mu H(s,t)\, ds\, dt + c\, e^{z(x)}$, where c is the left–hand side of

(5), $h(s,t) = |k(s,t)\, \alpha'_1(s)\, \beta'_1(t)|$ and $H(x) = \int_{[a,x]} h(s,t)\, ds\, dt$, hence

$H_{x_1x_2}(x) = h(x)$ a.e., and $\mu H(x) \le z(x)$. Consequently $\| T_1u - T_1v \|_\mu \le$

$(1/\mu + c) \, \| u - v \|_{\mu}$, with $1/\mu + c < 1$ for μ large enough, and therefore Banach's fixed point theorem yields the desired result. Q.E.D.

4. AN EXAMPLE

Let $J = [0, 1]$, $g = 1$, $k = \lambda > 0$, $\alpha(t) = \beta(t) = t + \gamma \chi_{[\tau,1]}(t)$ with $\gamma \neq 0$, $\tau \in (0, 1)$ and consider

$$u(x) = 1 + \lambda \int_{H(x)} u(\xi) \, d\alpha(\xi_1) \, d\alpha(\xi_2) \quad \text{in } J \times J$$

where $H(x) = [0, x_1] \times [0, x_2]$. Since α has a jump at τ, it is clear that u will be discontinuous on $\{\tau\} \times J \cup J \times \{\tau\}$. We now have the following observations.

1) <u>The case $x < \tau$ and $y < \tau$.</u> Here we have $d\alpha(t) = dt$, hence the equation above is equivalent to the hyperbolic problem $u_{xy} = \lambda u$, $u(0; y) = u(x; 0) = 1$, and the solution is obtained by means of successive approximation, starting with $u_0 = 1$,

$$\text{(7)} \qquad u(x, y) = \sum_{k \geq 0} \frac{(\lambda xy)^k}{(k!)^2} = I_0 (2 \sqrt{\lambda xy}) \quad \text{on } J \times J.$$

Recall that for $n \in \mathbb{N} \cup \{0\}$ the Bessel functions $J_n(.)$ and the modified Bessel functions $I_n(.)$ are related by $I_n(t) = i^{-n} J_n(it)$ with $i^2 = -1$, where

$$J_n(t) = \sum_{k \geq 0} \frac{(-1)^k}{k! \, (n+k)!} \left(\frac{t}{2}\right)^{n + 2k},$$

hence

$$I_n(t) = \sum_{k \geq 0} \frac{1}{k!\,(n+k)!} \left(\frac{t}{2}\right)^{n+2k} .$$

Also by means of successive approximation it is easy to verify that equation (with reasonable f)

$$(8) \qquad u(x) = f(x) + \lambda \int_{H(x)} u(\xi)\, d\xi \ \text{ on } J \times J$$

has resolvent

$$(9) \qquad R(x, \xi) = I_0\left(2\sqrt{\lambda (x_1 - \xi_1)(x_2 - \xi_2)}\right) \text{ for } x \in J^2 \text{ and } \xi \in H(x),$$

i.e. the solution u of (8) is given by

$$(10) \qquad u(x) = f(x) + \lambda \int_{H(x)} R(x, \xi)\, f(\xi)\, d\xi \ \text{ on } J^2 .$$

2) Integrals with respect to $d\alpha(.)$ For the measure μ defined by $\alpha(.)$ we have $\mu = \mu_0 + \gamma\,\delta_\tau$, where μ_0 is Lebesgue measure and δ_τ is Dirac at τ, hence $\mu(A) = \mu_0(A) + \gamma\chi_A(\tau)$ for measurable A belong to J. Therefore if $f : J \to \mathbb{R}$ is bounded measurable and right/left-continuous at τ, then respectively

$$\int_0^x f(t)\, d\alpha(t) = \int_0^x f(t)\, dt + \gamma f(\tau \pm 0)\, \chi_{[0,x]}(\tau) .$$

