

# SOME PERFORMANCE MEASURES FOR VACATION MODELS WITH A BATCH MARKOVIAN ARRIVAL PROCESS

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(Received November, 1993; revised January, 1994)

## ABSTRACT

We consider a single server infinite capacity queueing system, where the arrival process is a batch Markovian arrival process (BMAP). Particular BMAPs are the batch Poisson arrival process, the Markovian arrival process (MAP), many batch arrival processes with correlated interarrival times and batch sizes, and superpositions of these processes. We note that the MAP includes phase-type (PH) renewal processes and non-renewal processes such as the Markov modulated Poisson process (MMPP).

The server applies Kella's vacation scheme, i.e., a vacation policy where the decision of whether to take a new vacation or not, when the system is empty, depends on the number of vacations already taken in the current inactive phase. This exhaustive service discipline includes the single vacation  $T$ -policy,  $T(SV)$ , and the multiple vacation  $T$ -policy,  $T(MV)$ . The service times are i.i.d. random variables, independent of the interarrival times and the vacation durations. Some important performance measures such as the distribution functions and means of the virtual and the actual waiting times are given. Finally, a numerical example is presented.

**Key words:** Vacation Models, Batch Markovian Arrival Process, BMAP/G/1 Queue, Waiting Time Distributions, Matrix-Analytic Methodology.

**AMS (MOS) subject classifications:** Primary 60K25, Secondary 68M20, 90B22.

## 1. Introduction

In Matendo [11], we considered a BMAP/G/1 queueing system in which the server applies a general exhaustive service vacation policy. In that model, the server alternates between active and inactive states. During the active phase, the server continuously provides service to customers and during the inactive phase, which begins whenever the system becomes empty (exhaustive service), the server is unavailable to the customers. We refer to Loris-Teghem [5] and [6] for details. We note that this rather large class of exhaustive service vacation policies includes:

- the  $N$ -policy, according to which an inactive phase terminates as soon as at least  $N$  ( $N \geq 1$ ) customers accumulate;

- the  $T$ -policy
  - \* with single vacation,  $T(SV)$ , according to which, when the system gets empty, the server leaves for a time period called a vacation. Back in the system, the server resumes serving customers as soon as at least one customer is present;
  - \* with multiple vacations,  $T(MV)$ , according to which, when the system gets empty, the server takes repeated vacations until, back from such a vacation, he finds at least one customer in the system;
- combinations of these policies such as the  $(T(SV); N)$ -policy, the  $(T(MV); N)$ -policy, Kella's vacation scheme (Kella [4]), etc.

Using the matrix-analytic methodology, we obtained computational results for the queue length at a post departure or inactive phase termination epoch, at a post-departure epoch and at an arbitrary epoch. We noted that the embedded Markov renewal process at service completion or inactive phase termination epochs is of  $M/G/1$ -type.

In this paper, we consider a special case of the above model in which the server applies Kella's vacation scheme, i.e., a vacation policy where the decision of whether to take a new vacation or not, when the system is empty, depends on the number of vacations already taken in the current inactive phase. The aim of this paper is to obtain computational results for the virtual and the actual waiting time distributions.

Tractable expressions for the stationary queue length and waiting time distributions for the infinite capacity MAP/G/1 queue with multiple vacations are given in Lucantoni, Meier-Hellstern and Neuts [7]. Independent of our work, these results have been recently generalized to the BMAP/G/1 queue with server vacations (Ferrandiz [3], Schellhaas [13]). The particular case of a batch SPP/G/1 queue with multiple vacations is considered in Takine and Hasegawa [15]. We note that the SPP (switched Poisson process) is a particular case of the two-state MMPP. We also note that the finite capacity MAP/G/1 queue with server vacations is considered in Blondia [2].

Section 2 contains the description of the model. In Section 3, we obtain the virtual waiting time distribution using the stationary queue length at service completion or inactive phase termination epochs. These results agree with those in Kella [4] for the  $M/G/1$  queue. The actual waiting time distributions are given in Section 4. Special cases are considered in Section 5. In particular, we show that factorization results and the relationship between the virtual and actual waiting time distributions (observed in Lucantoni, Meier-Hellstern and Neuts [7] and in Takine and Hasegawa [15]), also hold for the BMAP/G/1 queue with multiple vacations. In Section 6, we present a numerical example.

