A THEOREM ON MANY FIXED POINTS FOR NONLINEAR OPERATOR[1]

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ABSTRACT

Multiple fixed points of weakly inward mappings are investigated by means of ordinary differential equations in abstract spaces.

Key words: Fixed point, weakly inward mapping, ordinary differential equation in abstract space.

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1. INTRODUCTION

Since the concept of weakly inward mappings was introduced by B.R. Halpern and G.M. Bergman in [5], many papers about weakly inward mappings have been published. This concept has wide applications to differential equations in Banach spaces, modern calculus, theory of fixed points, optimizationt theory, etc.

The purpose of this paper is to prove a theorem of three fixed points for weakly inward mappings. For the sake of convenience, we first recall some definitions and notations.

DEFINITION. Let E be a real Banach space, D a closed convex set of E. Suppose that A is a mapping from D into E. Then A is said to be a weakly inward mapping on D if

$$Ax \in \overline{I_D(x)}$$

for each $x \in D$, where

$$I_D(x) = \left\{(1-t)x + ty: \ t \geq 0 \ and \ y \in D\right\}.$$

$I_D(x)$ is called the inward set of x with respect to D and $\overline{I_D(x)}$ is the closure of $I_D(x)$.

LEMMA (cf. [2]). *Let E be a real Banach space, D a closed convex set of E, and A: $D \rightarrow E$ a mapping. Then A is a weakly inward mapping on D if and only if*

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$$\lim_{t \to +0} \frac{1}{t} d(x+t(Ax-x),\ D)=0,\ x \in D, \tag{1}$$

where $d(x+t(Ax-x),\ D)$ *denotes the distance between* $x+t(Ax-x)$ *and* D.

2. MAIN RESULT

We now state the main result of this paper.

THEOREM. *Let* E *be a real Banach space,* D *a bounded closed convex set of* E, *and* $A\colon\ D \to E$ *a strict-set-contraction and weakly inward mapping. Suppose that* A *satisfies a local Lipschitz condition; and there exist two relatively open convex sets* D_1 *and* D_2 *of* D, $D_1 \cap D_2 \neq \phi$, *such that* A *is weakly inward on* $\overline{D_1}$ *adn* $\overline{D_2}$ *respectively. Suppose further that* A *has no fixed points on* ∂D_1 *and* ∂D_2 *(where* ∂D_1 *and* ∂D_2 *are the boundary of* D_1 *and* D_2 *with respect to* D *respectively). Then* A *has at least three fixed points on* D.

PROOF. Consider the IVP

$$u' = Au - u,\ u(0) = x \in D \tag{2}$$

in Banach space E. It can be proved that IVP(2) has a unique solution $u(t,\ x)$ on $[0,\ +\infty)$ for each $x \in D$, and

$$u(t,\ x) \in D \text{ for any } t \geq 0. \tag{3}$$

Define, for each $t \geq 0$,

$$U(t)x = u(t,x),\ x \in D. \tag{4}$$

Then $U(t)$ is an operator and from (3) it follows that

$$U(t)D \subset D. \tag{5}$$

We know, in the same way, that

$$U(t)\overline{D_1} \subset \overline{D_1} \text{ and } U(t)\overline{D_2} \subset \overline{D_2}, \tag{6}$$

since A is weakly inward on $\overline{D_1}$ and $\overline{D_2}$ respectively. It follows from $A - I$ satisfying local Lipschitz condition that $U(t)$ is continuous by continuous dependence of solutions on the initial values.

It is possible to show, by using the same arguments as those of the proof of Theorem 4.8 in [3], that

$$\alpha(U(t)(B)) \leq e^{-(1-k)t}\alpha(B)$$

for any $t \geq 0$ and $B \subset D$, where $\alpha(\cdot)$ is Kratowskii's measure of noncompactness and $0 \leq k < 1$ is the coefficient of strict-set-contraction of A (i.e., $\alpha(A(B)) \leq k \cdot \alpha(B)$ for any $B \subset D$). From this it follows that $U(t)$: $D \to D$ is strict-set-contraction for each $t \geq 0$.

