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A SYSTEM OF IMPULSIVE DEGENERATE NONLINEAR PARABOLIC FUNCTIONAL-DIFFERENTIAL INEQUALITIES

LUDWIK BYSZEWSKI
Cracow University of Technology
Institute of Mathematics
Cracow 31-155, POLAND

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ABSTRACT

A theorem about a system of strong impulsive degenerate nonlinear parabolic functional-differential inequalities in an arbitrary parabolic set is proved. As a consequence of the theorem, some theorems about impulsive degenerate nonlinear parabolic differential inequalities and the uniqueness of a classical solution of an impulsive degenerate nonlinear parabolic differential problem are established.

Key words: Impulsive Parabolic Problems, Diagonal Systems, Functional-Differential Inequalities, Impulsive Conditions, Uniqueness Criterion, Arbitrary Parabolic Sets.

AMS (MOS) subject classifications: 35K65, 35R10, 35K85, 35K60, 35K99.

1. Introduction

In this paper we prove a theorem about strong inequalities for the following diagonal system of degenerate nonlinear parabolic functional-differential inequalities

$$F_i(t,x,u(t,x),u_t^i(t,x),u_x^i(t,x),u_{xx}^i(t,x),u)$$
$$> F_i(t,x,v(t,x),v_t^i(t,x),v_x^i(t,x),v_{xx}^i(t,x),v) \quad (i=1,\ldots,m), \tag{1.1}$$

where $(t,x) \in D \setminus \bigcup_{j=1}^{s} (\{t_j\} \times \mathbb{R}^n)$, $t_0 < t_1 < \ldots < t_s < t_0 + T$ and D is a relatively arbitrary set more general than the cylindrical domain $(t_0, t_0+T) \times \Omega_0 \subset \mathbb{R}^{n+1}$. In the expressions

$$F_i(t,x,w(t,x),w_t^i(t,x),w_x^i(t,x),w_{xx}^i(t,x),w) \quad (i=1,\ldots,m)$$

the symbol w denotes a function

$$w: \widetilde{D} \ni (t,x) \to w(t,x) = (w^1(t,x),\ldots,w^m(t,x)) \in \mathbb{R}^m,$$

where $\widetilde{D}$ is an arbitrary set such that $\bar{D} \cap [[t_0, t_0+T) \times \mathbb{R}^n] \subset \widetilde{D} \subset (-\infty, t_0+T) \times \mathbb{R}^n$, $w_x^i(t,x) := grad_x w^i(t,x)$ $(i=1,\ldots,m)$ and $w_{xx}^i(t,x) := \left[\dfrac{\partial^2 w^i(t,x)}{\partial x_j \partial x_k}\right]_{n \times n}$ $(i=1,\ldots,m)$. We assume that the limits $w(t_j^-, x)$, $w(t_j^+, x)$ $(j = 1, \ldots, s)$ exist for all admissible $x \in \mathbb{R}^n$, they are finite, all different and $w(t_j, x) := w(t_j^+, x)$ $(j = 1, \ldots, s)$ for all admissible $x \in \mathbb{R}^n$.

System (1.1) is studied together with impulsive and boundary inequalities. The impulsive inequalities are of the form

$$u(t_j, x) - u(t_j^-, x) \leq v(t_j, x) - v(t_j^-, x) \quad (j = 1, \ldots, s). \tag{1.2}$$

As a consequence of the theorem about the strong inequalities for system (1.1), we establish theorems about impulsive degenerate nonlinear parabolic differential inequalities and the uniqueness of a classical solution of an impulsive degenerate nonlinear parabolic differential problem.

The results obtained in the paper are direct generalizations of those given by the author in [2]. To prove the results of this paper, theorems of [2] are used. The paper is a continuation of author's publication [3] about impulsive parabolic problems. The impulsive conditions in the present paper are quite different from those considered in [3]. They are similar to the impulsive conditions used by Bainov, Kamont and Minchev in [1].

2. Preliminaries

We use the notation: $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $\mathbb{R}_+ = [0, \infty)$. For any vectors $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$, $\tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_m) \in \mathbb{R}^m$ we write $z \leq \tilde{z}$ if $z_i \leq \tilde{z}_i$ $(i = 1, \ldots, m)$.

By Ω we denote an arbitrary open subset of $(t_0, t_0 + T) \times \mathbb{R}^m$, where $t_0 \in \mathbb{R}$ and $T \in \mathbb{R}_+ \setminus \{0\}$, such that the projection of Ω on the t-axis is the interval $(t_0, t_0 + T)$.