## 2. Model Description

We consider a single server infinite capacity queueing system in which arrivals occur according to a BMAP with  $m$  phases and with coefficient matrices  $\{D_k, k \geq 0\}$ . The nonsingular  $m \times m$  matrix  $D_0$ , with negative diagonal elements, nonnegative off-diagonal elements and row sums less than or equal to zero, governs transitions in the *phase process* that do not generate arrivals, and for  $k \geq 1$ , the nonnegative  $m \times m$  matrices  $D_k$  govern transitions that correspond to arrivals of batches of size  $k$ . We refer to Lucantoni [8] and [9] for more details. We recall that the phase process is assumed to be an irreducible, positive recurrent Markov process on the state space  $\{1, \dots, m\}$  with stationary probability vector  $\underline{\pi}$ . Note that the matrix

$$D = \sum_{k \geq 0} D_k \quad (D \neq D_0)$$

is the generator of this Markov process. So,

$$D\underline{e} = 0, \quad \underline{\pi}D = 0 \quad \text{and} \quad \underline{\pi} \underline{e} = 1,$$

where  $\underline{e}$  is a column vector of 1's. The stationary arrival rate of the process is

$$\lambda^* = \underline{\pi} \sum_{k \geq 1} k D_k \underline{e}.$$

The server applies Kella's vacation scheme (Kella [4]) where the decision of whether to take a new vacation or not, when the system is empty, depends on the number of vacations already taken in the current inactive phase, as follows.

Upon returning from the  $(i - 1)$ st consecutive vacation ( $i \geq 1$ ) in a given inactive phase, the server becomes active immediately if he finds at least one customer waiting. Otherwise, he decides to take another vacation with probability  $\sigma_i$  or remains in the system instead with probability  $q_i = 1 - \sigma_i$ , (note that the  $i = 1$  case corresponds to the possible first vacation once the system becomes empty). In the latter case, the server remains inactive, inspecting the queue, until at least one customer is present. Vacation lengths are assumed to be i.i.d. random variables with distribution function  $U(\cdot)$ , Laplace-Stieltjes transform (L.S.T.)  $\tilde{U}(\cdot)$ , finite expectation  $E[U]$  and second order moment  $E[U^2]$ . Special cases are the  $T(SV)$ -policy (where  $\sigma_1 = 1$  and  $\sigma_i = 0, i \geq 2$ ), the  $T(MV)$ -policy (where  $\sigma_i = 1, i \geq 1$ ) and the ordinary (i.e., without vacations) model (where  $q_1 = 1$ ).

Customers are served in the order of their arrivals, (customers within a batch are preordered for service or served in random order). The service times are i.i.d. random variables, independent of the interarrival times and the vacation durations, with distribution function  $S(\cdot)$ , L.S.T.  $\tilde{S}(\cdot)$ , finite expectation  $E[S]$  and second order moment  $E[S^2]$ .

As in Matendo [11], the traffic intensity,  $\rho \equiv \lambda^* E[S] < 1$  and the sequences of  $m \times m$  matrices  $\{U_k\}_{k \geq 0}$  and  $\{C_k\}_{k \geq 1}$  are the matrix probability densities of the number of arrivals during a vacation and an inactive phase, respectively.

The matrix generating functions

$$\tilde{U}(z, 0) \equiv U^*(z) = \sum_{k \geq 0} U_k z^k \quad \text{and} \quad C(z) = \sum_{k \geq 1} C_k z^k, \quad |z| \leq 1$$

are given by

$$U^*(z) = \int_0^\infty \exp[D(z)t] dU(t), \tag{1}$$

and

$$C(z) = b[U^*(z) - U_0] + a[D(z) - D_0], \tag{2}$$

where

$$a = \sum_{r \geq 0} \left( \prod_{l=1}^r \sigma_l \right) q_{r+1} U_0^r (-D_0)^{-1}, \quad b = \sum_{r \geq 0} \left( \prod_{l=1}^{r+1} \sigma_l \right) U_0^r,$$

and

$$D(z) = \sum_{k \geq 0} D_k z^k.$$

### 3. Virtual Waiting Time Distribution

In this section, we relate the stationary virtual waiting time (i.e., the length of time a customer arriving at an arbitrary time instant would have to wait before entering service) to the steady-state probability vectors  $\{\underline{x}_i, i \geq 0\}$  of the queue length process at service completion or inactive phase termination epochs (see Matendo [11]), using the Markov renewal theory. We omit the details and refer to Neuts [12] and Lucantoni, Meier-Hellstern and Neuts [7] for similar calculations.

Let  $W(x) = \{W_1(x), \dots, W_m(x)\}$ , where  $W_j(x)$  is the steady-state joint probability that at an arbitrary time the arrival process is in phase  $j$  and that a *virtual* customer who arrived at that time would have been waiting at most a length of time  $x$  before entering service. Let  $\tilde{W}(s)$  denote the L.S.T. of  $W(\cdot)$ .

