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# EXISTENCE OF SOLUTIONS FOR SECOND-ORDER EVOLUTION INCLUSIONS

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**ABSTRACT**

In this paper we examine second-order nonlinear evolution inclusions and prove two existence theorems; one with a convex-valued orientor field and the other with a nonconvex-valued field. An example of a hyperbolic partial differential inclusion is also presented.

**Key words:** Evolution Triple, Monotone Operator, Hemicontinuous Operator, Symmetric Operator, Fixed Point, Sobolev Space, Program, Average Turnpike Property, Separation Theorem.

**AMS subject classification:** 34G20.

## 1. Introduction

In this paper we study the existence of solutions for second order nonlinear evolution inclusions. Our work here complements the existence results of [7], where we considered first order nonlinear evolution inclusions. We present two existence results. One in which the multivalued term (orientor field) is convex valued and the other with a nonconvex valued orientor field. At the end of the paper, we work in detail an example of a hyperbolic partial differential inclusion, illustrating the applicability of our result.

## 2. Mathematical Preliminaries

Let $T = [0,r]$ and $Y$ a separable Banach space. Throughout this paper we will be using the following notation: $P_{f(c)}(Y) = \{A \subseteq Y\text{: nonempty, closed (and convex)}\}$. A multifunction (set-valued function), $F: T \rightarrow P_f(Y)$ is said to be measurable if for all $x \in Y$, the $\mathbb{R}_+$-valued function $t \rightarrow d(x, F(t)) = inf\{ \| x-y \| : y \in F(t)\}$ is measurable. By $S_F^p (1 \leq p \leq \infty)$, we will denote the set of selectors of $F(\cdot)$ that belong to the Lebesgue-Bochner space $L^p(Y)$; i.e. $S_F^p = \{f \in L^p(Y)\text{:}\ f(t) \in F(t)\ a.e.\}$. It is easy to check using Aumann's selection theorem (see for example Wagner [8], theorem 5.10), that $S_F^p$ is nonempty if and only if the $\mathbb{R}_+$-valued function $t \rightarrow inf\{ \| x \| : x \in F(t)\}$ belongs to $L_+^p$.

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Let $H$ be a separable Banach space and $X$ a dense subspace of $H$, carrying the structure of a separable, reflexive Banach space, which embeds in $H$ continuously. Identifying $H$ with its dual (pivot space), we have $X \rightarrow H \rightarrow X^*$, with all embeddings being continuous and dense. Such a triple of spaces is known in the literature as "evolution triple" (or "Gelfand triple" or "spaces in normal position"). We will also assume that the above embeddings are compact, a condition that is very often satisfied in applications. By $\|\cdot\|$ (resp. $|\cdot|$, $\|\cdot\|_*$), we will denote the norm of $X$ (resp. of $H, X^*$). Also by $\langle\cdot,\cdot\rangle$ we will denote the duality brackets for the pair $(X, X^*)$ and by $(\cdot,\cdot)$ the inner product of $H$. The two are compatible in the sense that $\langle\cdot,\cdot\rangle|_{X\times H} = (\cdot,\cdot)$. To have a concrete example in mind let $Z \subseteq \mathbb{R}^N$ be a bounded domain, $X = W_0^{m,p}(Z)$, $H = L^2(Z)$ and $X^* = W_0^{m,p}(Z)^* = W^{-m,q}(Z)$, $2 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. From the well-known Sobolev's embedding theorem we know that $(X, H, X^*)$ is an evolution triple and furthermore all embeddings are compact. Let $W(T) = \{x \in L^2(X): \dot{x} \in L^2(X^*)\}$. The derivative in this definition is taken in the sense of vector valued distributions. Equipped with the norm $\|x\|_{W(T)} = [\|x\|^2_{L^2(X)} + \|\dot{x}\|_{L^2(X^*)}]^{1/2}$, $W(T)$ becomes a separable reflexive Banach space. Furthermore if $X$ is a Hilbert space, then $W(T)$ is too, with inner product $(x,y)_{W(T)} = (x,y)_{L^2(X)} + (\dot{x},\dot{y})_{L^2(X^*)}$, $x, y \in W(T)$. Note that the elements in $W(T)$ are up to a Lebesgue-null subset of $T$, equal to an $X^*$-valued absolutely continuous function, and, therefore the derivative $\dot{x}(\cdot)$, is also the strong derivative of the function $x: T \rightarrow X^*$. Also, it is well-known that $W(T)$ embeds continuously into $C(T, H)$. Thus, every equivalence class in $W(T)$, has a unique representative in $C(T, H)$. Furthermore, since we have assumed that $X \rightarrow H$ compactly, we have that $W(T) \rightarrow L^2(H)$ compactly. Recently, Nagy [3] proved that if $X$ is a Hilbert space too, then $W(T) \rightarrow C(T, H)$ compactly. For further details on evolution triples and the abstract Sobolev space $W(T)$ we refer to the book of Zeidler [9] and, in particular, chapter 23.