3) The case $x \geq \tau$ and $y < \tau$ (left- continuous solutions). Now we have

$$d\alpha(\xi_2) = d\xi_2 , \int_{H(x)} u(\xi)\, d\alpha(\xi_1)\, d\xi_2 = \int_{H(x)} u(\xi)\, d\xi + \gamma \int_0^{x_2} u(\tau - 0, \xi_2)\, d\xi_2$$

and $u(\tau - 0,t) = I_0(2\sqrt{\lambda\tau t})$ by (7), hence $\int_0^y u(\tau - 0,t)\,dt = \sqrt{\frac{y}{\lambda\tau}}\,I_1(2\sqrt{\lambda\tau y})$,

and therefore (10) implies

$$u(x) = 1 + \lambda\int_{H(x)} u(\xi)\,d\xi + \gamma\sqrt{\frac{\lambda x_2}{\tau}}\,I_1\left(2\sqrt{\lambda\tau x_2}\right) = g(x_2) + \lambda\int_{H(x)} R(x,\xi)\,g(\xi_2)\,d\xi$$

with $g(t) = 1 + \gamma\sqrt{\frac{\lambda t}{\tau}}\,I_1(2\sqrt{\lambda\tau t})$. Now a simple calculation using in particular (10) with $f = 1$, yields

$$\text{(11)} \qquad u(x,y) = I_0(2\sqrt{\lambda xy}) + \gamma\sqrt{\frac{\lambda y}{\tau}}\,I_1(2\sqrt{\lambda\tau y})$$

$$+ \lambda\gamma\sqrt{\frac{x}{\tau}}\int_0^y \sqrt{\frac{t}{y-t}}\,I_1(2\sqrt{\lambda(y-t)x})\,I_1(2\sqrt{\lambda\tau t})\,dt\,.$$

We shall show later that this integral can be expressed as a series over all I_n.

4) The case $x \geq \tau$ and $y < \tau$ (right-continuous solution). Since $u(\tau, y) = u(\tau + 0, y)$, letting $x_1 = \tau + \varepsilon$ and $\varepsilon \to 0^+$ in

$$\text{(12)} \qquad u(x) = 1 + \lambda\int_{H(x)} u(\xi)\,d\xi + \lambda\gamma\int_0^y u(\tau, t)\,dt\,,$$

we get

$$u(\tau, y) = g(y) + \lambda\gamma\int_0^y u(\tau, t)\,dt$$

with

$$g(y) = 1 + \lambda \int_0^\tau \int_0^y u(s,t)\,dt\,ds = 1 + \lambda \int_0^\tau \int_0^y I_0(2\sqrt{\lambda st})\,dt\,ds$$

by (7), hence

$$\lambda\gamma \int_0^y u(\tau,t)\,dt = \lambda\gamma \int_0^y e^{\lambda\gamma(y-t)} g(t)\,dt,$$

and therefore (12) yields (after simple partial integrations)

$$u(x) = e^{\lambda\gamma x_2} + \lambda \int_{H(x)} u(\xi)\,d\xi + \lambda \int_{H(\tau,y)} (e^{\lambda\gamma(y-t)} - 1)\, I_0(2\sqrt{\lambda st})\,dt\,ds,$$

in particular,

$$(13) \quad u(\tau,y) = e^{\lambda\gamma y} + \lambda \int_{H(\tau,y)} e^{\lambda\gamma(y-t)} I_0(2\sqrt{\lambda st})\,dt\,ds$$

and for $x > \tau$

$$u(x,y) = u(\tau,y) + \lambda \int_\tau^x \int_0^y u(s,t)\,dt\,ds,$$

hence with (9)

$$(14) \quad u(x,y) = u(\tau,y) + \lambda \int_\tau^x \int_0^y I_0(2\sqrt{\lambda(x-s)(y-t)})\, u(\tau,t)\,dt\,ds$$

$$= u(\tau,y) + \sum_{k\geq 0} \frac{[\lambda(x-\tau)]^{k+1}}{k!\,(k+1)!} \int_0^y (y-t)^k u(\tau,t)\,dt.$$

