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AN ABSTRACT INVERSE PROBLEM[1]

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ABSTRACT

In this paper we consider an inverse problem that corresponds to an abstract integrodifferential equation. First, we prove a local existence and uniqueness theorem. We also show that every continuous solution can be locally extended in a unique way. Finally, we give sufficient conditions for the existence and a stability of the global solution.

Key words: inverse problem, abstract integrodifferential equation, existence, uniqueness, stability.

AMS (MOS) subject classification: 35R.

1. INTRODUCTION

Let X, Y be two Banach spaces, and let $A: D(A) \subset X \to X$ be a linear operator. Let $T > 0, F_1, F_2: [0,T] \times X \times Y \to X, L: X \to Y, v: [0,T] \to Y$, and $x \in X$ be given data.

We consider the following problem: find $(u,p): [0,T] \to X \times Y$ such that

$$u'(t) = Au(t) + F_1(t, u(t), p(t)) + \int_0^t F_2(s, u(s), p(t-s))ds, 0 \leq t \leq T, \tag{1}$$

$$u(0) = x, \tag{2}$$

$$Lu(t) = v(t), 0 \leq t \leq T. \tag{3}$$

Such a problem has been considered previously by Prilepko, Orlovskii in [6,7], Lorenzi, Sinestrari in [4], and the author in [1].

The local existence and uniqueness result is obtained by Prilepko, Orlovskii for the case $F_2 = 0$, and by Lorenzi, Sinestrari for the case Y is a subspace of $L(X), F_1(t,u,p) = pBx$, and $F_2(t,u,p) = pBu$, where B is some given linear operator in X. The stability problem has been studied by Lorenzi and Sinestrari in [5].

In [1] the author treats the case of $Y = C[0,T]^n (n \geq 1), F_1(t, u, (p_1, \ldots, p_n)) = \sum_1^n p_i y_i$, y_i in $X (1 \leq i \leq n)$ and $F_2 = 0$. Then a global existence and uniqueness theorem is obtained.

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The present work is concerned with a generalization of those results.

Throughout this paper we assume:

$(H1)$ A is a closed linear operator with a dense domain generating a strongly continuous semigroup e^{At}. Without loss of generality, we suppose that e^{At} is equibounded:

$$\| e^{At} \| \leq M, t \geq 0 \text{ for some } M \geq 1.$$

$(H2)$ $x \in D(A)$,

$(H3)$ $L \in L(X,Y)$,

$(H4)$ $v \in C^1([0,T]:Y)$, and $v(0) = Lx$.

$(H5,1)$ F_1 and AF_1 are continuous in $[0,T] \times D(A) \times Y$.

For each $r > 0$, there exist positive continuous real valued functions $g_{1,i}(r, \cdot)$, $i = 0,1$ such that

$(H5,2)$ $\| F_1(t,u_1,p_1) \|_{D(A)} \leq g_{1,0}(r,t)$,

$(H5,3)$ $\| F_1(t,u_1,p_1) - F_1(t,u_2,p_2) \|_{D(A)} \leq g_{1,1}(r,t)(\| u_1 - u_2 \|_{D(A)} + \| p_1 - p_2 \|_Y)$,

for each $(u_i, p_i) \in \{(u,p) \in D(A) \times Y, \| u \|_{D(A)} + \| p \|_Y \leq r\}$, $i = 1,2$, and $t \in [0,T]$.

$(H6,1)$ $\int_0^t F_2$ and $A \int_0^t F_2$ are continuous in $[0,T] \times D(A) \times Y$.