Let $Z$ and $V$ be Hausdorff topological spaces. A multifunction $G: Z \rightarrow 2^V \setminus \{\emptyset\}$ is said to be *upper semicontinuous* (*u.s.c.*) (resp. *lower semicontinuous* (*l.s.c.*)), if for every open set $U \subseteq V$, the set $G^+(U) = \{z \in Z: G(z) \subseteq U\}$ (resp. the set $G^-(U) = \{z \in Z: G(z) \cap U \neq \emptyset\}$) is open in $Z$. Other equivalent definitions and further properties of such multifunctions can be found in the book of Klein-Thompson [2].

## 3. Existence Theorems

Let $T = [0, r]$ and $(X, H, X^*)$ be an evolution triple of spaces with all embeddings assumed to be compact. We will be considering the following second order nonlinear evolution inclusion:

$$\left\{ \begin{array}{c} \ddot{x}(t) + A(t, \dot{x}(t)) + Bx(t) \in F(t, x(t)) \ a.e. \\ x(0) = x_0 \in X, \ \dot{x}(0) = x_1 \in H. \end{array} \right\} \qquad (*)$$

By a solution of $(*)$, we understand a function $x \in C(T, X)$ such that $\dot{x} \in W(T)$ and an $f \in S^2_{F(\cdot, x(\cdot))}$ such that $\ddot{x}(t) + A(t, \dot{x}(t)) + Bx(t) = f(t)$ *a.e.* with $x(0) = x_0$ and $\dot{x}(0) = x_1$. Recall (see Section 2), that $W(T) \rightarrow C(T, H)$ and so the initial condition $\dot{x}(0) = x_1 \in H$ makes sense.

First we prove an existence theorem for $(*)$, for the case where the multivalued perturbation term $F(t, x)$ is convex-valued. To this end, we will need the following hypotheses on the data of $(*)$.

$\underline{H(A)}$: $A: T \times X \to X^*$ is a map such that
($\underline{1}$) $t \to A(t,v)$ is measurable,
($\underline{2}$) $v \to A(t,v)$ is monotone, hemicontinuous (i.e. for all $v$, $v' \in X$, $\langle A(t,v) - A(t,v'), v - v' \rangle \geq 0$ (monotonicity) and for all vectors $v$, $y, x \in X$, the map $\lambda \to \langle A(t, v+\lambda y), x\rangle$ is continuous on $[0,1]$ (demicontinuity)),
($\underline{3}$) $\langle A(t,v), v\rangle \geq c \| v \|^2$ *a.e.* with $c > 0$,
($\underline{4}$) $\| A(t,v) \|_* \leq a(t) + b \| v \|$ *a.e.* with $a(\cdot) \in L^2_+$, $b > 0$.

$\underline{H(B)}$: $B \in \mathcal{L}(X, X^*)$, $\langle Bx, y\rangle = \langle x, By\rangle$ for all $x, y \in X$ (i.e. $B$ is symmetric) and $\langle Bx, x\rangle \geq c' \| x \|^2$ with $c' > 0$.

$\underline{H(F)_1}$: $F: T \times H \to P_{fc}(H)$ is a multifunction such that
($\underline{1}$) $t \to F(t,x)$ is measurable,
($\underline{2}$) $x \to F(t,x)$ is *u.s.c.* from $H$ into $H_w$,
($\underline{3}$) $| F(t,x) | = sup\{ | v | : v \in F(t,x)\} \leq a_1(t) + b_1 | x |$ *a.e.* with $a_1(\cdot) \in L^2_+$, $b_1 > 0$.

We will denote the solution set of $(*)$ by $S(x_0, x_1) \subseteq C(T, X)$.

**Theorem 3.1:** *$\underline{\text{If}}$ hypotheses $H(A)$, $H(B)$, $H(F)_1$ hold and $x_0 \in X$, $x_1 \in H$, $\underline{\text{then}}$ $S(x_0, x_1)$ is a nonempty and compact subset of $C(T, X)$.*

**Proof:** First we will derive some *a priori* bounds for the solutions of $(*)$. Let $x(\cdot) \in C(T,X)$ be such a solution. Then, by the definition, for some $f \in S^2_{F(\cdot, x(\cdot))}$, we have

$$\ddot{x}(t) + A(t, \dot{x}(t)) + B(x(t)) = f(t) \ a.e.$$

it yields
$$\langle \ddot{x}(t), \dot{x}(t)\rangle + \langle A(t, \dot{x}(t)), \dot{x}(t)\rangle + \langle Bx(t), \dot{x}(t)\rangle = (f(t), \dot{x}(t)) \ a.e. \tag{1}$$

Since $\dot{x} \in W(T)$, from proposition 23.23 (*iv*), p. 422 of Zeidler [9], we know that

$$\langle \ddot{x}(t), \dot{x}(t)\rangle = \frac{1}{2}\frac{d}{dt} | \dot{x}(t) |^2. \tag{2}$$

Also because of hypothesis $H(A)$ ($\underline{3}$), we have that

$$\langle A(t, \dot{x}(t)), \dot{x}(t)\rangle \geq c \| \dot{x}(t) \|^2 \ a.e. \tag{3}$$