Next, by D we denote the subset of the set $\bar{\Omega} \cap [(t_0, t_0 + T) \times \mathbb{R}^n]$ satisfying the condition that for any $(\tilde{t}, \tilde{x}) \in D$ there exists a number $\rho > 0$ such that

$$\{(t, x): (t - \tilde{t})^2 + \sum_{i=1}^{n} (x_i - \tilde{x}_i)^2 < \rho, \ t < \tilde{t}\} \subset \Omega.$$

It is clear that $\Omega \subset D$.

We define the sets

$$S_t := \{x \in \mathbb{R}^n: (t, x) \in \bar{D}\} \text{ for } t \in [t_0, t_0 + T]$$

and

$$\sigma_t := \bar{D} \cap (\{t\} \times \mathbb{R}^n) \text{ for } t \in [t_0, t_0 + T].$$

By s we denote a fixed number belonging to $\mathbb{N}$.

Let $t_1, t_2, \ldots, t_s$ be arbitrary fixed real numbers such that

$$t_0 < t_1 < \ldots < t_s < t_0 + T.$$

We introduce the following sets:

$$D_j := D \cap [(t_j, t_{j+1}) \times \mathbb{R}^n] \quad (j = 0, 1, \ldots, s - 1),$$

$$D_s := D \cap [(t_s, t_0 + T) \times \mathbb{R}^n],$$

$$D_* := \bigcup_{j=0}^{s} D_j \text{ and } \sigma_* := \bigcup_{j=1}^{s} \sigma_{t_j}.$$

Let $\widetilde{D}$ be an arbitrary set such that

$$\bar{D} \cap [[t_0, t_0 + T) \times \mathbb{R}^n] \subset \widetilde{D} \subset (-\infty, t_0 + T) \times \mathbb{R}^n, \quad t_0 + T \leq \infty.$$

By Σ we denote the part of $\partial D \backslash (\sigma_{t_0} \cup \sigma_* \cup \sigma_{t_0 + T})$ disjoint with D.

Assumption (A): For each $i \in \{1, \ldots, m\}$ let Σ_i be a subset (possibly empty) of Σ and let, for each $(t,x) \in \Sigma_i$, $\ell_i(t,x)$ be a direction. We assume that ℓ_i is orthogonal to the t-axis and some open segment, with one extremity at (t,x), of the ray with origin at (t,x) in the direction of ℓ_i is contained in D.

Given a subset E of $\sigma_{t_0} \cup \sigma_* \cup \Sigma$ [E of σ_*] and a function $\omega: D_* \to \mathbb{R}$, we say that ω has *finite t-right-hand* [*t-left-hand*] *sided limits* in $E \cup \{\infty\}$ if for every $(\widetilde{t}, \widetilde{x}) \in E$ and every $\widetilde{t} \in P(E)$, and for each sequence $(t^\nu, x^\nu) \in D_*$ such that $t^\nu > \widetilde{t}$ [$t^\nu < \widetilde{t}$], $t^\nu \to \widetilde{t}$ and $(t^\nu, x^\nu) \to (\widetilde{t}, \widetilde{x})$ or $|x^\nu| \to \infty$, the limit $\lim_{\nu \to \infty} \omega(t^\nu, x^\nu)$ is finite; here $P(E)$ is the projection of E on the t-axis. Obviously, this limit does not depend on the choice of the sequence (t^ν, x^ν) and it will be denoted by $\omega(\widetilde{t}^+, \widetilde{x})$ and $\omega(\widetilde{t}^+, \infty)$ [$\omega(\widetilde{t}^-, \widetilde{x})$ and $\omega(\widetilde{t}^-, \infty)$], respectively.

Let E be a subset of $\sigma_{t_0} \cup \sigma_* \cup \Sigma$. If for $\widetilde{t} \in P(E)$ there is a sequence $(t^\nu, x^\nu) \in D_*$ such that $t^\nu > \widetilde{t}$, $t^\nu \to \widetilde{t}$ and $|x^\nu| \to \infty$ then we denote by $(\widetilde{t}, \infty)$ the class of all such sequences. By a function $\varphi: E \cup \{\infty\} \to \mathbb{R}$ we mean a function defined for $(t,x) \in E$ and (t, ∞) with $t \in P(E)$.

