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INCLUSION RELATIONS BETWEEN CLASSES OF HYPERGEOMETRIC FUNCTIONS[1]

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ABSTRACT

The use of hypergeometric functions in univalent function theory received special attention after the surprising application of such functions by de Branges in the proof of the 70-year old Bieberbach Conjecture. In this paper we consider certain classes of analytic functions and examine the distortion and containment properties of generalized hypergeometric functions under some operators in these classes.

Key words: Generalized hypergeometric functions, analytic functions.

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1. INTRODUCTION

The expansions and generating functions involving associated Lagurre, Jacobi Polynomials, Bessel functions and their generalizations such as hypergeometric functions of one and several variables occur frequently in the seemingly diverse fields of Physics, Engineering, Statistics, Probability, Operations Research and other branches of applied mathematics (see e.g. Exton [7] and Schiff [19]). The use of hypergeometric functions in univalent function theory received special attention after the surprising application of such functions by de Branges [6] in the proof of the Bieberbach Conjecture [2]; also see [1].

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Merkes-Scott [11], Carlson-Shaffer [4] and Ruscheweyh-Singh [18] studied the starlikeness of certain hypergeometric functions. Miller-Mocanu [12] found a univalence criterion for such functions. The second author and Silvia [8] and more recently Noor [14] studied the behavior of certain hypergeometric functions under various operators. In the present paper we consider certain classes of analytic functions and examine the distortion and containment properties of generalized hypergeometric functions under some operators in these classes.

Let p and q be natural numbers such that $p \leq q+1$. For α_j $(j=1,2,\ldots,p)$ and β_k $(k=1,2,\ldots,q)$ complex numbers such that $\beta_k \neq 0, -1, -2, \ldots$ $(k=1,2,\ldots,q)$, let ${}_pF_q(z)$, that is

$$ {}_pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n(\beta_2)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \tag{1.1} $$

denote the generalized hypergeometric function. Here $(\lambda)_n$ is the Pochhammer symbol defined by

$$ (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \left[\begin{array}{ll} 1, & \text{if } n=0 \\ \lambda(\lambda+1)\ldots(\lambda+n-1), & \text{if } n=1,2,\ldots. \end{array} \right. $$

It is known [9, p. 43] that the series given by (1.1) converges absolutely for $|z| < \infty$ if $p < q+1$, and for $z \in U = \{z: |z| < 1\}$ if $p = q+1$. Thus for $p \leq q+1$,

$$ {}_pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; z) \in \mathcal{A}, $$

where $\mathcal{A}$ denotes the class of functions that are analytic in U.

For ${}_pF_q(z)$ defined by (1.1), let

$$ {}_pG_q(z) := z\, {}_pF_q(z) = z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1} \cdots (\alpha_p)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1} \cdots (\beta_q)_{n-1}\,(1)_{n-1}} z^n \tag{1.2} $$

for all $z \in U$, where $p \leq q+1$.

Let S be the family of functions of the form

$$ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in U) \tag{1.3} $$

that are analytic and univalent in U. For $f \in S$, define the convolution operator Λ by

$$\Lambda(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; f) := {}_pG_q^* f. \tag{1.4}$$

The operator "$*$" denotes the Hadamard product or convolution of two power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n$$

that is

$$(f^* g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Thus for f of the form (1.3) in S, we may write

$$\Lambda(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; f)$$

$$= z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1} \cdots (\alpha_p)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1} \cdots (\beta_q)_{n-1}(1)_{n-1}} \, a_n z^n (z \in U), \tag{1.5}$$

where $p \leq q+1$.

We recall that a function Φ is convex in U if it is univalent conformal mapping of U onto a convex domain. It is well-known that Φ is convex in U if and only if

$$Re\left(1 + \frac{z\Phi''(z)}{\Phi'(z)}\right) > 0 \qquad (z \in U)$$

and $\Phi' \neq 0$. Also, a function f is said to be close-to-convex in U if there exists a convex function Φ in U such that $Re(f'(z)/\Phi'(z)) > 0$ $(z \in U)$. For $0 \leq \alpha < 1$, we define the following subclasses of $\mathcal{A}$:

$$\mathcal{A}(p, q; \alpha) := \{ {}_pF_q \in \mathcal{A} : Re\, {}_pF_q(z) > \alpha \},$$

$$R(\alpha, r) := \{ f \in \mathcal{A} : Re f'(z) > \alpha \text{ for } |z| < r \leq 1 \},$$

$$S(\alpha) := \{ f \in \mathcal{A} : Re(f(z)/z) > \alpha \},$$

$$T := \{ f \in S : f(z) = z - \sum_{k=2}^{\infty} a_k z^k \;\; z \in U,\, a_k \geq 0 \},$$

$$T^*(\alpha) := \{ f \in T : Re(zf'(z)/f(z)) > \alpha \},$$

$$C(\alpha) := \{ f \in T : Re(1 + zf''(z)/f'(z)) > \alpha \}.$$

2. MAIN RESULTS

We use the following lemma due to Chen [5] to prove our first theorem.