Now,

$$u(\tau,y) = e^{\lambda\gamma y} + \sum_{k\geq 0} \frac{(\lambda\tau)^{k+1}}{k!\,(k+1)!}\alpha_k \quad \text{with} \quad \alpha_k = \int_0^y e^{\lambda\gamma(y-t)} t^k\,dt,$$

hence $\alpha_k = \frac{1}{\lambda\gamma}(k\,\alpha_{k-1} - y^k)$, and this yields

$$\alpha_k = -\sum_{i=0}^{k} \frac{k!}{(k-i)!} \frac{y^{k-i}}{(\lambda\gamma)^{i+1}} + \frac{k!}{(\lambda\gamma)^{k+1}} e^{\lambda\gamma y} .$$

Therefore,

$$\text{(15)} \qquad u(\tau, y) = e^{\lambda\gamma y} + \sum_{k\geq 0} \frac{\left(\frac{\tau}{\gamma}\right)^{k+1}}{(k+1)!} \left(e^{\lambda\gamma y} - \sum_{i=0}^{k} \frac{(\lambda\gamma y)^i}{i!} \right)$$

$$= \sum_{k\geq 0} \gamma^k \left(\frac{\lambda y}{\tau}\right)^{\frac{k}{2}} I_k(2\sqrt{\lambda\tau y}) .$$

From (14) we now get

$$u(x, y) = u(\tau, y) + \sum_{i,j,k\geq 0} \frac{(x-\tau)^{k+1} \gamma^i \tau^j (\lambda y)^{i+j+k+1}}{k!\,(k+1)!\,j!\,(i+j)!} \int_0^1 (1-t)^k t^{i+j}\, dt .$$

Let $\beta_{m,k} = \int_0^1 t^m (1-t)^k\, dt$ for $m, k \geq 0$. Then $\beta_{m,0} = \frac{1}{m+1}$ and

$\beta_{m+1,k} = \frac{m+1}{k+1} \beta_{m,k+1}$, hence

$$\text{(16)} \qquad \beta_{m,k} = \frac{k!\,m!}{(m+k+1)!} ,$$

and therefore

$$(17) \quad u(x, y) = \sum_{i,j,k\geq 0} \frac{(x-\tau)^k \gamma^i \tau^j (\lambda y)^{i+j+k}}{k!\, j!\, (i+j+k)!},$$

$$u(x, y) = \sum_{m,k\geq 0} \frac{\gamma^m}{k!} (x-\tau)^k \sqrt{\frac{\lambda y}{\tau^{m+k}}}\, I_{m+k}(2\sqrt{\lambda\tau y}).$$

Notice that for $x = \tau$ only $k = 0$ remains and we get (15). Notice also that the last term in (11) for the left-continuous solution can be calculated. Similarly, namely with $\beta_{k,m}$ from (16)

$$\lambda\gamma\sqrt{\frac{x}{\tau}} \int_0^y \frac{\sqrt{t}}{\sqrt{y-t}} I_1(2\sqrt{\lambda(y-t)x})\, I_1(2\sqrt{\lambda\tau t})\, dt$$

$$= \lambda^2 \gamma x \sum_{j,k\geq 0} \frac{\lambda^{k+j} x^k \tau^j y^{k+j+2}}{k!\,(k+1)!\,j!\,(j+1)!} \beta_{j+1,k}.$$

Hence, after simple rearrangements,

$$(18) \quad u(x,y) = I_0(2\sqrt{\lambda xy}) + \gamma\sqrt{\frac{\lambda y}{\tau}} \sum_{k\geq 0} \frac{x^k}{k!} \left(\frac{\lambda y}{\tau}\right)^{\frac{k}{2}} I_{k+1}(2\sqrt{\lambda\tau y})$$

is the left continuous solution. Its jump at τ is

$$(19) \quad u(\tau+0, y) - u(\tau, y) = \gamma\sqrt{\frac{\lambda y}{\tau}} \sum_{k\geq 0} \frac{(\lambda\tau y)^{\frac{k}{2}}}{k!} I_{k+1}(2\sqrt{\lambda\tau y}),$$

while the jump of the right-continuous solution is given by

$$(20) \quad u(\tau, y) - u(\tau - 0, y) = \gamma \sqrt{\frac{\lambda y}{\tau}} \sum_{k \geq 0} \gamma^k \left(\frac{\lambda y}{\tau}\right)^{\frac{k}{2}} I_{k+1}(2\sqrt{\lambda \tau y}),$$

which is usually different for $y \neq 0$.