We now show that a positive number c can be found such that $U(t)$ has no fixed points on the boundary of D_1 for any $0 < t \leq c$. In fact, if it is the contrary, then $t_n > 0$ ($n= 1, 2, \ldots$) can be found such that $t_n \to 0$ and $U(t_n)$ has fixed point x_{t_n} on the boundary of D_1. Therefore, we have

$$u(t_n, x_{t_n})=U(t_n)x_{t_n}=x_{t_n} \in \partial D_1, \ n=1, 2, \ldots.$$

Let $u_n(s)=u(s, x_{t_n})$ for $s \geq 0$. Then it follows from the proof of Theorem 4.8 in [3] that there exists a subsequence $\left\{u_{n_i}(s)\right\}$ of $\left\{u_n(s)\right\}$ which uniformly converges to $z \in E$ on $[0, +\infty)$ as n_i goes to infinity, where z is a fixed point of A. Therefore,

$$\lim_{n_i \to \infty} x_{t_{n_i}} = \lim_{n_i \to \infty} u(t_{n_i}, x_{t_{n_i}})= \lim_{n_i \to \infty} u_{n_i}(t_{n_i})=z$$

and $z \in \partial D_1$. This is in contradiction with A having no fixed points on the boundary of D_1. This implies that a positive number c can be found such that $U(t)$ has no fixed points on the boundary of D_1 for any $0 < t \leq c$. In the same way we can show that a positive number $c\prime$ can be found such that $U(t)$ has no fixed points on the boundary of D_2 for any $(0) < t \leq c\prime$. Let $c^*=min\{c, c\prime\}$. Then $U(t)$ has no fixed points on the boundary of D_1 and the boundary of D_2 for any $0 < t \leq c^*$. And
from this and (5), (6) it follows that, for any $0 < t \leq c^*$,

$$i(U(t), D_1, D)=1, \tag{7}$$

$$i(U(t), D_2, D)=1, \tag{8}$$

$$i(U(t), D, D)=1, \tag{9}$$

where $i(\cdot, \cdot, \cdot,)$ is the fixed point index, whose definition and properties can be found in [6]. Let $D_3=D \backslash (\overline{D_1 \cup D_2})$. Then by (7), (8) and (9) we have

$$i(U(\mathrm{t}), D_3, D)=1-1-1=-1.$$

Therefore, by the solvability of fixed point index, $U(t)$ has fixed points in D_1, D_2 and $D_3$for each $0 < t \leq c^*$, respectively.

We finally show that A has a fixed point in D_3. In fact, if we select $t_m > 0 (m=1, 2,\ldots)$ such that $t_m \to 0$ as m goes to infinity, and let x_{t_m} be the fixed point of $U(t_m)$, then

$$u(t_m, x_{t_m})=U(t_m)x_{t_m}=x_{t_m} \in D_3.$$

Denote $u_m(s)=u(s, x_{t_m})$ for $s \geq 0$. Then it follows from the proof of Theorem 4.8 in [3] that a subsequence $\left\{u_{m_k}(s)\right\}$ of $\left\{u_m(s)\right\}$ can be found such that $u_{m_k}(s)$ uniformly converges to $z_1 \in E$ on $[0, +\infty)$ as m_k goes to infinity and z_1 is a fixed point of A. This implies

$$\lim_{m_k \to \infty} x_{t_{m_k}} = \lim_{m_k \to \infty} u(t_{m_k}, x_{t_{m_k}})= \lim_{m_k \to \infty} u_{m_k}(t_{m_k})=z_1,$$

so $z_1 \in \overline{D_3}$, and hence it follows from A having no fixed points on the boundary of D_3, which is the union of the boundary of D_1 and the boundary of D_2, that z_1 is in D_3. Thus, A has a fixed point in D_3. We can prove, by the same reasoning, that A has fixed points in D_1 and D_2 respectively. Hence A has three fixed points on D and the proof is completed.

REMARK 1. We can only conclude that A has two fixed points in D under the conditions of this Theorem by using Theorem 4.8 in [3], i.e., A has fixed points in D_1 and D_2 respectively, and the third fixed point of A in D can not be obtained.