Let

$$\tilde{M}(s) = D(\tilde{S}(s)), \quad \tilde{M}_1(s) = U^*(\tilde{S}(s)), \quad \text{and} \quad \tilde{M}_2(s) = C(\tilde{S}(s)).$$

For  $\nu = 1, 2$ , define

$$\begin{aligned} \tilde{M}^{(\nu)}(0) &= \left. \frac{d^\nu}{ds^\nu} \tilde{M}(s) \right|_{s=0}, & \tilde{M}_1^{(\nu)}(0) &= \left. \frac{d^\nu}{ds^\nu} \tilde{M}_1(s) \right|_{s=0}, \\ \tilde{M}_2^{(\nu)}(0) &= \left. \frac{d^\nu}{ds^\nu} \tilde{M}_2(s) \right|_{s=0}, & D^{(\nu)}(1) &= \left. \frac{d^\nu}{dz^\nu} D(z) \right|_{z=1}, \\ (U^*)^{(\nu)}(1) &= \left. \frac{d^\nu}{dz^\nu} (U^*)(z) \right|_{z=1} & \text{and} & C^{(\nu)}(1) = \left. \frac{d^\nu}{dz^\nu} C(z) \right|_{z=1}. \end{aligned}$$

Then,

$$\begin{aligned} \tilde{M}^{(1)}(0) &= -E[S]D^{(1)}(1), \\ \tilde{M}_1^{(1)}(0) &= -E[S](U^*)^{(1)}(1), \\ \tilde{M}_2^{(1)}(0) &= -E[S]C^{(1)}(1), \\ \tilde{M}^{(2)}(0) &= (E[S])^2 D^{(2)}(1) + E[S^2]D^{(1)}(1), \\ \tilde{M}_1^{(2)}(0) &= (E[S])^2 (U^*)^{(2)}(1) + E[S^2](U^*)^{(1)}(1), \end{aligned}$$

and

$$\tilde{M}_2^{(2)}(0) = (E[S])^2 C^{(2)}(1) + E[S^2]C^{(1)}(1).$$

We also define

$$m_1 = \tilde{M}^{(1)}(0)\underline{e} \quad \text{and} \quad m_2 = \tilde{M}^{(2)}(0)\underline{e}.$$

After some laborious calculations, the joint transform of the virtual waiting time and the phase of the BMAP,  $\tilde{W}(s)$ , appears to be related to the vectors  $\underline{x}_i, i \geq 0$  by:

$$\left[ \tilde{W}(s) - (E^*)^{-1} \underline{x}_0 a \right] \left[ sI + \tilde{M}(s) \right]$$

$$= (E^*)^{-1} \underline{x}_0 \left\{ b \left[ \tilde{M}_1(s) - \tilde{U}(s)I \right] + \left[ I - \tilde{M}_2(s) \right] \right\}, \quad (3)$$

and

$$\tilde{W}(0) = \underline{\pi},$$

where  $E^*$  is the fundamental mean of the embedded Markov renewal process at service completion or inactive phase termination epochs (Matendo [11]).

**Remark 1:** The virtual waiting time distribution is given by  $W(x)\underline{e}$ , with the L.S.T.  $\tilde{W}(s)\underline{e}$ .

**Mean virtual waiting time:** By differentiating in (3), setting  $s = 0$ , noting that  $\tilde{M}(0) = D$  and  $\tilde{W}(0) = \underline{\pi}$ , adding  $\tilde{W}^{(1)}(0)\underline{e} \underline{\pi}$  to both sides, and recalling that  $\underline{\pi}(\underline{e} \underline{\pi} + D)^{-1} = \underline{\pi}$ , the first moment vector  $\tilde{W}^{(1)}(0)$  results in

$$\begin{aligned} \tilde{W}^{(1)}(0) = \tilde{W}^{(1)}(0)\underline{e} \underline{\pi} - \underline{\pi} &- \left\{ \underline{\pi} \tilde{M}^{(1)}(0) - (E^*)^{-1} \underline{x}_0 \left\{ a \left( I + \tilde{M}^{(1)}(0) \right) \right. \right. \\ &\left. \left. + b \left( E[U]I + \tilde{M}_1^{(1)}(0) \right) - \tilde{M}_2^{(1)}(0) \right\} \right\} (\underline{e} \underline{\pi} + D)^{-1}. \end{aligned} \quad (4)$$

Further, by differentiating twice in (3), setting  $s = 0$ , postmultiplying by  $\underline{e}$ , using (4), and noting that  $\underline{\pi} m_1 = -\rho$ , the mean virtual waiting time  $-\tilde{W}^{(1)}(0)\underline{e}$  results in

$$\begin{aligned} -2\tilde{W}^{(1)}(0)\underline{e} (1 - \rho) = -2 \left\{ \underline{\pi} + \left\{ \underline{\pi} \tilde{M}^{(1)}(0) - (E^*)^{-1} \underline{x}_0 \left[ a \left( I + \tilde{M}^{(1)}(0) \right) \right. \right. \right. \\ \left. \left. + b \left( E[U]I + \tilde{M}_1^{(1)}(0) \right) - \tilde{M}_2^{(1)}(0) \right\} \right\} (\underline{e} \underline{\pi} + D)^{-1} \Big\} m_1 \\ + \left[ \underline{\pi} - (E^*)^{-1} \underline{x}_0 a \right] m_2 \\ - (E^*)^{-1} \underline{x}_0 \left\{ b \left[ \tilde{M}_1^{(2)}(0) - E[U^2]I \right] - \tilde{M}_2^{(2)}(0) \right\} \underline{e}. \end{aligned} \quad (5)$$

## 4. Actual Waiting Time Distributions

### 4.1 Actual waiting time distribution of (the first customer in) an arriving batch

Let  $W_{b,j}(x)$  denote the steady-state joint probability that an arriving batch has to wait at most a time  $x$  before entering service, and that immediately after that arrival epoch, the arrival process is in phase  $j$  ( $j = 1, \dots, m$ ).