For each $r > 0$, there exist positive continuous real valued functions $g_{2,i}(r, \cdot)$, $i = 0,1$, such that

$(H6,2)$ $$\| \int_0^t F_2(s,u_1(s),p_1(t-s))ds \|_{D(A)} \leq \int_0^t g_{2,0}(r,s)ds,$$

$(H6,3)$ $$\| \int_0^t (F_2(s,u_1(s),p_1(t-s)) - F_2(s,u_2(s),p_2(t-s)))ds \|_{D(A)}$$
$$\leq \int_0^t g_{2,1}(r,s)(\| u_1(s) - u_2(s) \|_{D(A)} + \| p_1(s) - p_2(s) \|_Y)ds,$$

for each $(u_i, p_i) \in \{(u,p) \in C([0,T]:D(A) \times Y), \sup_{0 \leq s \leq t} (\| u(s) \|_{D(A)} + \| p(s) \|_Y) \leq r\}$, $i = 1,2$, and $t \in [0,T]$.

There exist continuous function $H_1: [0,T] \times Y \times Y \to Y$ with the following properties. For each $r > 0$ there exist positive continuous real valued functions $C(r, \cdot)$ such that

$(H7,1)$ $\| H_1(t,u_1,p_1) - H_1(t,u_2,p_2) \|_Y \leq C(r,t)(\| u_1 - u_2 \|_{D(A)} + \| p_1 - p_2 \|_Y)$, for each $(u_i, p_i) \in \{(u,p) \in Y \times Y, \| u \|_Y + \| p \|_Y \leq r\}, i = 1,2$, and $t \in [0,T]$.

$K: p \to H_1(t,v(t),p)$ has an inverse $\Phi(t, \cdot)$ continuous map $t \to \Phi(t,w)$, and there exist positive continuous real valued function k, such that

$(H7,2)$ $\| \Phi(t,w_1) - \Phi(t,w_2) \|_Y, t \in [0,T], w_i \in Y, i = 1,2.$

$(H7,3)$ $LF_1(t,u,p) = H_1(t,Lu,p), (u,p) \in D(A) \times Y$, and $t \in [0,T]$.

2. EXISTENCE OF THE LOCAL SOLUTION

In this section we prove that the local solution of our inverse problem is obtained by a fixed point theorem. Let

$$a(t) = M \| x \|_{D(A)} + \| \Phi(t,0) \|_Y + k(t) \| v'(t) - Le^{At}Ax \|_Y, t \in [0,T], r_0 = 2 \sup_{0 \le t \le T} a(t),$$

$$g_i(r_0,t,s) = M(1 + k(t) \| L \|)(g_{1,i}(r_0,s) + (t-s)g_{2,i}(r_0,s)) + k(t) \| L \| g_{2,i}(r_0,s), \qquad 0 \le s \le t \le T,$$

$i = 0,1$, and let $T_0 \in [0,T]$ be such that

$$T_0 \sup_{0 \le s \le t \le T} g_0(r_0,t,s) \le \frac{r_0}{2}, \text{ and } T_0 \sup_{0 \le s \le t \le T} g_1(r_0,t,s) = \gamma < 1.$$

Let $Z(T_0) = C([0,T_0]: D(A) \times Y)$ equipped with the norm

$$\| (u,p) \|_{Z(T_0)} = \sup_{0 \le t \le T_0} (\| u(t) \|_{D(A)} + \| p(t) \|_Y).$$

Then, we define the mapping

$$\Psi: Z(T_0) \to Z(T_0): (u,p) \to (U,P),$$

where

$$U(t) = e^{At}x + \int_0^t e^{A(t-s)} F_1(s,u(s),p(s))ds$$

$$+ \int_0^t e^{A(t-s)} \int_0^s F_2(\sigma,u(\sigma),p(s-\sigma))d\sigma ds,$$

$$P(t) = \Psi(t, v'(t) - Le^{At}Ax - \int_0^t LF_2(s,u(s),p(t-s))ds$$

$$- \int_0^t Le^{A(t-s)}AF_1(s,u(s),p(t-s))ds$$

$$- \int_0^t Le^{A(t-s)}A \int_0^s F_2(\sigma,u(\sigma),p(s-\sigma))d\sigma ds), 0 \le t \le T_0.$$

Proposition 1. There exists a unique (u_0,p_0) in $B(r_0,T_0)$ satisfying $(u_0,p_0) = \Psi(u_0,p_0)$, where $B(r_0,T_0)$ denotes the closed ball of $Z(T_0)$ with the center 0 and radius r_0.