5. The case $x < \tau$ and $y \geq \tau$ Since the problem is symmetric, it is obvious that we get left/right-continuous solutions by writing x for y and y for x in the right-hand sides of (17) and (18). Notice also that in all cases considered so far we have no convergence problems, i.e. all series converge (even fast).

6. The case $x \geq \tau$ and $y \geq \tau$ (left-continuous solution).

Now we have

$$(21) \quad u(x,y) = g(x, y) + \lambda \int_0^x \int_0^y u(s, t)\, ds\, dt$$

with

$$(22) \quad g(x, y) = 1 + \lambda\gamma \int_0^x u(s, \tau - 0)\, ds + \lambda\gamma \int_0^y u(\tau - 0, t)\, dt + \lambda\gamma^2 u(\tau - 0, \tau - 0).$$

Using (18) and $u(x, y) = I_0(2\sqrt{\lambda x y})$ for $x \leq \tau$ and $y \leq \tau$, we obtain $u(\tau - 0, \tau - 0) = I_0(2\sqrt{\lambda}\tau)$ and

$$(23) \quad \int_0^x u(s, \tau - 0)\, ds = \int_0^x I_0(2\sqrt{\lambda s \tau})\, ds$$

$$+ \gamma \sum_{k \geq 0} \frac{\lambda^{\frac{k+1}{2}}}{(k+1)!} I_{k+1}(2\sqrt{\lambda}\, \tau)\, (x^{k+1} - \tau^{k+1}).$$

Using the corresponding formula for $x < \tau$ and $y\tau$, we get that $\int_0^y u(\tau - 0)\, dt$ is the right-hand side of (23) with y for x, hence

$$g(x, y) = 1 + \lambda\gamma^2 I_0(2\sqrt{\lambda}\tau) - 2\lambda\gamma^2 \sum_{k\geq 0} \frac{(\tau\sqrt{\lambda})^{k+1}}{(k+1)!} I_{k+1}(2\sqrt{\lambda}\tau)$$

$$+ \lambda\gamma \left\{ \int_0^x + \int_0^y \right\} I_0(2\sqrt{\lambda\tau s})\, ds + \lambda\gamma^2 \sum_{k\geq 0} \frac{\lambda^{\frac{k+1}{2}}}{(k+1)!} I_{k+1}(2\sqrt{\lambda}\tau)(x^{k+1} + y^{k+1}),$$

and therefore $u(x, y) = g(x, y) + \lambda \int_{H(x, y)} R(x, \xi)\, g(\xi)\, d\xi$ gives an explicit formula for $u(x, y)$.

7. The case $x \geq \tau$ and $y \geq \tau$ (right-continuous solution). We have again (21) and (22) with $\tau - 0$ replaced by $\tau + 0$. To determine these unknown limits, we let $x \to \tau + 0$ and $y \to \tau + 0$ in (21) to obtain $u(\tau, \tau)$. This yields

$$u(\tau, \tau) = 1 + \lambda\gamma \int_0^\tau u(s, \tau + 0)\, ds + \lambda\gamma \int_0^\tau u(\tau + 0, t)\, dt$$

$$+ \lambda\gamma^2 u(\tau, \tau) + \lambda \int_0^\tau \int_0^\tau u(s, t)\, ds\, dt.$$

Notice that these (Lebesgue) integrals are known, for example $\int_0^\tau u(\tau + 0, t)\, dt$ by (15), but if $\lambda\gamma^2 = 1$ we get an equation which is not solvable (except for trivial cases); hence we need $\lambda\gamma^2 \neq 1$ and get $u(\tau, \tau)$. Letting only $y \to \tau^+$ in (21) we get

$$u(x, \tau) = u(\tau, \tau) + \lambda\gamma \int_\tau^x u(s, \tau)\, ds + \lambda \int_\tau^x \int_0^\tau u(s, t)\, dt\, ds,$$

where $u(\tau, \tau)$ and the second integral are already known. Hence we can determine

$u(.,\tau)$ on $[\tau, 1]$ like in section 4, and similarly we get $u(\tau, .)$, hence $g(x, y)$ and therefore $u(x, y)$.