By $PC_m(\widetilde{D})$ we denote the space of mappings

$$w: \widetilde{D} \ni (t,x) \to w(t,x) = (w^1(t,x), \ldots, w^m(t,x)) \in \mathbb{R}^m,$$

such that, for every $i \in \{1, \ldots, m\}$, w^i is continuous in $(D \cup \Sigma_i) \backslash \sigma_*$, has finite t-right-hand sided limits $w^i(t^+, x)$, $w^i(t^+, \infty)$ in $\sigma_{t_0} \cup \sigma_* \cup (\Sigma \backslash \Sigma_i) \cup \{\infty\}$, has finite t-left-hand sided limits $w^i(t^-, x)$, $w^i(t^-, \infty)$ in $\sigma_* \cup \{\infty\}$, and $w^i(t,x) := w^i(t^+, x)$ for $(t,x) \in \sigma_{t_0} \cup \sigma_* \cup (\Sigma \backslash \Sigma_i)$ and $w^i(t, \infty) := w^i(t^+, \infty)$ for $t \in P[\sigma_{t_0} \cup \sigma_* \cup (\Sigma \backslash \Sigma_i)]$.

For $w, \widetilde{w} \in PC_m(\widetilde{D})$ and for every fixed $t < t_0 + T$, we write $w \overset{t}{\leq} \widetilde{w}$ if $w^i(\tau, x) \leq \widetilde{w}^i(\tau, x)$ for $(\tau, x) \in \widetilde{D}$, $\tau \leq t$ $(i = 1, \ldots, m)$. Given the sets Σ_i $(i = 1, \ldots, m)$ and the directions ℓ_i $(i = 1, \ldots, m)$ satisfying Assumption (A), a function $w \in PC_m(\widetilde{D})$ is said to *belong to* $PC^{1,2}_{m,\Sigma}(\widetilde{D})$ if w^i_t, w^i_x, w^i_{xx} $(i = 1, \ldots, m)$ are continuous in D_* and the derivatives $\frac{dw^i}{d\ell_i}$ $(i = 1, \ldots, m)$ are finite on Σ_i $(i = 1, \ldots, m)$, respectively.

By $M_{n \times n}(\mathbb{R})$ we denote the space of real square symmetric matrices $r = [r_{jk}]_{n \times n}$. For each $i \in \{1, \ldots, m\}$ by F_i we denote the mapping

$$F_i: D_* \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \times M_{n \times n}(\mathbb{R}) \times PC^{1,2}_{m,\Sigma}(\widetilde{D})$$

$$\ni (t, x, z, p, q, r, w) \to F_i(t, x, z, p, q, r, w) \in \mathbb{R},$$

where $q = (q_1, \ldots, q_n)$ and $r = [r_{jk}]_{n \times n}$.

We use the notation

$$F_i[t, x, w] := F_i(t, x, w(t,x), w^i_t(t,x), w^i_x(t,x), w^i_{xx}(t,x), w) \quad (i = 1, \ldots, m)$$

for all $(t,x) \in D_*$ and $w \in PC^{1,2}_{m,\Sigma}(\widetilde{D})$.

By Z we denote a fixed subset of $PC^{1,2}_{m,\Sigma}(\widetilde{D})$. Functions u and v belonging to Z are called *solutions* of the system

$$F_i[t,x,u] > F_i[t,x,v] \quad (i = 1,\ldots,m) \tag{2.1}$$

in D_*, if they satisfy (2.1) for all $(t,x) \in D_*$.

The functions F_i $(i = 1,\ldots,m)$ are said to be *parabolic* with respect to $w \in PC^{1,2}_{m,\Sigma}(\widetilde{D})$ in D_* if for every $r = [r_{jk}]$, $\widetilde{r} = [\widetilde{r}_{jk}] \in M_{n \times n}(\mathbb{R})$ and $(t,x) \in D_*$ the following implications hold:

$$r \leq \widetilde{r} \Rightarrow F_i(t,x,u(t,x),u_t^i(t,x),u_x^i(t,x),r,u)$$

$$\leq F_i(t,x,u(t,x),u_t^i(t,x),u_x^i(t,x),\widetilde{r},u) \quad (i = 1,\ldots,m), \tag{2.2}$$

where $r \leq \widetilde{r}$ means that the inequality $\sum_{j,k=1}^{n} (r_{jk} - \widetilde{r}_{jk})\lambda_j \lambda_k \leq 0$ is satisfied for each $(\lambda_1,\ldots,\lambda_n) \in \mathbb{R}^n$.