Proof. We claim that Ψ is a strict contraction from $B(r_0,T_0)$ into itself. Hence, according to the fixed point theorem, there is a unique (u_0,p_0) in $B(r_0,T_0)$ such that $(u_0,p_0)=\Psi(u_0,p_0)$. Let (u_i,p_i) in $B(r_0,T_0),(U_i,P_i)=\Psi(u_i,p_i), i=1,2,$ and t in $[0,T_0]$.
We have then

$$\| U_1(t) \|_{D(A)} \leq M \| x \|_{D(A)} + M\int_0^t \| F_1(s,u(s),p(s)) \|_{D(A)} ds$$
$$+ M\int_0^t \| \int_0^s F_2(\sigma,u(\sigma),p(s-\sigma))d\sigma \|_{D(A)} ds.$$

Using $(H5)$ and $(H6)$ we obtain

$$\| U_1(t) \|_{D(A)} \leq M \| x \|_{D(A)} + M\int_0^t g_{1,0}(r_0,s)ds + M\int_0^t\int_0^s g_{2,0}(r_0,\sigma)d\sigma ds$$

$$\leq M \| x \|_{D(A)} + M\int_0^t (g_{1,0}(r_0,s)+(t-s)g_{2,0}(r_0,s))ds.$$

From $(H7,2)$ we deduce

$$\| P_1(t) \|_Y \leq \| \Phi(t,0) \|_Y + k(t) \| v'(t) - Le^{At}Ax$$

$$- \int_0^t LF_2(s,u(s),p(t-s))ds - \int_0^t Le^{A(t-s)}AF_1(s,u(s),p(t-s))ds$$

$$- \int_0^t Le^{A(t-s)}A\int_0^s F_2(\sigma,u(\sigma),p(s-\sigma))d\sigma ds) \|_Y.$$

Hence

$$\| P_1(t) \|_Y \leq \| \Phi(t,0) \|_Y + k(t) \| v'(t) - Le^{At}Ax \| + \| L \| k(t)\int_0^t g_{2,0}(r_0,s)ds$$

$$+ M \| L \| k(t)\int_0^t (g_{1,0}(r_0,s)+(t-s)g_{2,0}(r_0,s))ds.$$

Thus

$$\| U_1(t) \|_{D(A)} + \| P_1(t) \|_Y \leq \| x \|_{D(A)} + \| \Phi(t,0) \|_Y + k(t) \| v'(t) - Le^{At}Ax \|$$

$$+ \| L \| k(t)\int_0^t g_{2,0}(r_0,s)ds$$

$$+ M(1+ \| L \| k(t))\int_0^t (g_{1,0}(r_0,s)+(t-s)g_{2,0}(r_0,s))ds$$

$$\leq a(t) + \int_0^t g_1(r_0,t,s)ds.$$

This implies that

$$\| (U_1, P_1) \|_{Z(T_0)} \leq r_0.$$

On the other hand, in the same way as above, it is easily seen that

$$\| U_1(t) - U_2(t) \|_{D(A)} + \| P_1(t) - P_2(t) \|_Y$$

$$\leq \int_0^t g_2(r_0, t, s)(\| u_1(s) - u_2(s) \|_{D(A)} + \| p_1(s) - p_2(s) \|_Y) ds$$

$$\leq \gamma \sup_{0 \leq s \leq t} (\| u_1(s) - u_2(s) \|_{D(A)} + \| p_1(s) - p_2(s) \|_Y) ds.$$

It follows that

$$\| (U_1, P_1) - (U_2, P_2) \|_{Z(T_0)} \leq \gamma \| (u_1, p_1) - (u_2, p_2) \|_{Z(T_0)}.$$

Our claim is proven.

Proposition 2. (u, p) is a solution of the inverse problem (1)−(3) in $[0, T]$ iff $(u, p) = \Psi(u, p)$.