Let  $\lambda^0 = \underline{\pi}(-D_0\underline{e})$  denote the stationary arrival rate of groups. Then, the L.S.T.  $\tilde{W}_b(s)$  of the row vector  $W_b(x)$  (with components  $W_{b,j}(x)$ ,  $j = 1, \dots, m$ ) is given by

$$\begin{aligned} \tilde{W}_b(s) &= (\lambda^0)^{-1} \tilde{W}(s) \sum_{k \geq 1} D_k \\ &= (\lambda^0)^{-1} \tilde{W}(s) [D - D_0], \end{aligned} \quad (6)$$

where  $\tilde{W}(s)$  is given by (3).

Therefore, the actual waiting time distribution  $W_b(x)\underline{e}$  of an arriving batch is given by

$$\begin{aligned} W_b(x)\underline{e} &= (\lambda^0)^{-1}W(x)\sum_{k \geq 1} D_k \underline{e} \\ &= (\lambda^0)^{-1}W(x)(-D_0 \underline{e}), \end{aligned} \quad (7)$$

with the L.S.T.  $\tilde{W}_b(s)\underline{e}$  given by

$$\tilde{W}_b(s)\underline{e} = (\lambda^0)^{-1}\tilde{W}(s)(-D_0 \underline{e}). \quad (8)$$

From (8), the mean actual waiting time  $-\tilde{W}_b^{(1)}(0)\underline{e}$  of an arriving batch is given by

$$-\tilde{W}_b^{(1)}(0)\underline{e} = (\lambda^0)^{-1}\tilde{W}^{(1)}(0)(D_0 \underline{e}), \quad (9)$$

where  $\tilde{W}^{(1)}(0)$  is given by (4).

#### 4.2 Actual waiting time distribution of an arbitrary customer

Consider an arbitrary tagged customer in an arriving group (i.e., a randomly selected customer of an arriving group). Let  $W_{c,j}(x)$  (with L.S.T.  $\tilde{W}_{c,j}(s)$ ) denote the steady-state joint probability that the tagged customer has to wait at most a time  $x$  before entering service, and that immediately after the arrival epoch, the arrival process is in phase  $j$  ( $j = 1, \dots, m$ ).

It can be shown, after some calculations, that the row-vector  $\tilde{W}_c(s)$  (with components  $\tilde{W}_{c,j}(s)$ ,  $j = 1, \dots, m$ ) is given by

$$\begin{aligned} \tilde{W}_c(s) &= (\lambda^*)^{-1}\tilde{W}(s)\sum_{k \geq 1} D_k \frac{1 - \tilde{S}^k(s)}{1 - \tilde{S}(s)} \\ &= \frac{1}{\lambda^*(1 - \tilde{S}(s))}\tilde{W}(s)[D - D(\tilde{S}(s))]. \end{aligned} \quad (10)$$

Therefore, the L.S.T.  $\tilde{W}_c(s)\underline{e}$  of the waiting time distribution of an arbitrary customer is given by

$$\tilde{W}_c(s)\underline{e} = \frac{1}{\lambda^*(1 - \tilde{S}(s))}\tilde{W}(s)(-D(\tilde{S}(s))\underline{e}). \quad (11)$$

It follows that the corresponding mean waiting time  $-\tilde{W}_c^{(1)}(0)\underline{e}$  is given by

$$-\tilde{W}_c^{(1)}(0)\underline{e} = -(\lambda^*)^{-1}\tilde{W}^{(1)}(0)D^{(1)}(1)\underline{e} + \frac{E[S]\pi D^{(2)}(1)\underline{e}}{2\lambda^*}. \quad (12)$$

**Remark 2:** When the input process is a MAP with parameter matrices  $D_0$  and  $D_1$  and i.i.d. batch sizes, with probability distribution  $\{d_k\}_{k \geq 1}$ , generating function  $d(z)$ , first and second factorial moment  $d^{(1)}$  and  $d^{(2)}$  respectively, then

$$\lambda^* = d^{(1)}\lambda^0 \quad \text{and} \quad D(z) = D_0 + D_1d(z).$$

Thus (11) and (12) reduce to

$$\tilde{W}_c(s)\underline{e} = \tilde{W}_b(s)\underline{e} \frac{1 - d(\tilde{S}(s))}{d^{(1)}(1 - \tilde{S}(s))} \quad (13a)$$

and

$$-\tilde{W}_c^{(1)}(0)\underline{e} = -\tilde{W}_b^{(1)}(0)\underline{e} + E[S]\frac{d^{(2)}}{2d^{(1)}}. \quad (13b)$$

We note that the second factor in (13a) is the L.S.T. of the waiting time distribution of the tagged customer within the batch. Therefore, we obtain the factorization of the waiting time distribution of the tagged customer into the convolution product of the waiting time distribution of the (first customer in the) batch and the waiting time distribution of the tagged customer within the batch.