Lemma 1: *If $f \in S(\alpha)$, then*

$$Ref'(z) \geq \begin{cases} \dfrac{1+2(2\alpha-1)r+(2\alpha-1)r^2}{(1+r)^2} & if\ 0 \leq r < 1/2, \\[2ex] \dfrac{\alpha-2\alpha r^2+(2\alpha-1)r^4}{(1-r^2)^2} & if\ 1/2 \leq r < 1. \end{cases}$$

Theorem 1: *Let*

$${}_{p+1}G_{q+1}(z) = z\,{}_{p+1}F_{q+1}(\alpha_1, \alpha_2, \ldots, \alpha_p, 1+\gamma; \beta_1, \beta_2, \ldots, \beta_q, \gamma; z).$$

If

$${}_pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; z) \in \mathcal{A}(p, q; \alpha),$$

then

(*i*) *For* $0 \leq |z| = r < 1/2$, $z^{-1}{}_{p+1}G_{q+1} \in \mathcal{A}(p+1, q+1; X)$, *where*

$$X = \frac{1}{\gamma}\left(\alpha\gamma + \alpha - 1 + \frac{2(1-\alpha)}{(1+r)^2}\right), \quad \gamma \geq 1,$$

(*ii*) *For* $1/2 \leq |z| = r < 1$, $z^{-1}{}_{p+1}G_{q+1} \in \mathcal{A}(p+1, q+1; Y)$, *where*

$$Y = \alpha - \frac{(1-\alpha)r^4}{\gamma(1-r^2)^2}, \qquad \gamma \geq 1.$$

Proof: We observe that

$$(1-1/\gamma)\ {}_pG_q(z) + \frac{1}{\gamma}z\ {}_pG_q'(z) = {}_{p+1}G_{q+1}(z).$$

Thus by an application of Lemma 1, we have

$$Re(z^{-1}\ {}_{p+1}G_{q+1}(z)) = (1-1/\gamma)Re(z^{-1}\ {}_pG_q(z)) + \frac{1}{\gamma}Re(\ {}_pG_q'(z))$$

$$\geq (1-1/\gamma)\alpha + \frac{1+2(2\alpha-1)r+(2\alpha-1)r^2}{\gamma(1+r)^2}$$

$$= X, \text{ if } 0 \leq |z| = r < 1/2,$$

and

$$Re(z^{-1}\ {}_{p+1}G_{q+1}(z)) \geq (1-1/\gamma)\alpha + \frac{\alpha-2\alpha r^2+(2\alpha-1)r^4}{\gamma(1-r^2)^2}$$

$$= Y, \text{ if } 1/2 \leq |z| = r < 1.$$

To prove our next theorem we need the following lemma due to MacGregor [10].

Lemma 2: *If $f \in R(0,1)$, then*

$$|f'(z)| \leq \frac{1+|z|}{1-|z|}$$

and

$$-|z| + 2log(1+|z|) \leq |f(z)| \leq -|z| - 2log(1-|z|).$$

Theorem 2: *Let*

$$\Psi_\gamma({}_pF_q(z)) = \frac{\gamma}{z^{\gamma-1}} \int_0^z t^{\gamma-1}{}_pF_q(t)\, dt \qquad (\gamma \geq 1).$$

If $\psi_\gamma({}_pF_q) \in R(0,1)$, then

(*i*) $|{}_pF_q(z)| \leq -1 + \frac{2}{\gamma}(\frac{1}{1-|z|} + \frac{1-\gamma}{|z|} log(1-|z|),$

(*ii*) $|{}_pF_q(z)| \geq -1 - \frac{2}{\gamma}(\frac{1}{1-|z|} + \frac{1-\gamma}{|z|} log(1+|z|)).$

Proof: Note that

$$(1-1/\gamma)\Psi_\gamma({}_pF_q(z)) + \frac{1}{\gamma}\, z(\Psi'_\gamma({}_pF_q(z)) = z\ {}_pF_q(z). \tag{2.1}$$