Proof. It is well known that the solution of Cauchy problem (1) and (2) is given by $u(t) = U(t)$. Therefore, it suffices to show

$$Lu(t) = v(t) \text{ iff } p(t) = \Psi(t, v'(t) - \int_0^t LF_2(s, u(s), p(t-s)) ds - LAu(t))$$

for each t in $[0, T]$.

First, we differentiate $Lu(t) = v(t)$ to obtain

$$Lu'(t) = L\{Au(t) + F_1(t, u(t), p(t)) + \int_0^t F_2(s, u(s), p(t-s)) ds\} = v'(t).$$

Hence

$$H_1(t, v(t), p(t)) = LF_1(t, u(t), p(t))$$

$$= v'(t) - \int_0^t LF_2(s, u(s), p(t-s)) ds - LAu(t).$$

Using $(H7, 2)$ we get

$$p(t) = \Psi(t, v'(t) - \int_0^t LF_2(s, u(s), p(t-s)) ds - LAu(t)).$$

Conversely, this last equality implies that

$$H_1(t,v(t),p(t)) = v'(t) - L\{\int_0^t F_2(s,u(s),p(t-s))ds - Au(t)\}$$

$$- v'(t) - L\{u'(t) - F_1(t,u(t),p(t))\}$$

$$= v'(t) - Lu'(t) + H_1(t,Lu(t),p(t)).$$

Thus

$$\frac{d}{dt}(v(t)) = H_1(t,v(t),p(t)) - H_1(t,Lu(t),p(t)).$$

Integrating and using the fact that $v(0) = Lu(0) = Lx$, we obtain

$$v(t) - Lu(t) = \int_0^t (H_1(s,v(s),p(s)) - H_1(s,Lu(s),p(s)))ds.$$

But, $(H7,1)$ leads to

$$\| v(t) - Lu(t) \|_Y = \int_0^t C(R,s) \| v(s) - Lu(s) \|_Y ds,$$

where

$$R = max(\sup_{0 \leq t < T} (\| Lu(t) \|_Y + \| p(t) \|_Y), \sup_{0 \leq t \leq T} (\| v(t) \|_Y + \| p(t) \|_Y)).$$

Hence, by using Gronwall's inequality, it follows that

$$v(t) - Lu(t) = 0, \ 0 \leq t \leq T.$$

Now, we combine propositions 1 and 2 to deduce the following local existence and uniqueness theorem for the inverse problem $(1)-(3)$.

Theorem 1. *Under the assumptions* $(H1)-(H7)$, *there exist* T_0 *in* $[0,T]$ *and* (u_0,p_0) *in* $C([0,T_0]:D(A) \times Y)$ *which is the unique solution of the inverse problem* $(1)-(3)$ *in* $[0,T_0]$.

Remark. Theorem 1 is still valued if we add to the right side of equality (1) a function $f:[0,T] \rightarrow X$ such that f and Af are continuous.

3. GLOBAL SOLUTION

We begin this section by showing that any solution (u_0,p_0) in $C([0,T_0]:D(A) \times Y)$ of the inverse problem $(1)-(3)$ in $[0,T_0]$ can be uniquely extended to a solution in $[0,T_0+T_1]$ for some $T_1 > 0$, whenever $0 < T_0 < T$.

If $\overline{T}$ is in $[0,min(T_0,T-T_0)]$, we consider the following inverse problem:

$$(4)\qquad u'(t) = Au(t) + K_1(t, u(t), p(t)) + \int_0^t K_2(s, u(s), p(t-s))ds + f(t),\ 0 \le t \le \overline{T}$$

$$(5)\qquad u(0) = x_1 = u_0(T_0)$$

$$(6)\qquad Lu(t) = w(t),\ 0 \le t \le \overline{T}$$

where

$$K_1(t, u(t), p(t)) = F_1(t + T_0, u(t), p(t)),\ 0 \le t \le \overline{T},$$

$$K_2(s, u(s), p(t-s)) = F_2(s, u_0(s), p(t-s)) + F_2(s + T_0, u(s), p_0(t-s)),\ 0 \le t \le \overline{T},$$