Observe that for a batch Poisson arrival process with rate  $\lambda$ , we have that  $D_0 = -\lambda$ ,  $D_k = \lambda d_k$ ,  $k \geq 1$ ,  $\lambda^0 = \lambda$ ,  $\tilde{M}^{(1)}(0) = -\rho$  and  $D(z) = -\lambda + \lambda d(z)$ .

In the case of single arrivals,  $\lambda^* = \lambda$  and  $D(z) = -\lambda + \lambda z$ . Note that (see Matendo [10] and [11]) in the light of

$$C(z) = b[U^*(z) - U_0] + [1 - (1 - U_0)b]z$$

$$\left( \text{so that } C^{(1)}(1) = 1 - (1 - U_0)b + \lambda b E[U] \text{ and } C^{(2)}(1) = \lambda^2 b E[U^2] \right)$$

and

$$(E^*)^{-1} x_0 = \frac{\lambda(1 - \rho)}{C^{(1)}(1)},$$

(13a) and (13b) become

$$\begin{aligned} \tilde{W}_c(s) &= \tilde{W}_b(s) \\ &= \tilde{W}(s) \\ &= \frac{(1 - \rho)s}{[s - \lambda + \lambda \tilde{S}(s)]} \frac{\lambda b[1 - \tilde{U}(s)] + s[1 - (1 - U_0)b]}{s C^{(1)}(1)} \end{aligned}$$

and

$$\begin{aligned} -\tilde{W}_c^{(1)}(0) &= -\tilde{W}_b^{(1)}(0) \\ &= -\tilde{W}^{(1)}(0) \\ &= \frac{\lambda E[S^2]}{2(1 - \rho)} + \frac{\lambda b E[U^2]}{2C^{(1)}(1)}, \end{aligned}$$

which agree with the results obtained by Kella [4].

## 5. Some Special Cases

In this section, we particularize the results to the BMAP/G/1 queue with a single vacation and the BMAP/G/1 queue with multiple vacations, respectively. In the notations relative to the waiting times, the subscripts  $sv$ ,  $mv$  and  $nv$  will refer to the queue with single vacation, multiple vacations and without vacations, respectively.

### 5.1 BMAP/G/1 queue with a single vacation

Let  $\sigma_1 = 1$  and  $\sigma_i = 0$ ,  $i \geq 2$ . Then  $a = U_0 (-D_0)^{-1}$  and  $b = I$ .

Using (2), we have

$$\tilde{M}_2(s) = \tilde{M}_1(s) + U_0(-D_0)^{-1}\tilde{M}(s), \quad (14a)$$

so that

$$\tilde{M}_2^{(\nu)}(0) = \tilde{M}_1^{(\nu)}(0) + U_0(-D_0)^{-1}\tilde{M}^{(\nu)}(0), \quad \nu = 1, 2. \quad (14b)$$

Substituting (14a) into (3), we obtain

$$\tilde{W}_{sv}(s)[sI + D(\tilde{S}(s))] = (E^*)^{-1}\underline{x}_0[I - \tilde{U}(s)I + sU_0(-D_0)^{-1}]. \quad (15)$$

Moreover, substituting (14b) into (4) and (5) respectively, we get

$$\tilde{W}_{sv}^{(1)}(0) = \tilde{W}_{sv}^{(1)}(0)\underline{e} \quad \underline{\pi} - \underline{\pi}$$

$$-\left\{ \underline{\pi} \tilde{M}^{(1)}(0) - (E^*)^{-1}\underline{x}_0[U_0(-D_0)^{-1} + E[U]I] \right\} (\underline{e} \quad \underline{\pi} + D)^{-1} \quad (16)$$

and

$$\begin{aligned} & -2\tilde{W}_{sv}^{(1)}(0)\underline{e}(1-\rho) \\ & = 2\left\{ \rho - \left\{ \underline{\pi} \tilde{M}^{(1)}(0) - (E^*)^{-1}\underline{x}_0[U_0(-D_0)^{-1} + E[U]I] \right\} (\underline{e} \quad \underline{\pi} + D)^{-1} m_1 \right\} \\ & \quad + \underline{\pi} m_2 + (E^*)^{-1}\underline{x}_0 \underline{e} E[U^2]. \end{aligned} \quad (17)$$