Since

$$Re(\Psi'_\gamma({}_pF_q(z)) > 0,$$

we obtain from (2.1) and Lemma 2

$$|{}_pF_q(z)| \leq (1-1/\gamma)\,|z^{-1}\Psi_\gamma({}_pF_q(z))| + \frac{1}{\gamma}\ |\Psi'_\gamma({}_pF_q(z))|$$

$$\leq -(1-1/\gamma) - \frac{2(1-1/\gamma)}{|z|} log(1-|z|) + \frac{1}{\gamma}\left(\frac{1+|z|}{1-|z|}\right)$$

$$= -1 + \frac{2}{\gamma}\left(\frac{1}{1-|z|} + \frac{1-\gamma}{|z|} log(1-|z|)\right),$$

and

$$|{}_pF_q(z)| \geq (1-1/\gamma)\ |z^{-1}\Psi_\gamma({}_pF_q(z))| - \frac{1}{\gamma}|\Psi'_\gamma({}_pF_q(z))|$$

$$\geq -(1-1/\gamma) + \frac{2(1-1/\gamma)}{|z|} log(1+|z|) - \frac{1}{\gamma}\left(\frac{1+|z|}{1-|z|}\right)$$

$$= -1 - \frac{2}{\gamma}\left(\frac{1}{1-|z|} + \frac{1-\gamma}{|z|}\log(1+|z|)\right).$$

Theorem 3: *If ${}_pF_q \in \mathcal{A}(p,q;\alpha)$, $0 \le \alpha < 1$, then for $\gamma \ge 1$,*

$$\left|{}_{p+1}F_{q+1}(\alpha_1, \alpha_2, \ldots, \alpha_p, \gamma; \beta_1, \beta_2, \ldots, \beta_q, \gamma+1; z)\right|$$
$$\le \frac{\gamma(1+(1-2\alpha)|z|}{1-|z|}.$$

Proof: We observe that

$$z_{p+1}F_{q+1}(\alpha_1, \alpha_2, \ldots, \alpha_p, \gamma; \beta_1, \beta_2, \ldots, \beta_q, \gamma+1; z) = \frac{\gamma}{z^{\gamma-1}}\int_0^z t^{\gamma-1}\ {}_pF_q(t)dt.$$

Now by applying a result of Owa [15] that if $f \in S(\alpha)$, then

$$\left|\frac{f(z)}{z}\right| \le \frac{1+(1-2\alpha)|z|}{1-|z|},$$

we obtain

$$\left|{}_{p+1}F_{q+1}(z)\right| = \frac{\gamma}{|z|^{\gamma}}\left|\int_0^z \frac{{}_pF_q(t)}{t^{1-\gamma}}dt\right|$$

$$\le \frac{\gamma}{|z|^{\gamma}}\ \frac{1}{|z|^{1-\gamma}}\ \frac{1+(1-2\alpha)|z|}{1-|z|}\ |z|$$
$$= \frac{\gamma((1+(1-2\alpha)|z|)}{1-|z|}.$$

Next we use a result of Nikolaeva and Repnina [13] concerning the convex combination of certain analytic functions to prove our Theorem 4.

Lemma 3: *Let $f \in R(0,1)$. Denote*

$$h_0 = \frac{\mu^3}{(1+\sqrt{1+3\mu^2})^2}, \qquad 0 \le \mu \le 1.$$

Let

$$F_\mu(z) = (1-\mu)f(z) + \mu z f'(z), \quad h = \alpha/(1-\alpha).$$

Then $F_\mu \in R(\alpha, r)$ where

$$r = \begin{cases} \sqrt{\dfrac{1-\mu+2\sqrt{\mu h}}{1+\mu+2\sqrt{\mu h}}}, & \text{if } h \ge h_0 \\[2ex] \dfrac{1+h}{\mu - h + \sqrt{\mu^2 - 2\mu h + 1}}, & \text{if } 0 \le h \le h_0. \end{cases}$$