$$f(t) = \int_t^{T_0} F_2(s, u_0(s), p_0(t + T_0 - s))ds,\ 0 \le t \le \overline{T}, \text{ and}$$

$$w(t) = v(t + T_0), 0 \le t \le \overline{T}.$$

Proposition 3. If (u_0, p_0) in $C([0, T_0]: D(A) \times Y)$ denotes any solution of the inverse problem $(1)-(3)$ in $[0, T_0]$, then there exist T_1 in $[0, min(T_0, T - T_0)]$ and (u, p) in $C([0, T_0 + T_1]: D(A) \times Y)$ such that $(u, p) = (u_0 p_0)$ in $[0, T_0]$, and (u, p) satisfies $(1)-(3)$ in $[0, T_0 + T_1]$.

Proof. It is not difficult to see that K_1, K_2, w have the same properties as F_1, F_2, and v, and that f and Af are continuous. It follows from Theorem 1 that there exist $T_1 \in]0, \overline{T}]$ and $(u_1, p_1) \in C([0, T_1]: D(A) \times Y)$, which is the unique solution of the inverse problem $(4)-(6)$ given by

$$u_1(t) = e^{At}x + \int_0^t e^{A(t-s)}K_1(s, u(s), p(s))ds + \int_0^t e^{A(t-s)}f(s)ds$$
$$+ \int_0^t e^{A(t-s)} \int_0^s K_2(\sigma, u(\sigma), p(s-\sigma))d\sigma ds,\ 0 \le t \le T_1,$$

$$p_1(t) = \Psi(t + T_0, w'(t) - LAu_1(t) - \int_0^t LK_2(s, u_1(s), p_1(t-s))ds - Lf(t)),\ 0 \le t \le T_1.$$

We have

$$p_1(0) = \Psi(T_0, w'(0) - LAu_1(0) - Lf(0))$$
$$= \Psi(T_0, v'(T_0) - LAu(T_0) - \int_0^{T_0} LF_2(s, u_0(s), p_0(T_0 - s))ds$$
$$= p(T_0).$$

One can easily check that

$$(u(t),p(t)) = \begin{cases} (u_0(t),p_0(t)), & 0 \le t \le T_0, \\ (u_1(t),p_1(t)), & T_0 < t \le T_1, \end{cases}$$

belongs to $C([0,T_0+T_1]:D(A)\times Y)$. It remains to show that (u,p) is a solution of the inverse problem $(1)-(3)$ in $[0,T_0+T_1]$. Since u_1 satisfies (4), we can deduce that

$$\begin{aligned} u'(t+T_1) &= u_1'(t) \\ &= Au_1(t)+F_1(t+T_0,u_1(t),p_1(t))+\int_0^t F_2(s,u_0(s),p_1(t-s))ds \\ &\quad +\int_0^t F_2(s+T_0,u_1(s),p_0(t-s))ds+\int_t^{T_0} F_2(s,u_0(s),p_0(t+T_0-s))ds \\ &= Au(t+T_0)+F_1(t+T_0,u(t+T_0),p(t+T_0))+\int_0^t F_2(s,u(s),p(t+T_0-s))ds \\ &\quad +\int_{T_0}^{t+T_0} F_2(s,u(s),p(t+T_0-s))ds+\int_t^{T_0} F_2(s,u(s),p(t+T_0-s))ds \\ &= Au(t+T_0)+F_1(t+T_0,u(t+T_0),p(t+T_0)) \\ &\quad +\int_0^{t+T_0} F_2(s,u(s),p(t+T_0-s))ds, 0 \le t \le T_1. \end{aligned}$$

On the other hand

$$Lu(t+T_0)=Lu_1(t)=w(t)=v(t+T_0), 0 \le t \le T_1.$$

Therefore we may conclude that (u,p) is a solution of the inverse problem $(1)-(3)$ in $[0,T_0+T_1]$.