3. Theorem about Impulsive Functional-Differential Inequalities

Theorem 3.1. *Assume that:*

1. *The functions F_i $(i = 1,\ldots,m)$ are weakly increasing with respect to $z_1,\ldots,z_{i-1}$, $z_{i+1},\ldots,z_m$ $(i = 1,\ldots,m)$, respectively. Moreover, F_i $(i = 1,\ldots,m)$ are weakly increasing with respect to w in the sense of the relation $\overset{t}{\leq}$ for all $t \in (t_0, t_0 + T)$ and*

$$F_i(t,x,z,p,q,r,w) \geq F_i(t,x,z,\widetilde{p},q,r,w) \quad (i = 1,\ldots,m)$$

for all $(t,x) \in D_*$, $z \in \mathbb{R}^m$, $p < \widetilde{p}$, $q \in \mathbb{R}^n$, $r \in M_{n \times n}(\mathbb{R})$, $w \in Z$.

2. *For the given sets $\Sigma_i (i = 1,\ldots,m)$ and the directions ℓ_i $(i = 1,\ldots,m)$ satisfying Assumption (A), for the given functions $a_i : \Sigma_i \to \mathbb{R}_+ (i = 1,\ldots,m)$ and for the given functions $\phi_i : \Sigma_i \times \mathbb{R} \to \mathbb{R} (i = 1,\ldots,m)$ of the variables (t,x,ξ) and weakly increasing with respect to ξ, functions u and v belonging to Z satisfy the inequalities*

$$u(t,x) < v(t,x) \ \ \textit{for } (t,x) \in \widetilde{D} \backslash \bar{D}, \tag{3.1}$$

$$u^i(t,x) < v^i(t,x) \textit{ for } (t,x) \in \sigma_{t_0} \cup (\Sigma \backslash \Sigma_i) \cup \{\infty\} \quad (i = 1,\ldots,m), \tag{3.2}$$

$$u(t,x) - u(t^-,x) < v(t,x) - v(t^-,x) \textit{ for } (t,x) \in \sigma_*, \tag{3.3}$$

$$\phi_i(t,x,u^i(t,x)) - \phi_i(t,x,v^i(t,x)) < a_i(t,x) \frac{d[u^i(t,x) - v^i(t,x)]}{d\ell_i} \tag{3.4}$$

$$\textit{for } (t,x) \in \Sigma_i \quad (i = 1,\ldots,m)$$

and the condition

$$u^i(t,x) \neq v^i(t,x) \textit{ for } (t,x) \in \Sigma_i \quad (i = 1,\ldots,m). \tag{3.5}$$

3. *F_i $(i = 1, \ldots, m)$ are parabolic with respect to u in D_* and u, v are solutions of system (2.1) in D_*.*

Then,

$$u(t,x) < v(t,x) \text{ for } (t,x) \in \tilde{D}. \tag{3.6}$$

Proof. To prove Theorem 3.1 consider the following problem:

$$\left.\begin{array}{c}
F_i[t,x,u] > F_i[t,x,v] \text{ for } (t,x) \in D_0 \;\; (i = 1, \ldots, m), \\[2mm]
u(t,x) < v(t,x) \text{ for } (t,x) \in (\tilde{D} \backslash \bar{D}) \cap ((-\infty, t_1) \times \mathbb{R}^n), \\[2mm]
u^i(t,x) < v^i(t,x) \text{ for } (t,x) \in \left[\sigma_{t_0} \cup (\Sigma \backslash \Sigma_i) \cup \{\infty\}\right] \\[2mm]
\cap \left[[t_0, t_1) \times \mathbb{R}^n\right] \;\; (i = 1, \ldots, m), \\[2mm]
\phi_i(t,x,u^i(t,x)) - \phi_i(t,x,v^i(t,x)) < a_i(t,x)\dfrac{d[u^i(t,x) - v^i(t,x)]}{d\ell_i} \\[2mm]
\text{for } (t,x) \in \Sigma_i \cap [(t_0, t_1) \times \mathbb{R}^n] \;\; (i = 1, \ldots, m).
\end{array}\right\} \tag{3.7}$$