Theorem 4: *If* ${}_pG_q \in R(\alpha, 1)$, $0 \le \alpha < 1$, *then the function*

$$_{p+1}G_{q+1}(z) = z_{p+1}F_{q+1}(\alpha_1, \alpha_2, \ldots, \alpha_p, 1+\gamma;\ \beta_1, \beta_2, \ldots, \beta_q, \gamma; z)$$

is in $R(\alpha, r)$ *where*

$$(i) \quad r = r(\alpha, \gamma) = \frac{(\gamma-1)\sqrt{(\alpha\gamma)}}{(\gamma+1)\sqrt{(1-\alpha)} + 2\sqrt{(\alpha\gamma)}},$$

if $\alpha \ge 1/(1+\gamma(\gamma+\sqrt{(\gamma^2+3)})$ *and*

$$(ii) \quad r = r(\alpha, \gamma) = \frac{\gamma}{1-\alpha(1+\gamma)+\sqrt{(\gamma(1-\alpha)(\gamma-\alpha\gamma-2\alpha))}},$$

if $0 \le \alpha < 1/(1+\gamma(\gamma+\sqrt{(\gamma^2+3)}))$ *for* $\gamma \ge 1$.

Proof: Noting that

$$_{p+1}G_{q+1}(z0 = (1-1/\gamma) \ {}_pG_q(z) + \frac{1}{\gamma} z \ {}_pG_q'(z),$$

and comparing with Lemma 3, the result follows by simple algebraic manipulations.

Corollary: *If* ${}_pG_q \in R(\alpha, 1)$, *then* $r = r(\alpha, \gamma)$ *is also the radius of univalence of* $_{p+1}G_{q+1}$ *for* $\gamma \ge 1$.

Theorem 5: *If*

$$z \ {}_pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; z)$$

is convex, close-to-convex, or starlike of order $\alpha (0 \le \alpha < 1)$, *then so is*

$$z \ {}_{p+1}F_{q+1}(\alpha_1, \alpha_2, \ldots, \alpha_p, \gamma+1; \beta_1, \beta_2, \ldots, \beta_q, \gamma+2; z),$$

for $Re(\gamma) > 0$.

Proof: Consider the general transform

$$\Phi_\gamma(f(z)) = \frac{\gamma+1}{z^\gamma} \int_0^z t^{\gamma-1} f(t)dt, \quad (Re(\gamma) > 0).$$

Then

$$\Phi_\gamma(z \ {}_pF_q(z)) = \frac{\gamma+1}{z^\gamma} \int_0^z t^{\gamma-1}(z \ {}_pF_q(t))dt$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_p)_n(\gamma+1)}{(\beta_1)_n(\beta_2)_n \cdots (\beta_q)_n(\gamma+n+1)} \ \frac{z^{n+1}}{n!}$$

$$=\left(\sum_{n=1}^{\infty} \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}\cdots(\alpha_p)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}\cdots(\beta_q)_{n-1}(1)_{n-1}} z^n\right) * \left(\sum_{n=1}^{\infty} \frac{\gamma+1}{\gamma+n} z^n\right)$$

$$z \; {}_pF_q(z) * H(z),$$

where

$$H(z) = \sum_{n=1}^{\infty} \frac{\gamma+1}{\gamma+n} z^n$$

is known [16] to be convex in U. It follows from the work of Ruscheweyh and Sheil-Small [17] that the function $\Phi_\gamma(z \; {}_pF_q)$ is convex, close-to-convex, or starlike of order α whenever f is such. Now the result follows because we may write

$$\Phi_\gamma(z \; {}_pF_q(z)) = z \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_p)_n(\gamma+1)_n}{(\beta_1)_n(\beta_2)_n\cdots(\beta_q)_n(\gamma+2)_n} \frac{z^n}{n!}$$

$$= z \; {}_{p+1}F_{q+1}(\alpha_1, \alpha_2, \ldots, \alpha_p, \gamma+1; \; \beta_1, \beta_2, \ldots, \beta_q, \gamma+2; z).$$

Our last theorem deals with hypergeometric functions with negative coefficients. Its proof uses the following lemma which is due to Silverman [20].

Lemma 4: *Let*

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0.$$

Then

(*i*) $f \in T^*(\alpha) \Leftrightarrow \sum_{n=2}^{\infty} (n-\alpha)a_n \leq 1-\alpha,$

(*ii*) $f \in C(\alpha) \Leftrightarrow \sum_{n=2}^{\infty} n(n-\alpha)a_n \leq 1-\alpha,$

(*iii*) $f \in T^*(\alpha) \Rightarrow |a_n| \leq \frac{1-\alpha}{n-\alpha} \; (n \geq 2),$

(*iv*) $f \in C(\alpha) \Rightarrow |a_n| \leq \frac{1-\alpha}{n(n-\alpha)} \; (n \geq 2).$