Proposition 4. Let $(u,p)\in C([0,T_{max}[:D(A)\times Y)$ be the maximal solution of the inverse problem $(1)-(3)$, where $0<T_{max}\le T$. If

$$\max_{0<t<T_{max}} \left(\sup_{0\le s\le t} (\| u(s)\|_{D(A)} + \| p(s)\|_Y)\right) < +\infty, \tag{7}$$

then $T_{max}=T$.

Proof. Clearly, from Proposition 2 (u,p) can be continuously extended to a solution in

$[0, T_{max}]$. If $T_{max} < T$, then, following the previous proposition, the solution in $[0, T_{max}]$ can be extended to a solution in $[0, T_{max} + \epsilon]$, for some $\epsilon > 0$. This contradicts the maximality of T_{max}.

Now, we will give a sufficient conditions to realize (7). For this purpose, we recall the following comparison theorem.

Theorem 2 [2]. *Let I be a real interval, and let $G: I \times I \times R^+ \to R^+$ be continuous such that $G(t,s,r)$ is monotone nondecreasing in r for each (t,s) in $I \times I$. Let b in $C(I)$, and let f in $C(I)$ denote the maximal solution of the integral equation*

$$f(t) = b(t) + \int_{t_0}^{t} G(t,s,f(s))ds, t \geq t_0.$$

If $g \in C(I)$ is such that

$$g(t) \leq b(t) + \int_{t_0}^{t} G(t,s,g(s))ds, t \geq t_0,$$

then $g(t) \leq f(t), t \geq t_0$.

Here, by a maximal solution we mean that any other solution $h \in C(I)$ must satisfy $h(t) \leq f(t), t \geq t_0$.

Before stating a global existence and uniqueness result for our inverse problem, we need to modify some assumptions on F_1, and F_2.

Instead of $(H5,2)$ and $(H6,2)$ we suppose that there exist $G_i(t,r): [0,T] \times R^+ \to R^+$ continuous and monotone nondecreasing in r for each t in $[0,T], i = 1,2$, such that

$(H5,2')$ $\quad \| F_1(t,u,p) \|_{D(A)} \leq G_1(t, \| u \|_{D(A)} + \| p \|_Y)$,

$(H6,2')$ $\quad \| \int_0^t F_2(s,u(s),p(t-s))ds \|_{D(A)} \leq \int_0^t G_2(s, \| u(s) \|_{D(A)} + \| p(s) \|_Y)ds.$

Set

$$G(t,s,r) = M(1 + k(t) \| L \|)(G_1(s,r) + (t-s)G_2(s,r)) + k(t) \| L \| G_2(s,r), 0 \leq s \leq t \leq T, i = 0,1.$$

Clearly, $G(t,s,r)$ is monotone nondecreasing in $r, 0 \leq s \leq t \leq T$.

Theorem 3. *Assume that $(H1)-(H7)$ are satisfied, where $(H5,2)$ and $(H6,2)$ are changed by $(H5,2')$ and $(H6,2')$. If the nonlinear Volterra integral equation:*

$$(8) \qquad r(t) = a(t) + \int_0^t G(t,s,r(s))ds, 0 \le t \le T,$$

has a continuous maximal solution in $[0,T]$*, then the inverse problem* $(1)-(3)$ *has a unique solution in* $[0,T]$.

Proof. Let r denote the continuous maximal solution of the integral equation (8). Proceeding in the manner of the proof of Proposition 1, we obtain

$$\| u(t) \|_{D(A)} + \| p(t) \|_Y \le a(t) + \int_0^t G(t,s, \| u(s) \|_{D(A)} + \| p(s) \|_Y)ds, 0 \le t \le T.$$

Thus, the condition (7) is satisfied.

The uniqueness of the global solution is just a consequence of the fact that the unique local solution allows a unique extension.

4. STABILITY RESULT

First of all, we give the exact assumptions under which the stability result will hold.