Further, using (15), expressions (6) and (10) are reduced to

$$\begin{aligned} \tilde{W}_{sv,b}(s) &= (\lambda^\circ)^{-1}(E^*)^{-1}\underline{x}_0[I - \tilde{U}(s)I + sU_0(-D_0)^{-1}] \\ & \quad [sI + D(\tilde{S}(s))]^{-1} [D - D_0], \end{aligned} \quad (18)$$

and

$$\begin{aligned} \tilde{W}_{sv,c}(s) &= \frac{1}{\lambda^*(1 - \tilde{S}(s))} \tilde{W}_{sv}(s) [D - D(\tilde{S}(s))] \\ &= \frac{1}{\lambda^*(1 - \tilde{S}(s))} \left\{ \tilde{W}_{sv}(s)[sI + D] - (E^*)^{-1}\underline{x}_0[I - \tilde{U}(s)I + sU_0(-D_0)^{-1}] \right\}. \end{aligned} \quad (19)$$

It follows that

$$\begin{aligned} \tilde{W}_{sv,b}(s)\underline{e} &= (\lambda^\circ)^{-1}(E^*)^{-1}\underline{x}_0[I - \tilde{U}(s)I + sU_0(-D_0)^{-1}] \\ & \quad [sI + D(\tilde{S}(s))]^{-1} (-D_0\underline{e}), \end{aligned} \quad (20)$$

and

$$\begin{aligned} \tilde{W}_{sv,c}(s)\underline{e} &= \frac{1}{\lambda^*(\tilde{S}(s) - 1)} \tilde{W}_{sv}(s) (-D(\tilde{S}(s))\underline{e}) \\ &= \frac{1}{\lambda^*(1 - \tilde{S}(s))} \left\{ s\tilde{W}_{sv}(s)\underline{e} - (E^*)^{-1}\underline{x}_0[I - \tilde{U}(s)I + sU_0(-D_0)^{-1}]\underline{e} \right\}. \end{aligned} \quad (21a)$$

From the second expression in (21a), we obtain the mean waiting time of an arbitrary customer as

$$-\tilde{W}_{sv,c}^{(1)}(0)\underline{e} = -\frac{\tilde{W}_{sv}^{(1)}(0)\underline{e}}{\rho} - (E^*)^{-1}\underline{x}_0 \underline{e} \frac{E[U^2]}{2\rho} - \frac{E[S^2]}{2E[S]}. \quad (21b)$$

**Remark 3:** (a) For the BMAP/G/1 queue without vacations (i.e.,  $U_0 = I$ ,  $\tilde{U}(s) = 1$ ), it can be easily shown (see Matendo [11]) that

$$(E^*)^{-1}\underline{x}_0(-D_0)^{-1} = (1-\rho)\underline{g},$$

where  $\underline{g}$  is the invariant probability measure of a transition probability matrix - usually denoted by  $G$  - which plays a key role in the analysis of Markov chains of  $M/G/1$  type.

It follows immediately from (15), (16), (17), (18) and (19) that

$$\tilde{W}_{nv}(s) = s(1-\rho)\underline{g}[sI + D(\tilde{S}(s))]^{-1}, \quad (22)$$

$$\tilde{W}_{nv}^{(1)}(0) = \tilde{W}_{nv}^{(1)}(0)\underline{e} \underline{\pi} - \underline{\pi} + [(1-\rho)\underline{g} - \underline{\pi} \tilde{M}^{(1)}(0)](\underline{e} \underline{\pi} + D)^{-1}, \quad (23)$$

$$-2\tilde{W}_{nv}^{(1)}(0)\underline{e}(1-\rho) = 2\left\{\rho + [(1-\rho)\underline{g} - \underline{\pi} \tilde{M}^{(1)}(0)](\underline{e} \underline{\pi} + D)^{-1}m_1\right\} + \underline{\pi} m_2, \quad (24)$$

$$\tilde{W}_{nv,b}(s) = (\lambda^o)^{-1}s(1-\rho)\underline{g}[sI + D(\tilde{S}(s))]^{-1}(D - D_0), \quad (25)$$

and

$$\begin{aligned} \tilde{W}_{nv,c}(s) &= \frac{1}{\lambda^*(1-\tilde{S}(s))} \tilde{W}_{nv}(s)[D - D(\tilde{S}(s))] \\ &= \frac{1}{\lambda^*(1-\tilde{S}(s))} \left\{ \tilde{W}_{nv}(s)[sI + D] - s(1-\rho)\underline{g} \right\}, \end{aligned} \quad (26)$$

which agree with the results obtained by Lucantoni [8], [9].<sup>1</sup>

From (25) and (26), we readily obtain that

$$\tilde{W}_{nv,b}(s)\underline{e} = (\lambda^o)^{-1}s(1-\rho)\underline{g}[sI + D(\tilde{S}(s))]^{-1}(-D_0\underline{e}), \quad (27)$$

and

$$\begin{aligned} \tilde{W}_{nv,c}(s)\underline{e} &= \frac{1}{\lambda^*(1-\tilde{S}(s))} \tilde{W}_{nv}(s)(-D(\tilde{S}(s))\underline{e}) \\ &= \frac{s[\tilde{W}_{nv}(s)\underline{e} - (1-\rho)]}{\lambda^*(1-\tilde{S}(s))}. \end{aligned} \quad (28a)$$