8. Successive Approximation (case $x \geq \tau$ and $y < \tau$). Consider

$$(24) \quad u_{n+1}(x, y) = 1 + \lambda \int_0^x \int_0^y u_n(s, t)\, dt\, d\alpha(s)$$

$$= 1 + \lambda \int_0^x \int_0^y u_n(s, t)\, dt\, ds + \lambda\gamma \int_0^y u_n(\tau, t)\, dt \chi_{[\tau, 1]}(x)$$

with $u_0(x, y) = 1$. Notice first that we have $u_n(x, y) = \sum_{k=0}^{n} \frac{(\lambda xy)^k}{k!^2}$, if we also $x < \tau$. By the consideration in section 4 one will expect that if (u_n) is convergent then the limit will be the right continuous solution given by (17). Now (17) suggests to write the u_n as

$$(25) \quad u_n(x, y) = \sum_{k=0}^{n} \frac{(x-\tau)^k}{k!} \varphi_{k,n} \chi_{[\tau, 1]}(x) + \sum_{k=0}^{n} \frac{(\lambda xy)^k}{k!^2} \chi_{[0, \tau]}(x) .$$

Inserting this into (23) we get for $x \geq \tau$

$$u_{n+1} = \sum_{k=0}^{n+1} \frac{(\lambda \tau y)^k}{k!^2} + \lambda \sum_{k=1}^{n+1} \frac{(x-\tau)^k}{k!} \int_0^y \varphi_{k-1,n}(t)\, dt + \lambda\gamma \int_0^y \varphi_{o,n}(t)\, dt$$

hence comparison of coefficients for equal powers of $x - \tau$ yields

(26) $$\phi_{0,\,n+1}(y) = \sum_{k=0}^{n+1} \frac{(\lambda\tau y)^k}{k!^2} + \lambda\gamma \int_0^y \phi_{0,\,n}(t)\,dt$$

(27) $$\phi_{k,\,n+1}(y) = \lambda \int_0^y \phi_{k-1,\,n}(t)\,dt \quad \text{for} \quad k = 1, \ldots, n+1$$

with $\phi_{0,0} = 1$. So we first obtain the $\phi_{0,n}$ by means of (26) and then the $\phi_{k,n}$ by means of (27). This yields (for $x \geq \tau$)

(28) $$u_1(x, y) = 1 + \lambda x y + \lambda \gamma y,$$

$$u_2(x, y) = u_1(x, y) + \frac{\lambda^2 y^2}{2!}\left(\frac{x^2}{2!} + \gamma x + \gamma^2\right)$$

and so on. Now it is obvious that the $(\phi_{k,\,n})$ are convergent (as in $n \to \infty$) to ϕ_k and, for example, ϕ_0 satisfies the equation obtained by letting $n \to \infty$ in (26), i.e.

$$\phi_0(y) = I_0(2\sqrt{\lambda\tau y}) + \lambda\gamma \int_0^y \phi_0(t)\,dt,$$

the solution of which is exactly $u(\tau, y)$ given by (15), and therefore it is clear that

$$u(x, y) = \sum_{k \geq 0} \frac{(x-\tau)^k}{k!} \phi_k(y)$$

is the function given by (17). A restriction on γ, λ will come again in case $x \geq \tau$ and $y \geq \tau$.

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REFERENCES

[1] Hildebrandt, T.H., *Theory of Integration*, Academic Press, (1963).

[2] McShane, E., *Integration*, Princeton University Press, (1944).

[3] Sneddon, I., *Elements of Partial Differential Equations*, McGraw-Hill, (1984).