We assume that $(H1)-(H5,1),(H6,1),(H7)$ are satisfied, and there exist $G_i(t,r):[0,T]\times R^+ \to R^+$ continuous and monotone nondecreasing in r for each t in $[0,T], i = 1,2,$ such that

$(H8,1)$ $\| F_1(t,u_1,p_1) - F_1(t,u_2,p_2) \|_{D(A)} \le G_1(t, \| u_1 - u_2 \|_{D(A)} + \| p_1 - p_2 \|_Y)$, for each (u_i, p_i) in $D(A)\times Y, i = 1,2,$ and $0 \le t \le T$.

$(H8,2)$ $$\| \int_0^t (F_2(s,u_1(s),p_1(t-s)) - F_2(s,u_2(s),p_2(t-s)))ds \|_{D(A)} \le \int_0^t G_2(s, \| u_1(s) - u_2(s) \|_{D(A)} + \| p_1(s) - p_2(s) \|_Y)ds$$

for each (u_i, p_i) in $C([0,T]:D(A)\times Y), i = 1,2,$ and $0 \le t \le T$.

$(v(t), H_1(t,v(t),p)) \to \Phi(t,K(p))$ has the following property:
there exist continuous $g:[0,T]\times R^+ \to R^+$, such that

$$\| \Phi_1(t,w_1) - \Phi_2(t,w_2) \|_Y \le g(t)(\| v_1(t) - v_2(t) \|_Y + \| w_1 - w_2 \|_Y),$$

for each v_i in $C([0,T]:Y), w_i \in Y, i = 1,2,$ and $0 \le t \le T$.

Here, $\Phi_i(t,\cdot)$ denotes the inverse of the mapping $K_i: p \to H_1(t, v_i(t), p)), (i = 1,2)$. We set

$$G(t,s,r) = M(1+g(t)\,\|L\|)(G_1(s,r)+(t-s)G_2(s,r)) + g(t)\,\|L\|\,G_2(s,r),$$
$$0 \le s \le t \le T, \quad i = 0,1.$$

Theorem 4. *Suppose that the assumptions listed below are satisfied for $x = x_i$, $v = v_i$, $i = 1,2$. Let (u_i, p_i) in $C([0,T]: D(A) \times Y)$ denote any solution of the inverse problem (1)–(3) corresponding to $x = x_i$, $v = v_i$, $i = 1,2$, and let*

$$r_0(t) = M(1+g(t)\,\|L\|)\,\|x_1 - x_2\|_{D(A)} + g(t)(\,\|v_1(t)-v_2(t)\|_Y + (\,\|v_1'(t)-v_2'(t)\|_Y).$$

If the maximal continuous solution, given its existence, of the Volterra integral equation

$$m(t) = r_0(t) + \int_0^t G(t,s,m(s))ds, \quad 0 \le t \le T, \tag{9}$$

satisfies the condition that there exists a constant $C > 0$, not depending on m, such that

$$m(t) \le C r_0(t), \quad 0 \le t \le T, \tag{10}$$

then

$$\|u_1(t)-u_2(t)\|_{D(A)} + \|p_1(t)\|_Y \le C r_0(t), \; 0 \le t \le T. \tag{11}$$

Proof. Let m denote the maximal solution of the integral equation (9), and let

$$r(t) = \|u_1(t)-u_2(t)\|_{D(A)} + \|p_1(t)-p_2(t)\|_Y, \; 0 \le t \le T.$$

It is easy to see that

$$r(t) \le r_0(t) + \int_0^t G(t,s,r(s))ds, \; 0 \le t \le T.$$

Using the comparison Theorem 2, we deduce that $r(t) \le m(t)$. Hence, (11) follows from (10).

Remark. We have $G(t,s,r) \le G(T,s,r)$. Then if $G(T,s,r)$ takes the form $G(T,s,r) = G(s)r$, the conclusion of Theorem 4.1 follows from Gronwall's inequality.

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