Theorem 6: *For*

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in T^*(\alpha),$$

the function

$$\Lambda(\alpha_1, \alpha_2, \ldots, \alpha_{p+1}, \beta_1, \beta_2, \ldots, \beta_p; f) \in T^*(\alpha) \Leftrightarrow$$

$$\alpha_i, \beta_j > 0 \quad (i, j = 1, 2, \ldots, p),\ \alpha_{p+1} > 0,\ \sum_{j=1}^{p}(\beta_j - \alpha_j) > \alpha_{p+1},$$

and

$$_{p+1}F_p(\alpha_1, \alpha_2, \ldots, \alpha_{p+1}; \beta_1, \beta_2, \ldots, \beta_p; 1) \leq 2. \tag{2.2}$$

Proof: For

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in T$$

and using (1.5), we have

$$\Lambda(\alpha_1, \alpha_2, \ldots, \alpha_{p+1}; \beta_1, \beta_2, \ldots, \beta_p; f) = z - \sum_{k=2}^{\infty} d_k z^k, \tag{2.3}$$

where

$$d_k = \frac{(\alpha_1)_{k-1}(\alpha_2)_{k-1} \ldots (\alpha_{p+1})_{k-1}}{(\beta_1)_{k-1}(\beta_2)_{k-1} \cdots (\beta_p)_{k-1}(1)_{k-1}}\ a_k \geq 0.$$

In view of Lemma 4, the function given by (2.3) is in $T^*(\alpha)$ if and only if

$$\sum_{k=2}^{\infty} \frac{k-\alpha}{1-\alpha}\ |\, d_k \,| \leq 1.$$

Since $|\, a_k \,| \leq (1-\alpha)/(k-\alpha)$ $(k \geq 2)$, by Lemma 4 (*iii*), we have

$$\sum_{k=2}^{\infty} \frac{k-\alpha}{1-\alpha}\ |\, d_k \,| = \sum_{k=2}^{\infty} \frac{k-\alpha}{1-\alpha}\ \frac{(\alpha_1)_{k-1}(\alpha_2)_{k-1} \cdots (\alpha_{p+1})_{k-1}}{(\beta_1)_{k-1}(\beta_2)_{k-1}(\beta_p)_{k-1}(1)_{k-1}}\ |\, a_k \,|$$

$$\leq \sum_{k=2}^{\infty} \frac{(\alpha_1)_{k-1}(\alpha_2)_{k-1} \cdots (\alpha_{p+1})_{k-1}}{(\beta_1)_{k-1}(\beta_2)_{k-1}(\beta_p)_{k-1}(1)_{k-1}}$$

$$= \left(\sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k \cdots (\alpha_{p+1})_k}{(\beta_1)_k(\beta_2)_k \cdots (\beta_p)_k k!}\right) - 1$$

$$= {}_{p+1}F_p(\alpha_1, \alpha_2, \ldots, \alpha_{p+1}; \beta_1, \beta_2, \ldots, \beta_p; 1) - 1.$$

The desired result follows because the $_{p+1}F_p$ series is absolutely convergent for $|\, z \,| = 1$ if

$$\sum_{k=1}^{\infty}((\beta_k - \alpha_k) - \alpha_{p+1}) > 0;$$

see [9, p. 44].

Remark 1: Using (*ii*) and (*iv*) in Lemma 4, we can similarly prove that Theorem 6 holds for $C(\alpha)$, that is,

$$\Lambda(\alpha_1, \alpha_2, \ldots, \alpha_{p+1}, \beta_1, \beta_2, \ldots, \beta_p; C(\alpha)) \subset C(\alpha)$$

if and only if the conditions of Theorem 6 hold.

Remark 2: In view of the generalization of the Gaussian summation formula for $p = 2, 3, \ldots$, determined in [3], the condition (2.2) may be expressed as

$$\sum_{k=0}^{\infty} \frac{(s)_k}{(\alpha_1 + s)_k(\alpha_2 + s)_k} A_k^{(p)} \leq \frac{2\Gamma(\alpha_3)\ldots\Gamma(\alpha_{p+1})\Gamma(\alpha_1 + s)\Gamma(\alpha_2 + s)}{\Gamma(\beta_1)\Gamma(\beta_2)\ldots\Gamma(\beta_p)\Gamma(s)},$$

where $A_k^{(p)}$'s are given by some lengthy expressions in [3] and

$$s = \sum_{j=1}^{p} (\beta_j - \alpha_j) - \alpha_{p+1} > 0.$$

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