Moreover, from (21b), or from the second expression in (28a), we get

$$-\tilde{W}_{nv,c}^{(1)}(0)\underline{e} = -\frac{\tilde{W}_{nv}^{(1)}(0)\underline{e}}{\rho} - \frac{E[S^2]}{2E[S]}. \quad (28b)$$

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<sup>1</sup>Except that in the preprint of Lucantoni [9] in our possession, the denominator of the second expression in (26) is  $\lambda^*(\tilde{S}(s) - 1)$  instead of  $\lambda^*(1 - \tilde{S}(s))$ .

(b) In the particular case of a batch Poisson arrival process with rate  $\lambda$ , we have (see Matendo [10])

$$(E^*)^{-1}x_0 = \frac{\lambda(1-\rho)}{\lambda E[U] + U_0}. \quad (29)$$

Substituting (29) into (15), (20) and (21a) yield

$$\tilde{W}_{sv}(s) = \tilde{W}_{sv,b}(s) = \tilde{W}_{nv}(s) \frac{\lambda - \lambda \tilde{U}(s) + sU_0}{s[U_0 + \lambda E[U]]}, \quad (30a)$$

$$\tilde{W}_{sv,c}(s) = \tilde{W}_{nv,c}(s) \frac{\lambda - \lambda \tilde{U}(s) + sU_0}{s[U_0 + \lambda E[U]]}, \quad (30b)$$

where

$$\tilde{W}_{nv}(s) = \frac{(1-\rho)s}{s - \lambda + \lambda d(\tilde{S}(s))}, \quad (31)$$

and

$$\tilde{W}_{nv,c}(s) = \tilde{W}_{nv}(s) \frac{1 - d(\tilde{S}(s))}{d^{(1)}(1 - \tilde{S}(s))} = \frac{s[\tilde{W}_{nv}(s) - (1-\rho)]}{\lambda d^{(1)}(1 - \tilde{S}(s))}. \quad (32)$$

It follows that

$$-\tilde{W}_{sv}^{(1)}(0) = -\tilde{W}_{nv}^{(1)}(0) + \frac{\lambda E[U^2]}{2[U_0 + \lambda E[U]]} \quad (33)$$

and

$$-\tilde{W}_{sv,c}^{(1)}(0) = -\tilde{W}_{nv,c}^{(1)}(0) + \frac{\lambda E[U^2]}{2[U_0 + \lambda E[U]]}, \quad (34)$$

where

$$-\tilde{W}_{nv}^{(1)}(0) = \frac{\rho E[S]d^{(2)} + \lambda E[S^2](d^{(1)})^2}{2d^{(1)}(1-\rho)}, \quad (35a)$$

and

$$-\tilde{W}_{nv,c}^{(1)}(0) = \frac{E[S]d^{(2)} + \lambda E[S^2](d^{(1)})^2}{2d^{(1)}(1-\rho)} = -\frac{\tilde{W}_{nv}^{(1)}(0)}{\rho} - \frac{E[S^2]}{2E[S]}. \quad (35b)$$

Observe that relations (30b) and (34) agree with the results obtained by Takagi ([14], pp. 143-144, relations (3.22a) and (3.22b)).

## 5.2 BMAP/G/1 queue with multiple vacations

Let  $\sigma_i = 1$ ,  $i \geq 1$ . Then  $a = 0$ ,  $b = (I - U_0)^{-1}$  and from (2) we have

$$\tilde{M}_2(s) = (I - U_0)^{-1}[\tilde{M}_1(s) - U_0] \quad (36a)$$

so that

$$\tilde{M}_2^{(\nu)}(0) = (I - U_0)^{-1}\tilde{M}_1^{(\nu)}(0), \quad \nu = 1, 2. \quad (36b)$$

Substituting (36a) and (36b) into (3) and (4), and noting that (Matendo [11])

$$(E^*)^{-1} \underline{x}_0 = \frac{(1-\rho)g(I-U_0)}{E[U]}, \quad (36c)$$

we have

$$\tilde{W}_{mv}(s) = \frac{1-\tilde{U}(s)}{sE[U]} \tilde{W}_{nv}(s), \quad (37)$$

and

$$\tilde{W}_{mv}^{(1)}(0) = \tilde{W}_{mv}^{(1)}(0) \underline{e} \pi - \pi + \left[ (1-\rho)g - \pi \tilde{M}^{(1)}(0) \right] (\underline{e} \pi + D)^{-1}. \quad (38)$$