According to the assumptions of Theorem 3.1 corresponding to problem (3.7), by Theorem 2.1 from [2] applied to set D_0, we obtain the inequality

$$u(t,x) < v(t,x) \text{ for } (t,x) \in D_0. \tag{3.8}$$

By (3.8) and by the fact that $u, v \in PC_m(\tilde{D})$,

$$u(t^-, x) \leq v(t^-, x) \text{ for } (t,x) \in \sigma_{t_1}. \tag{3.9}$$

From (3.3) and (3.9), we have

$$u(t,x) < v(t,x) \text{ for } (t,x) \in \sigma_{t_1}. \tag{3.10}$$

Inequalities (3.1), (3.8), (3.2) and (3.10) imply that

$$u^i(t,x) < v^i(t,x) \text{ for } (t,x) \in (\tilde{D} \backslash \Sigma_i) \cap [(-\infty, t_1] \times \mathbb{R}^n] \;\; (i = 1, \ldots, m). \tag{3.11}$$

By (3.11), (3.5) and the fact that $u^i (i = 1, \ldots, m)$ is continuous in Σ_i $(i = 1, \ldots, m)$, we get

$$u(t,x) < v(t,x) \text{ for } (t,x) \in \tilde{D} \cap [(-\infty, t_1] \times \mathbb{R}^n]. \tag{3.12}$$

Now, set the following problem:

$$\left.\begin{array}{c} F_i[t,x,u] > F_i[t,x,v] \text{ for } (t,x) \in D_1 \quad (i=1,\ldots,m), \\ u(t,x) < v(t,x) \text{ for } (t,x) \in \left[\widetilde{D} \cap ((-\infty,t_2) \times \mathbb{R}^n)\right] \backslash \bar{D}_1, \\ u^i(t,x) < v^i(t,x) \text{ for } (t,x) \in \left[\sigma_{t_1} \cup (\Sigma \backslash \Sigma_i) \cup \{\infty\}\right] \\ \cap \left[[t_1,t_2) \times \mathbb{R}^n\right] \quad (i=1,\ldots,m), \\ \phi_i(t,x,u^i(t,x)) - \phi_i(t,x,v^i(t,x)) < a_i(t,x)\dfrac{d[u^i(t,x)-v^i(t,x)]}{d\ell_i} \\ \text{for } (t,x) \in \Sigma_i \cap [(t_1,t_2) \times \mathbb{R}^n] \quad (i=1,\ldots,m). \end{array}\right\} \tag{3.13}$$

According to the assumptions of Theorem 3.1 corresponding to problem (3.13), by Theorem 2.1 from [2] applied to set D_1, we arrive at the inequality

$$u(t,x) < v(t,x) \text{ for } (t,x) \in D_1. \tag{3.14}$$

By (3.14) and by the fact that $u, v \in PC_m(\widetilde{D})$,

$$u(t^-,x) \le v(t^-,x) \text{ for } (t,x) \in \sigma_{t_2}. \tag{3.15}$$

From (3.3) and (3.15), we have

$$u(t,x) < v(t,x) \text{ for } (t,x) \in \sigma_{t_2}. \tag{3.16}$$

Inequalities (3.1), (3.12), (3.14), (3.2) and (3.16) imply that

$$u^i(t,x) < v^i(t,x) \text{ for } (t,x) \in \left[\widetilde{D} \cap [(-\infty,t_2] \times \mathbb{R}^n]\right] \backslash \left[\Sigma_i \cap [(t_1,t_2) \times \mathbb{R}^n]\right]. \tag{3.17}$$

By (3.17), (3.5) and the fact that u^i $(i=1,\ldots,m)$ is continuous in Σ_i $(i=1,\ldots,m)$, we get

$$u(t,x) < v(t,x) \text{ for } (t,x) \in \widetilde{D} \cap [(-\infty,t_2] \times \mathbb{R}^n]. \tag{3.18}$$

Repeating the above procedure $s-2$ times, we obtain

$$u(t,x) < v(t,x) \text{ for } (t,x) \in \sigma_{t_s} \tag{3.19}$$

and

$$u(t,x) < v(t,x) \text{ for } (t,x) \in \widetilde{D} \cap [(-\infty,t_s] \times \mathbb{R}^n]. \tag{3.20}$$