Using the product rule and the symmetry hypothesis on $B$, we get

$$\begin{aligned} \frac{d}{dt}\langle Bx(t), x(t)\rangle &= \langle B\dot{x}(t), x(t)\rangle + \langle Bx(t), \dot{x}(t)\rangle \\ &= 2\langle Bx(t), \dot{x}(t)\rangle. \end{aligned} \tag{4}$$

Substituting (2), (3) and (4) into (1) above, we finally have

$$\frac{1}{2}\frac{d}{dt} | \dot{x}(t) |^2 + c \| \dot{x}(t) \|^2 + \frac{1}{2}\frac{d}{dt}\langle Bx(t), x(t)\rangle \leq (f(t), \dot{x}(t)) \ a.e.$$

Integrating the above inequality, we get that

$$\frac{1}{2} | \dot{x}(t) |^2 - \frac{1}{2} | x_1 |^2 + c\int_0^t \| \dot{x}(s) \|^2 ds + \frac{1}{2}\langle Bx(t), x(t)\rangle - \frac{1}{2}\langle Bx_0, x_0\rangle \leq \int_0^t (f(s), \dot{x}(s)) ds$$

it yields

$$| \dot{x}(t) |^2 + 2c \int_0^t \| \dot{x}(s) \|^2 ds + c' \| x(t) \|^2 \leq M + 2 \int_0^t (f(s), \dot{x}(s)) ds \tag{5}$$

where $M = | x_1 |^2 + \| B \|_{\mathcal{L}} \| x_0 \|^2$.

Applying Cauchy's inequality with $\epsilon > 0$, we get

$$\int_0^t (f(s), \dot{x}(s)) ds \leq \int_0^t | f(s) | \cdot | \dot{x}(s) | ds$$

$$\leq \frac{\epsilon}{2} \int_0^t | f(s) |^2 ds + \frac{1}{2\epsilon} \int_0^t | \dot{x}(s) |^2 ds$$

$$\leq \frac{\epsilon}{2} \int_0^t (2a_1(s)^2 + 2b_1^2 | x(s) |^2) ds + \frac{1}{2\epsilon} \int_0^t | \dot{x}(s) |^2 ds$$

$$\leq \epsilon \int_0^t (a_1(s)^2 + b_1^2 | x(s) |^2) ds + \frac{1}{2\epsilon} \int_0^t \beta^2 \| \dot{x}(s) \|^2 ds$$

where $\beta > 0$ is such that $| \cdot | \leq \beta \| \cdot \|$. It exists since by hypothesis $X \rightarrow H$ continuously. So, we have

$$| \dot{x}(t) |^2 + 2c \int_0^t \| \dot{x}(s) \|^2 ds + c' \| x(t) \|^2$$

$$\leq M + \epsilon \| a_1 \|_2^2 + \epsilon b_1^2 \int_0^t | x(s) |^2 ds + \frac{\beta^2}{2\epsilon} \int_0^t \| \dot{x}(s) \|^2 ds.$$

Let $\frac{\beta^2}{2\epsilon} = 2c$ implies that $\epsilon = \frac{\beta^2}{4c}$. Then we have:

$$| \dot{x}(t) |^2 + \frac{c'}{\beta^2} | x(t) |^2 \leq M + \frac{\beta^2}{4c} \| a_1 \|_2^2 + \frac{\beta^2}{4c} b_1^2 \int_0^t | x(s) |^2 ds. \tag{$*$}$$

From $(*)$ by neglecting $| \dot{x}(t) |^2$ and using Gronwall's inequality, we get

$$| x(t) |^2 \leq \left( \frac{\beta^2}{c'} M + \frac{\beta^4}{4cc'} \| a_1 \|_2^2 \right) exp\left( \frac{\beta^2 b_1^2}{4cc'} r \right) = M_2^2, \ t \in T. \tag{6}$$

Using (6) and neglecting $\frac{c'}{\beta^2} | x(t) |^2$ in $(*)$, we obtain

$$|\dot{x}(t)|^2 \leq M + \frac{\beta^2}{4c} \| a_1 \|_2^2 + \frac{\beta^2}{4c} b_1^2 M_2^2 r = M_1^2,\ t \in T. \tag{7}$$

Coming back to (5) and using estimates (6) and (7) above, we get

$$\| \dot{x} \|_{L^2(X)} \leq \frac{1}{2c}(M + 2\| a_1 \|_2^2 + M_2^2 r + M_1^2 r) = M_3^2. \tag{8}$$

Finally, from (5) and (8), we deduce that

$$\| x(t) \|^2 \leq \frac{1}{c'}(M + 2\| a_1 \|_2^2 + 2b_1^2 M_2^2 r + M_1^2 r) = M_4^2. \tag{9}$$