3. EXAMPLE

Let us now show an example. Suppose that the real Banach space E is $\mathbb{R}^2$. Let

$$D=\{(x,\ y) \in \mathbb{R}^2\colon\ 0 \le x \le 5,\ 0 \le\ y \le 5\},$$

$$D_1=\{(x,\ y) \in \mathbb{R}^2\colon\ 0 \le x < 1,\ 0 \le y < 1\},$$

$$D_2=D_2' \cup D_2'' \text{ and}$$

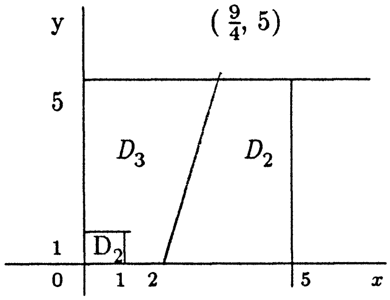
$$D_2'=\{(x,\ y) \in \mathbb{R}^2\colon\ 2 < x < 9/4,\ 0 \le y \le 20x-40\},$$

$$D_2''=\{(x,y) \in \mathbb{R}^2\colon\ 9/4 \le x \le 5,\ 0 \le y \le 5\}\backslash\{(9/4,\ 5)\}.$$

and let the operator A from D into $\mathbb{R}^2$ be

$$A(x,y)=(y-x(x-1)(x-5),-y).$$

The Theorem will be applied to show that A has three fixed points in D. In order to do this, we verify that A is weakly inward on D, $\overline{D_1}$ and $\overline{D_2}$ respectively and A has no fixed points on the boundary of D_1 and D_2 with respect to D.



In fact, it is easy to show that $A\colon\ D \to \mathbb{R}^2$ is weakly on D and on $\overline{D_1}$. We now show that A is weakly inward on $\overline{D_2}$. Clearly, it is sufficient to show that $A(x,\ y)$ is on the right side of the line $y=20x-40$ for $2 < x < 9/4$ and $y=20x-40$, and to verify the following:

$$\text{the } x \text{ coordinate of } A(2,0) \ge 2, \tag{10}$$

$$\text{the } y \text{ coordinate of } A(2,0) \ge 0, \tag{11}$$

$$\text{the } x \text{ coordinate of } A(9/4,\ 5) \ge 9/4 \tag{12}$$

$$\text{the } y \text{ coordinate of } A(9/4,\ 5) \le 5. \tag{13}$$

It is easy to check that (10) - (13) hold. We now show that $A(x,y)$ is on the right side of the line $y=20x-40$ for $2 < x < 9/4$ and $y=20x-40$. In order to do this, we should show

$$-y \leq 20y - 20x(x-1)(x-5) - 40 \tag{14}$$

holds for $2 < x < 9/4$ and $y=20x-40$. Substituting $y=20x-40$ in (14) gives

$$20x(x-1)(x-5) - 21(20x-40) + 40 \leq 0,\ 2 < x < 9/4. \tag{15}$$

Define a function $g(x)$ on $\mathbb{R}^1$ by

$$g(x) = 20x(x-1)(x-5) - 21(20x-40) + 40.$$

Then

$$g(2) = 40 \cdot (1) \cdot (-3) + 40 = -80 < 0,$$

and

$$g\prime(x) = 20x(x-1) + 20x(x-5) + 20(x-1)(x-5) - 420 = 20(3x^2 - 12x - 16).$$

We know that equation $g\prime(x)=0$ has two real solutions

$$x_1 = 2 - \sqrt{84}/3,\ x_2 = 2 + \sqrt{84}/3.$$

Since $g\prime(x)$ is a quadratic function that opens upward, we get

$$g\prime(x_1) = g\prime(x_2) = 0,$$

and

$$g\prime(x) < 0 \ for \ x_1 < x < x_2. \tag{16}$$

And hence,

$$g\prime(x) < 0 \ for \ 2 \leq x \leq 9/4 \tag{17}$$

by virtue of (16) and $x_1 < 2 < 9/4 < x_2$. From $g(2) < 0$ and $g\prime(x) < 0$ for $2 \leq x \leq 9/4$, it follows that

$$g(x) < 0 \ for \ 2 \leq x \leq 9/4.$$

This implies that (15) holds, i.e., (14) holds. Hence, $A(x, y)$ is on the right side of the line $y=20x-40$ for $2 < x < 9/4$ and $y=20x-40$. This yields that A is weakly inward on $\overline{D_2}$.

On the other hand, we can show that A has no fixed points on the boundary of D_1 and D_2 with respect to D. Therefore the conditions of the Theorem are fulfilled by the operator A, and hence A has three fixed points in D.

REMARK 2. The fixed points of operator A in the above example can be obtained by solving the equation $A(x, y)=(x, y)$ directly, which are $(0, 0)$ in D_1, $(3+\sqrt{3}, 0)$ in D_2 and $(3-\sqrt{3}, 0)$ in D_3.

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