Finally, consider the problem

$$\left.\begin{aligned}
&F_i[t,x,u] > F_i[t,x,v] \text{ for } (t,x) \in D_s \quad (i=1,\ldots,m),\\
&u(t,x) < v(t,x) \text{ for } (t,x) \in \widetilde{D}\backslash \bar{D}_s,\\
&u^i(t,x) < v^i(t,x) \text{ for } (t,x) \in \left[\sigma_{t_s} \cup (\Sigma\backslash\Sigma_i) \cup \{\infty\}\right]\\
&\cap\left[\,[t_s, t_0+T)\times\mathbb{R}^n\right] \quad (i=1,\ldots,m),\\
&\phi_i(t,x,u^i(t,x)) - \phi_i(t,x,v^i(t,x)) < a_i(t,x)\frac{d[u^i(t,x)-v^i(t,x)]}{d\ell_i}\\
&\text{for } (t,x) \in \Sigma_i \cap [(t_s, t_0+T)\times\mathbb{R}^n] \quad (i=1,\ldots,m).
\end{aligned}\right\} \tag{3.21}$$

According to the assumptions of Theorem 3.1 corresponding to problem (3.21), by Theorem 2.1 from [2] applied to set D_s, we get the inequality

$$u(t,x) < v(t,x) \text{ for } (t,x) \in D_s. \tag{3.22}$$

Inequalities (3.1), (3.20), (3.22), (3.2) and (3.19) imply that

$$u^i(t,x) < v^i(t,x) \text{ for } (t,x) \in \widetilde{D}\backslash\left[\Sigma_i \cap [(t_s, t_0+T)\times\mathbb{R}^n]\right] (i=1,\ldots,m). \tag{3.23}$$

By (3.23), (3.5) and the fact that $u^i (i=1,\ldots,m)$ is continuous in Σ_i $(i= 1,\ldots,m)$, we have

$$u(t,x) < v(t,x) \text{ for } (t,x) \in \widetilde{D}. \tag{3.24}$$

4. Theorems about Impulsive Differential Inequalities

From the proof of Theorem 3.1 it is easy to see that the following theorem is true:

Theorem 4.1. *Assume that:*

1. $\widetilde{D} = \bar{D}$ *and the functions*

$$G_i: D_* \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \times M_{n\times n}(\mathbb{R}) \ni (t,x,z,p,q,r) \to G_i(t,x,z,p,q,r) \in \mathbb{R}$$

are weakly increasing with respect to $z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_m$ $(i=1,\ldots,m)$, *respectively, and*

$$G_i(t,x,z,p,q,r) \geq G_i(t,x,z,\widetilde{p},q,r) \qquad (i=1,\ldots,m)$$

for all $(t,x) \in D_*$, $z \in \mathbb{R}^m$, $p < \widetilde{p}$, $q \in \mathbb{R}^n$, $r \in M_{n\times n}(\mathbb{R})$.

2. *For the given sets* $\Sigma_i (i=1,\ldots,m)$ *and the directions* $\ell_i (i=1,\ldots,m)$ *satisfying Assumption (A), for the given functions* $a_i: \Sigma_i \to \mathbb{R}_+$ $(i=1,\ldots,m)$ *and for the given functions* $\phi_i: \Sigma_i \times \mathbb{R} \to \mathbb{R}$ $(i=1,\ldots,m)$ *of the variables* (t,x,ξ) *and weakly increasing with respect to* ξ, *functions* u *and* v *belonging to* $Z \subset PC^{1,2}_{m,\Sigma}(\bar{D})$ *satisfy inequalities* (3.2)-(3.4).

3. $G_i (i=1,\ldots,m)$ *are parabolic with respect to* u *in* D_*, *and* u,v *are solutions of the system*

$$G_i(t,x,u(t,x),u_t^i(t,x),u_x^i(t,x),u_{xx}^i(t,x))$$

$$> G_i(t,x,v(t,x),v^i_t(t,x),v^i_x(t,x),v^i_{xx}(t,x)) \quad (i=1,\ldots,m)$$

in D_*.

Then

(*i*) $u^i(t,x) < v^i(t,x)$ *for* $(t,x) \in (\bar{D}\backslash\Sigma_i) \cap ([t_0, t_0+T)\times\mathbb{R}^n)$ $(i=1,\ldots,m)$

and

(*ii*) $u^i(t,x) \le v^i(t,x)$ *for* $(t,x) \in \Sigma_i$ $(i=1,\ldots,m)$.