Finally, let $p \in L^2(X)$ and denote by $((\cdot,\cdot))_0$ the duality brackets for the pair $(L^2(X), L^2(X^*) = L^2(X)^*)$. Also let $\widehat{A}: L^2(X) \to L^2(X^*)$ be the Nemitsky operator corresponding to the map $A(t,x)$; i.e. $(\widehat{A}x)(t) = A(t,x(t))$. Then we have:

$$((\ddot{x}, p))_0 \leq |((\widehat{A}(\dot{x}), p))_0| + |((Bx, p))_0| + ((f,p))_0$$
$$\leq [\| \widehat{A}(x) \|_{L^2(X^*)} + \| Bx \|_{L^2(X^*)} + \| f \|_{L^2(X^*)}] \| p \|_{L^2(X)}$$
$$\leq [\| a \|_2 + bM_3 + \| B \|_{\mathcal{L}} M_r r^{1/2} + \beta' \| a_1 \|_2 + \beta' b_1 M_2 r^{1/2}] \| p \|_{L^2(X)},$$

where $\beta' > 0$ is such that $\| \cdot \|_* \leq \beta' | \cdot |$. It exists since $H \to X^*$ continuously. Since $p \in L^2(X)$ was arbitrary, we deduce that there exists $M_5 > 0$ such that for all $x \in S(x_0, x_1)$, we have

$$\| \ddot{x} \|_{L^2(X^*)} \leq M_5. \tag{10}$$

From (8) and (10) above, we deduce that the set

$$S'(x_0, x_0) = \{\dot{x} \in W(T): x \in S(x_0, x_1)\}$$

is bounded, hence relatively weakly compact in $W(T)$.

Now introduce the following modification of the original orientor field $F(t,x)$:

$$\widehat{F}(t,x) = \begin{cases} F(t,x) & \text{if } |x| \leq M_2 \\ F(t, \frac{M_2 x}{|x|}) & \text{if } |x| > M_2. \end{cases}$$

Observe that $\widehat{F}(t,x) = F(t, p_{M_2}(x))$, where $p_{M_2}(\cdot)$ is the $M_2$-radial retraction in $H$. Since $p_{M_2}(\cdot)$ is Lipschitz continuous, we have, using hypothesis $H(F)_1$, that $t \to \widehat{F}(t,x)$ is measurable while $x \to \widehat{F}(t,x)$ is *u.s.c.* from $H$ into $H_w$. Furthermore, note that $|\widehat{F}(t,x)| \leq a(t) + bM_2 = \phi(t)$ *a.e.*, with $\phi(\cdot) \in L_+^2$. Let $K = \{h \in L^2(H): h(t)| \leq \phi(t) a.e.\}$. This set, endowed with the relative weak $L^2(H)$-topology, is compactly metrizable. In what follows, this will be the topology considered on $K$. Let $\gamma: K \to C(T,X)$ be the map which to each $h \in K$, assigns the unique solution of the initial value problem $\ddot{x}(t) + A(t, \dot{x}(t)) + Bx(t) = h(t)$, $x(0) = x_0$, $\dot{x}(0) = x_1$ (see Zeidler [9], theorem 33.A, p. 224). We claim that $\gamma(\cdot)$ is continuous. To this end, let $h_n \to h$ in $K$ and let $x_n = \gamma(h_n)$. Recall that $\{\dot{x}_n\}_{n \geq 1} \subseteq W(T)$ is relatively weakly compact. Hence, by passing to a subsequence if necessary, we may assume that $\dot{x}_n \xrightarrow{w} y$ in $W(T)$. Let $x = \gamma(h)$. We need to show that $y = \widehat{x}$. We have:

$$\langle \ddot{x}_n(t) - \ddot{x}(t), \dot{x}_n(t) - \dot{x}(t)\rangle + \langle A(t, \dot{x}_n(t)) - A(t, \dot{x}(t)), \dot{x}_n(t) - \dot{x}(t)\rangle$$
$$+ \langle Bx_n(t) - Bx(t), \dot{x}_n(t) - \dot{x}(t)\rangle$$
$$= (h_n(t) - h(t), \dot{x}_n(t) - \dot{x}(t)) \ a.e.$$

Exploiting the fact that $A(t, \cdot)$ is monotone and using the integration by parts formula for functions in $W(T)$ (see Zeidler [9], proposition 23.23, p. 422), we get

$$\frac{1}{2}\frac{d}{dt} | \dot{x}_n(t) - \dot{x}(t) |^2 + \langle B(x_n(t) - x(t)), \dot{x}_n(t) - \dot{x}(t)\rangle \leq (h_n(t) - h(t), \dot{x}_n(t) - \dot{x}(t)) \ a.e.$$

But, as before, exploiting the symmetry of the operator $B$, we have

$$\langle B(x_n(t) - x(t)), \ \dot{x}_n(t) - \dot{x}(t)\rangle = \frac{1}{2}\frac{d}{dt}\langle B(x_n(t) - x(t)), x_n(t) - x(t)\rangle.$$

So we get:

$$\frac{1}{2}\frac{d}{dt} | \dot{x}_n(t) - \dot{x}(t) |^2 + \frac{1}{2}\frac{d}{dt}\langle B(x_n(t) - x(t)), x_n(t) - x(t)\rangle \leq (h_n(t) - h(t), \dot{x}_n(t) - \dot{x}(t)) \ a.e.$$