It follows from (37), (6) and (10) that

$$\tilde{W}_{mv,b}(s) = \frac{1-\tilde{U}(s)}{sE[U]} \tilde{W}_{nv,b}(s), \quad (39)$$

and

$$\tilde{W}_{mv,c}(s) = \frac{1-\tilde{U}(s)}{sE[U]} \tilde{W}_{nv,c}(s). \quad (40)$$

From (37), (39) and (40), we obtain

$$\tilde{W}_{mv}(s) \underline{e} = \frac{1-\tilde{U}(s)}{sE[U]} \tilde{W}_{nv}(s) \underline{e}, \quad (41)$$

$$\tilde{W}_{mv,b}(s) \underline{e} = \frac{1-\tilde{U}(s)}{sE[U]} \tilde{W}_{nv,b}(s) \underline{e}, \quad (42)$$

and

$$\tilde{W}_{mv,c}(s) \underline{e} = \frac{1-\tilde{U}(s)}{sE[U]} \tilde{W}_{nv,c}(s) \underline{e}. \quad (43)$$

Therefore, the various mean waiting times are given by

$$-\tilde{W}_{mv}^{(1)}(0) \underline{e} = -\tilde{W}_{nv}^{(1)}(0) \underline{e} + \frac{E[U^2]}{2E[U]}, \quad (44)$$

$$-\tilde{W}_{mv,b}^{(1)}(0) \underline{e} = -\tilde{W}_{nv,b}^{(1)}(0) \underline{e} + \frac{E[U^2]}{2E[U]}, \quad (45)$$

and

$$-\tilde{W}_{mv,c}^{(1)}(0) \underline{e} = -\tilde{W}_{nv,c}^{(1)}(0) \underline{e} + \frac{E[U^2]}{2E[U]}. \quad (46)$$

**Remark 4:** (a) Observe that Lucantoni, Meier-Hellstern and Neuts [7] and Takine and Hasegawa [15] obtained some factorization or decomposition results for the MAP/G/1 queue with multiple vacations and for the batch SPP/G/1 queue with multiple vacations, respectively. They showed that the actual (virtual) waiting time distribution is the convolution of the residual vacation time and the actual (virtual) waiting time distributions in the corresponding model without vacations. Therefore, (37) and (39)-(43) extend these factorization results to the case of BMAP input. We also mention that the relationship

$$\tilde{W}_{mv}(s) \underline{e} = \rho \frac{1-\tilde{S}(s)}{sE[S]} \tilde{W}_{mv,c}(s) \underline{e} + (1-\rho) \frac{1-\tilde{U}(s)}{sE[U]}, \quad (47)$$

between the virtual and the actual waiting time distributions, established in those papers, also holds for the BMAP. This follows from (11), (22) and (37).

(b) In the particular case of a batch Poisson arrival process with rate  $\lambda$ , (43) and (46) reduce to the results obtained by Baba [1].

## 6. A Numerical Example

We assume that the input stream is a two-state MMPP (Markov modulated Poisson process) with i.i.d. batch arrivals. The infinitesimal generator and the arrival rate matrix of the MMPP are given by

$$D = \begin{pmatrix} -1.0 & 1.0 \\ 2.0 & -2.0 \end{pmatrix} \text{ and } \Lambda = \text{diag}(4.0, 1.0).$$

The batch size distribution is geometric with parameter 0.4. Thus  $\lambda^* = 7.5$  and  $\lambda^o = 3.0$ . The service time distribution is phase-type with representation  $-\text{diag}(6.0, 10.0)$  and  $(0.25, 0.75)$  (ie., a hyperexponential distribution). This yields the traffic intensity  $\rho$  of 0.875. The vacation length is exponential with parameter  $\mu$ . For this problem, the various mean waiting times, for the ordinary model, the  $T(SV)$ -model and the  $T(MV)$ -model, are given in the appendix.

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**APPENDIX: Mean waiting times**

<b>Ordinary model</b>		
virtual	first customer in a batch	arbitrary customer
2.91343	2.97249	3.20582

$\mu$	<b>T(SV)-model</b>			<b>T(MV)-model</b>		
	virtual	first customer in a batch	arbitrary customer	virtual	first customer in a batch	arbitrary customer
0.1	12.8982	12.9576	13.1909	12.91343	12.97249	13.20582
0.5	4.85093	4.911	5.14433	4.91343	4.97249	5.20582
1	3.81585	3.87633	4.10966	3.91343	3.97249	4.20582
2	3.29055	3.35111	3.58444	3.41343	3.47249	3.70582
10	2.94026	2.99968	3.23302	3.01343	3.07249	3.30582
50	2.91461	2.97369	3.20703	2.93343	2.99249	3.22582
100	2.91372	2.97279	3.20613	2.92343	2.98249	3.21582
1000	2.91343	2.97249	3.20582	2.91443	2.97349	3.20682
$\infty$	2.91343	2.97249	3.20582	2.91343	2.97249	3.20582