Moreover,

(*iii*) $u(t,x) < v(t,x)$ *for* $(t,x) \in \bar{D} \cap ([t_0, t_0+T)\times\mathbb{R}^n)$

if (3.5) *holds.*

As a consequence of Theorem 4.1, we obtain the following theorem:

Theorem 4.2. *Assume that:*

1. $\widetilde{D} = \bar{D}$ *and the function*

$$G: D_* \times \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times M_{n\times n}(\mathbb{R}) \ni (t,x,z,p,q,r) \to G(t,x,z,p,q,r) \in \mathbb{R}$$

is weakly decreasing with respect to z *and* p *in* D_*.

2. *For the given set* $\widehat{\Sigma} \subset \Sigma$ *and the direction* ℓ *satisfying Assumption* (A), *for the given function* $a: \widehat{\Sigma} \to \mathbb{R}_+$ *and for the given function* $\phi: \widehat{\Sigma}\times\mathbb{R}\to\mathbb{R}$ *of the variables* (t,x,ξ) *and strictly increasing with respect to* ξ, *functions* u *and* v *belonging to* $Z \subset PC^{1,2}_{1,\Sigma}(\bar{D})$ *satisfy the inequalities*

$$u(t,x) \le v(t,x) \text{ for } (t,x) \in \sigma_{t_0} \cup (\Sigma\backslash\widehat{\Sigma}) \cup \{\infty\},$$

$$u(t,x) - u(t^-,x) \le v(t,x) - v(t^-,x) \text{ for } (t,x) \in \sigma_*$$

and

$$\phi(t,x,u(t,x)) - \phi(t,x,v(t,x)) \le a(t,x)\frac{d[u(t,x)-v(t,x)]}{d\ell} \text{ for } (t,x) \in \widehat{\Sigma}.$$

3. G *is parabolic with respect to* u *in* D_* *and* u, v *are solutions of the inequality*

$$G(t,x,u(t,x),u_t(t,x),u_x(t,x),u_{xx}(t,x))$$
$$> G(t,x,v(t,x),v_t(t,x),v_x(t,x),v_{xx}(t,x)) \tag{4.1}$$

in D_*.

Then

$$u(t,x) \le v(t,x) \text{ for } (t,x) \in D_*. \tag{4.2}$$

Proof. Let $\epsilon > 0$ and let

$$v^\epsilon(t,x) := \begin{cases} v(t,x)+\epsilon \text{ for } (t,x) \in \bar{D}_0\backslash\sigma_{t_1}, \\ v(t,x)+2\epsilon \text{ for } (t,x) \in \bar{D}_1\backslash\sigma_{t_2}, \\ \cdots\cdots\cdots\cdots \\ v(t,x)+s\epsilon \text{ for } (t,x) \in \bar{D}_{s-1}\backslash\sigma_{t_s}, \\ v(t,x)+(s+1)\epsilon \text{ for } (t,x) \in \bar{D}_s\backslash\sigma_{t_0+T}. \end{cases} \tag{4.3}$$

By (4.1), (4.3) and by the fact that G is weakly decreasing with respect to z, we obtain

$$G(t,x,u(t,x),u_t(t,x),u_x(t,x),u_{xx}(t,x))$$

$$-G(t,x,v^\epsilon(t,x),v_t^\epsilon(t,x),v_x^\epsilon(t,x),v_{xx}^\epsilon(t,x))$$

$$> G(t,x,v(t,x),v_t(t,x),v_x(t,x),v_{xx}(t,x))$$

$$-G(t,x,v^\epsilon(t,x),v_t^\epsilon(t,x),v_x^\epsilon(t,x),v_{xx}^\epsilon(t,x))$$

$$\geq 0 \text{ for } (t,x)\in D_*.$$

Moreover, from assumption 2 of Theorem 4.2 and from (4.3) it follows that

$$u(t,x) < v^\epsilon(t,x) \text{ for } (t,x)\in \sigma_{t_0}\cup(\Sigma\backslash\hat{\Sigma})\cup\{\infty\},$$

$$\phi(t,x,u(t,x)) - \phi(t,x,v^\epsilon(t,x))$$

$$< \phi(t,x,u(t,x)) - \phi(t,x,v(t,x))$$

$$\leq a(t,x)\frac{d[u(t,x)-v(t,x)]}{d\ell}$$

$$= a(t,x)\frac{d[u(t,x)-v^\epsilon(t,x)]}{d\ell} \text{ for } (t,x)\in\hat{\Sigma}$$

and

$$u(t_j,x) - u(t_j^-,x) \leq v(t_j,x) - v(t_j^-,x)$$

$$< [v(t_j,x)+(j+1)\epsilon] - [v(t_j^-,x)+j\epsilon]$$

$$= v^\epsilon(t_j,x) - v^\epsilon(t_j^-,x) \text{ for } x\in S_{t_j} \quad (j=1,2,\ldots,s).$$

Then we have the inequality

$$u(t,x) < v^\epsilon(t,x) \text{ for } (t,x)\in D_*$$

because functions u and v^ϵ satisfy all the assumptions of Theorem 4.1. Hence (4.2) holds.