Integrating and recalling that $x_n(0) = x(0) = x_0$, $\dot{x}_n(0) = \dot{x}(0) = x_1$, we have:

$$\frac{1}{2} | \dot{x}_n(t) - \dot{x}(t) |^2 + \frac{1}{2}\langle B(x_n(t) - x(t)), x_n(t) - x(t)\rangle \leq \int_0^t (h_n(s) - h(s), \dot{x}_n(s) - \dot{x}(s))ds$$

which yields

$$\frac{c'}{2} \| x_n(t) - x(t) \|^2 \leq \int_0^t (h_n(s) - h(s), \dot{x}_n(s) - \dot{x}(s))ds$$

which yields

$$\| x_n(t) - x(t) \|^2 \leq \frac{2}{c'} \int_0^t (h_n(s) - h(s), \dot{x}_n(s) - \dot{x}(s))ds.$$

Note that $h_n \xrightarrow{w} h$ in $L^2(H)$ and $\dot{x}_n \xrightarrow{w} y$ in $W(T)$. Since $W(T) \rightarrow L^2(H)$ compactly, we have that $\dot{x}_n \xrightarrow{s} y$ in $L^2(H)$. Thus we have:

$$\int_0^t (h_n(s) - h(s), \dot{x}_n(s) - \dot{x}(s))ds$$
$$= \int_0^t (h_n(s) - h(s), \dot{x}_n(s) - y(s))ds + \int_0^t (h_n(s) - h(s), y(s) - \dot{x}(s))ds \rightarrow 0 \text{ as } n \rightarrow \infty.$$

So $x_n(t) \xrightarrow{s} x(t)$ in $X$ yields $\dot{x} = y \in W(T)$. Now note that

$$\| x_n(t) - x(t) \|^2 \leq \frac{2}{c'} \| h_n - h \|_{L^2(H)} \| \dot{x}_n - \dot{x} \|_{L^2(H)}.$$

Since $h_n \xrightarrow{w} h$ in $K$, we have $\| h_n - h \|_{L^2(H)} \leq N$ for all $n \geq 1$ and some $N > 0$. Thus

$$\| x_n(t) - x(t) \|^2 \leq \frac{2}{c'} N \| \dot{x}_n - \dot{x} \|_{L^2(H)} \to 0$$

which implies that $\gamma(\cdot)$ is indeed continuous as claimed.

Let $R: K \to 2^K$ be the multifunction defined by

$$R(h) = S^2_{\widehat{F}(\cdot, \gamma(h)(\cdot))}.$$

First we will show that $R(\cdot)$ has nonempty values. Let $s_n(\cdot)$ be simple functions such that $s_n(t) \xrightarrow{s} \gamma(h)(t)$ *a.e.* in $H$. Then because of hypothesis $H(F)_1(\underline{1})$, for each $n \geq 1$, $t \to \widehat{F}(t, s_n(t))$ is measurable. Apply Aumann's selection theorem to get $f_n: T \to H$ measurable such that $f_n(t) \in \widehat{F}(t, x_n(t))$ *a.e.*, $n \geq 1$. Note that $| f_n(t) | \leq \phi(t)$ *a.e.* with $\phi(\cdot) \in L^2_+$. Hence by passing to a subsequence if necessary, we may assume that $f_n \xrightarrow{w} f$ in $L^2(H)$. Then theorem 3.1 of [6], tells us that

$$f(t) \in \overline{conv}\ w\text{-}\overline{lim}\{f_n(t)\}_{n \geq 1}$$

$$\subseteq \overline{conv}\ w\text{-}\overline{lim}\widehat{F}(t, s_n(t))$$

$$\subseteq \widehat{F}(t, \gamma(t)(t))\ a.e.$$

The last inclusion follows from the fact that $\widehat{F}(t, \cdot)$ is *u.s.c.* from $H$ into $H_w$ and since $s_n(t) \xrightarrow{s} \gamma(h)(t)$ *a.e.* in $H$. Therefore $f \in S^2_{\widehat{F}(\cdot, p(h)(\cdot))}$ and so we have established that the values of the multifunction $R(\cdot)$ are nonempty. Also since $F(t,x)$ is $P_{fc}(H)$-valued, it is clear that for every $h \in K$, $R(h) \in P_{fc}(K)$. Furthermore using theorem 4.2 of [6] and recalling that $\gamma(\cdot)$ is continuous on $K$ into $C(T,X)$, we get that $R(\cdot)$ is *u.s.c.* Apply the Kakutani-KyFan fixed point theorem to get $h \in R(h)$. Then $x = \gamma(h)$ is a solution of $(*)$, with $F(t,x)$ replaced by $\widehat{F}(t,x)$. But as in the beginning of the proof, with the same *a priori* estimation, we can show that $| x(t) | \leq M_2$ for all $t \in T$ implies that $\widehat{F}(t, x(t)) = F(t, x(t))$ and this yields that $x(\cdot)$ solves $(*)$.

Finally to establish the compactness of $S(x_0, x_1)$ in $C(T,X)$, note that $S(x_0, x_1) \subseteq \gamma(K)$ and the latter is compact in $C(T,X)$ since $\gamma: K \to C(T,X)$ is continuous. So it suffices to show that $S(x_0, x_1)$ is closed in $C(T,X)$. So let $\{x_n\}_{n \geq 1} \subseteq S(x_0, x_1)$ and assume that $x_n \to x$ in $C(T,X)$. Then by definition $x_n = \gamma(f_n)$ with $f_n \in S^2_{F(\cdot)(x_n(\cdot))}$. Note that because of hypothesis $H(F)_1(\underline{3})$ $| f_n(t) | \leq a_1(t) + b_1 \widehat{N}$, where $\widehat{N} = sup \| x_n \|_{C(T,X)}$. So we may assume that $f_n \xrightarrow{w} f$ in $L^2(H)$ implies that $\gamma(f_n) \to \gamma(f)$ in $C(T,X)$ which yields $x = \gamma(f)$ and from theorem 3.1 of [6], we have that $f(t) \in \overline{conv}\ w - \overline{lim}\ \{f_n(t)\}_{n \geq 1} \subseteq \overline{conv}\ w - \overline{lim} F(t, x_n(t)) \subseteq F(t, x(t))$ *a.e.* which yields $x \in S(x_0, x_1)$. <u>*Q.E.D.*</u>

Now we consider the case where the multivalued perturbation term $F(t,x)$ is not necessarily convex-valued. We will need the following hypothesis on the orientor field $F(t,x)$.

<u>$H(F_2)$</u>: $F: T \times H \to P_f(H)$ is a multifunction such that

- ($\underline{1}$) $(t,x) \to F(t,x)$ is graph measurable; i.e. $GrF = \{(t,x,y) \in T \times H \times H : y \in F(t,x)\} \in B(T) \times B(H)$, with $B(T)$ (resp. $B(H)$), being the Borel $\sigma$-field of $T$ (resp. of $H$) (recall that measurability of $F(\cdot, \cdot)$ implies graph measurability).
- ($\underline{2}$) $x \to F(t,x)$ is *l.s.c.*
- ($\underline{3}$) $| F(t,x) | = sup\{ | y | : y \in F(t,x)\} \leq a_1(t) + b_1 | x |$ *a.e.* with $a_1(\cdot) \in L^2_+$, $b_1 > 0$.

**Theorem 3.2:** *<u>If</u> hypotheses $H(A)$, $H(B)$, $H(F)_2$ hold and $x_0 \in X$, $x_1 \in H$, <u>then</u>* $S(x_0, x_1) \neq \emptyset$.

**Proof:** As in the proof of theorem 3.1, let $\widehat{F}(t,x) = F(t, p_{M_2}(x))$ (it is clear that the same *a priori* estimation is valid in the present situation). Then given that $p_{M_2}(\cdot)$ is Lipschitz continuous, we have that $(t,x) \to \widehat{F}(t,x)$ is graph measurable, $x \to \widehat{F}(t,x)$ is *l.s.c.* and furthermore note that $|\widehat{F}(t,x)| \leq a_1(t) + b_1 M_2 = \phi(t)$ *a.e.* with $\phi(\cdot) \in L^2_+$.

Let $V \subseteq L^1(H)$ be defined by $V = \{h \in L^1(H)\colon |h(t)| \leq \phi(t) \text{ a.e.}\}$. From proposition 3.1 of [5], we know that $V$, equipped with the relative weak $L^1(H)$-topology, is compact metrizable. Consider the multifunction $\Gamma: V \to P_f(L^1(H))$ defined by $\Gamma(h) = S^1_{\widehat{F}(\cdot,\gamma(h)(\cdot))}$. It is easy to check using the continuity of $\gamma(\cdot)$ and theorem 4.1 of [6], that $\Gamma(\cdot)$ is *l.s.c.* ( note that if $h_n \xrightarrow{w} h$ in $V \subseteq L^1(H)$, then $h_n \xrightarrow{w} h$ in $L^2(H)$, since $\phi(\cdot) \in L^2_+$). So, we can apply Fryszkowski's continuous selection theorem [1], to get $k: V \to V$ continuous such that $k(h) \in R(h)$. Applying the Schauder-Tichonov fixed point theorem, we get $h \in V$ such that $h = k(h)$. Then $x = p(h)$ solves $(*)$ with $F(t,x)$ replaced by $\widehat{F}(t,x)$. But as before we can check that $|x(t)| \leq M_2$ which implies $\widehat{F}(t,x(t))$ implies that $F(t,x(t))$ which yields $x \in S(x_0,x_1)$. <u>*Q.E.D.*</u>

## 4. An Example

In this section we present an example of a nonlinear hyperbolic partial differential inclusion illustrating the applicability of our work.