Remark 4.1. From the proof of Theorem 4.2 it is easy to see that if function G from Theorem 4.2 is strictly decreasing with respect to z and weakly decreasing with respect to p in D_* then Theorem 4.2 is true if strong inequality (4.1) is replaced by the weak inequality

$$G(t,x,u(t,x),u_t(t,x),u_x(t,x),u_{xx}(t,x))$$

$$\geq G(t,x,v(t,x),v_t(t,x),v_x(t,x),v_{xx}(t,x)), \quad (t,x)\in D_*.$$

Theorem 4.2 and Remark 4.1 imply the following theorem about the uniqueness of a classical solution of a mixed impulsive parabolic differential problem:

Theorem 4.3. *Assume that:*

1. $\widetilde{D}=\bar{D}$ *and the function* G *from Theorem 4.2 is strictly decreasing with respect to* z *and weakly decreasing with respect to* p *in* D_*.

2. *The set* $\widehat{\Sigma}\subset\Sigma$ *and the direction* ℓ *satisfy Assumption* (A), $a:\widehat{\Sigma}\to\mathbb{R}_+$ *is a given function, the function* $\phi:\widehat{\Sigma}\times\mathbb{R}\to\mathbb{R}$ *of the variables* (t,x,ξ) *is strictly increasing with respect to* ξ, *and* $f:\sigma_{t_0}\cup(\Sigma\backslash\widehat{\Sigma})\cup\{\infty\}\to\mathbb{R}$, $g:\sigma_*\to\mathbb{R}$, $h:\widehat{\Sigma}\to\mathbb{R}$ *are given functions.*

Then in the class of all functions w *belonging to* $PC^{1,2}_{1,\Sigma}(\bar{D})$ *and such that function* G *is parabolic with respect to* w *in* D_* *there exists at most one function satisfying the following mixed impulsive parabolic differential problem:*

$$G(t,x,w(t,x),w_t(t,x),w_x(t,x),w_{xx}(t,x))=0,\quad (t,x)\in D_*,$$

$$w(t,x)=f(t,x),\quad (t,x)\in\sigma_{t_0}\cup(\Sigma\backslash\widehat{\Sigma})\cup\{\infty\},$$

$$w(t,x)-w(t^-,x)=g(t,x),\ (t,x)\in\sigma_*,$$

$$g(t,x,w(t,x))-a(t,x)\frac{dw(t,x)}{d\ell}=h(t,x),\quad (t,x)\in\widehat{\Sigma}.$$

5. Remarks

Remark 5.1. Since the functions F_i $(i=1,\dots,m)$ from Theorem 3.1 are weakly decreasing with respect to p then these functions may be, particularly, defined by the following formulae:

$$F_i(t,x,z,p,q,r,w):=f_i(t,x,z,q,r,w)-c_i(t,x)p\quad (i=1,\dots,m),$$

where $(t,x)\in D_*$, $z\in\mathbb{R}^m$, $p\in\mathbb{R}$, $q\in\mathbb{R}^n$, $r\in M_{n\times n}(\mathbb{R})$, $w\in Z$, and $c_i(t,x)\geq 0$ $(i=1,\dots,m)$ for $(t,x)\in D_*$.

The same remarks are true for functions $G_i(i=1,\dots,m)$ and G from Theorems 4.1-4.3.

Therefore, the *degenerate* parabolic problems from this paper are more general than the parabolic problems, in the normal form with respect to p, corresponding to the considered degenerate parabolic problems.

Remark 5.2. Theorems 4.2 and 4.3 are formulated only for the *differential* parabolic problems and for $m=1$ because assuming, simultaneously, that $F_i(i=1,\dots,m)$ from Theorem 3.1 are weakly increasing with respect to $z_1,\dots,z_{i-1},z_{i+1},\dots,z_n$, w and weakly decreasing with respect to $z_1,\dots,z_n$, w we can consider only the differential problems, where $m=1$.

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