So let $T = [0,r]$ and $Z$ a bounded domain in $\mathbb{R}^N$, with smooth boundary $\Gamma = \partial Z$. We will consider the following initial-boundary value problem of hyperbolic type with multivalued terms.

$$\left\{ \begin{array}{c} \dfrac{\partial^2 x}{\partial t^2} - \Delta x - \sum_{i=1}^{N} D_i(k(t, |Dx_t|^2) D_i x_t) \in [f_1(t,z,x(t,z)), f_2(t,z,x(t,z))] \\ x|_{T \times \Gamma} = 0, x(0,z) = x_0(z), x_t(0,z) = x_1(z). \end{array} \right\} \qquad (**)$$

Here $D_i = \frac{\partial}{\partial z_i}$ $i = 1,\ldots,N$, $Dx = (D_1 x_1, \ldots, D_N x) = grad(x)$, $DxDy = \sum_{i=1}^{N} D_i x D_i y$ and $|Dx|^2 = \sum_{i=1}^{N} |D_i x|^2$.

We will need the following hypotheses on the data of $(**)$:

<u>$H(k)$:</u> $k: T \times \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that
- (<u>1</u>) $t \to k(t,\mu)$ is measurable,
- (<u>2</u>) $\mu \to k(t,\mu)$ is continuous,
- (<u>3</u>) $0 \leq k(t,\lambda^2) \leq L$ for all $(t,\lambda) \in T \times \mathbb{R}_+$, with $L > 0$ and $k(t,0) = 0$,
- (<u>4</u>) $k(t,\lambda^2)\lambda - k(t,\mu^2)\mu \geq d(\lambda - \mu)$ for all $\lambda, \mu \in \mathbb{R}_+, \lambda \geq \mu$ and for some $d > 0$.

<u>$H(f)$:</u> $f_1, f_2: T \times Z \times \mathbb{R} \to \mathbb{R}$ are measurable functions such that $x \to f_1(t,z,x)$, $-f_2(t,z,x)$ are *l.s.c.* and $|f_i(t,z,x)| \leq a_1(t,z) + b_1(z)|x|$ *a.e.* $i = 1,2$ with $a_1(\cdot,\cdot) \in L^2(T \times Z)$, $b_1(\cdot) \in L^\infty(Z)$ and $f_1 < f_2$.

<u>$A_0$:</u> $x_0(\cdot) \in H^1_0(Z)$, $x_1(\cdot) \in L^2(Z)$.

In this case, $X = H^1_0(Z)$, $H = L^2(Z)$ and $X^* = H^1_0(Z)^* = H^{-1}(Z)$. We know that $(X,H,X^*)$ is an evolution triple with all embeddings being compact (Sobolev embedding theorem). Consider the following Dirichlet forms:

$$a_1(t,x,y) = \int_Z \sum_{i=1}^{N} k(t, |Dx|^2) D_i x D_j y dz = \int_Z k(t, |Dx|^2) DxDy dz$$

and

$$a_2(x,y) = \int_Z \sum_{i=1}^{N} D_i x D_i y dz = \int_Z DxDy dz$$

for all $x, y \in H_0^1(Z)$.

Using hypothesis $H(k)$ ($\underline{3}$), we get

$$|a_1(t,x,y)| \leq L \| x \|_{H_0^1(Z)} \| y \|_{H_0^1(Z)}.$$

So there exists a nonlinear operator $A: T \times X \to X^*$ such that

$$\langle A(t,z), y \rangle = a_1(t,x,y).$$

From Fubini's theorem we have that $t \to a_1(t,x,y)$ is measurable which implies that $t \to A(t,x)$ is weakly measurable. But $H^{-1}(Z)$ is a separable Hilbert space. So the Pettis measurability theorem tells us that $t \to A(t,x)$ is measurable. Also if $x_n \to x$ in $H_0^1(Z)$, then by passing to a subsequence if necessary, we will have that $|Dx_n(z)|^2 \to |Dx(z)|^2$ *a.e.* and since by hypothesis $H(k)(\underline{2})$ $k(t,\cdot)$ is continuous we have $k(t, |Dx_n(z)|^2) \to k(t, |Dx(z)|^2)$ for all $t \in T$ and almost all $z \in Z$. Also $D_i x_n \xrightarrow{s} D_i x$ in $L^2(Z)$. Thus $\int_Z k(t, |Dx_n|^2) Dx_n Dy dz \to \int_Z k(t, |Dx|^2) Dx Dy dz$ implies that $A(t,x_n) \xrightarrow{w} A(t,x)$ which yields $A(t,\cdot)$ is demicontinuous, this hemicontinuous. Also we have

$$\langle A(t,x) - A(t,y), x - y \rangle = \int_Z (k(t, |Dx|^2)Dx - k(t, |Dy|^2)Dy)(Dx - Dy)dz.$$

Then, because of hypothesis $H(k)(\underline{2})$ and lemma 25.26 ($\underline{b}$), p. 524 of Zeidler [9], we have

$$\langle A(t,x) - A(t,y), x - y \rangle \geq c \| x - y \|^2_{H_0^1(Z)}, c > 0$$

which yields that $A(t,\cdot)$ is strongly monotone.

Also since $k(t,0) = 0$ (by hypothesis $H(k)(\underline{3})$), we have $A(t,0) = 0$ yields that $A(t,\cdot)$ is coercive; i.e., $\langle A(t,x), x \rangle \geq c \| x \|^2_{H_0^1(Z)}$. Thus, we satisfied hypothesis $H(A)$.

Next note that by the Cauchy-Schwartz inequality, we have

$$|a_2(x,y)| \leq \| x \|_{H_0^1(Z)} \| y \|_{H_0^1(Z)}.$$

So there exists a continuous linear operator $B: X \to X^*$ such that

$$\langle Bx, y \rangle = a_2(x,y).$$

Clearly $\langle Bx, y \rangle = \langle x, By \rangle$; i.e. $B$ is symmetric and by Poincaré's inequality, we have $\langle Bx, x \rangle \geq c' \| x \|^2_{H_0^1(Z)}$, $c' > 0$. Therefore, we satisfied hypothesis $H(B)$.

Next let $F: T \times L^2(Z) \to P_{fc}(L^2(Z))$ be defined by

$$F(t,x) = \{h \in L^2(Z): f_1(t,z,x(z)) \leq h(z) \leq f_2(t,z,x(z)) \ a.e.\}.$$

Let $\eta: T \times Z \times \mathbb{R} \to P_{fc}(\mathbb{R})$ be defined by $\eta(t,z,x) = [f_1(t,z,x), f_2(t,z,x)]$. Because of hypothesis $H(f)$, we deduce that $\eta(\cdot,\cdot,\cdot)$ is measurable while $\eta(t,z,\cdot)$ is *u.s.c.* (see Klein-Thompson [2], p. 74). Note that $F(t,x) = S^2_{\eta(t,\cdot,x(\cdot))}$. So, from theorem 4.2 of [6], we have that $F(t,\cdot)$ is *u.s.c.* from $H$ into $H_w$, while clearly $t \to F(t,z)$ is measurable. Also, $|F(t,x)| = sup\{|y|_{L^2(Z)}: y \in F(t,x)\} \leq \widehat{a}_1(t) + \widehat{b}_1 |x|_{L^2(Z)}$, with $\widehat{a}_1(t) = \| a(t,\cdot) \|_{L^2(Z)}$, $\widehat{b}_1 = \| b \|_{L^\infty(Z)}$. Thus, we satisfied hypothesis $H(F)_1$. Finally, let $\widehat{x}_0 = x_0(\cdot) \in H_0^1(Z)$, $\widehat{x}_1 = x_1(\cdot) \in L^2(Z)$.

Rewrite (**) in the following equivalent nonlinear evolution inclusion form:

$$\left\{ \begin{array}{c} \dot{x}(t) + A(t, \dot{x}(t)) + Bx(t) \in F(t, x(t)) \\ x(0) = \widehat{x}_0, \dot{x}(0) = x_1. \end{array} \right\} \qquad (**)'$$

**Theorem 4.1:** *If hypotheses $H(k)$, $H(f)$ and $H_0$ hold, then (**) has a solution $x \in C(T, H_0^1(Z))$ such that $\frac{\partial x}{\partial t} \in L^2(T, H_0^1(Z)) \cap C(T, L^2(Z))$ and $\frac{\partial^2 x}{\partial t^2} \in L^2(T, H^{-1}(Z))$. Also, the solution set is compact in $C(T, H_0^1(Z))$.*

Now suppose that (**) corresponds to an optimal control problem; i.e.

$$f_1(t, z, x) = f(t, z, x)u_1(z)$$

and

$$f_2(t, z, x) = f(t, z, x)u_2(z)$$

with a function $f: T \times Z \times \mathbb{R} \to \mathbb{R}_+$ such that $(t, z) \to f(t, z, x)$ is measurable, $x \to f(t, z, x)$ is continuous and $|f(t, z, x)| \leq a_1(t, z) + b_1(z)x$ *a.e.*, with $a_1(\cdot, \cdot) \in L^2(T \times Z)$, $b_1(\cdot) \in L^\infty(Z)$. The control constraint set is defined as

$$U(t, z) = \{v \in \mathbb{R}: u_1(z) \leq v \leq u_2(z)\}$$

with $0 < u_1(z) < u_2(z) \leq M$ *a.e.*

We are also given a cost functional $J(x) = \int_0^b \int_Z L(t, z, x(t, z))dzdt$ to be minimized over all ad missible trajectories. Assume that $L: T \times Z \times \mathbb{R} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a measurable integrand such that $L(t, z, \cdot)$ is *l.s.c.* and $\phi(t, z) - M(z)|x| \leq L(t, z, x)$ *a.e.* with $\phi(\cdot, \cdot) \in L^1(T \times Z)$, $M(\cdot) \in L_+^\infty(Z)$. Then, $J(\cdot)$ is *l.s.c.* on $C(T, H_0^1(Z))$, and so, using theorem 4.1 above, we deduce that this distributed parameter optimal control problem has a solution. Analogous results for parabolic systems can be found in [4].

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