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BROWNIAN LOCAL TIMES

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ABSTRACT

In this paper explicit formulas are given for the distribution functions and the moments of the local times of the Brownian motion, the reflecting Brownian motion, the Brownian meander, the Brownian bridge, the reflecting Brownian bridge and the Brownian excursion.

Key words: Distribution Functions, Moments, Local Times, Brownian Motion, Brownian Meander, Brownian Bridge, Brownian Excursion.

AMS (MOS) subject classifications: 60J55, 60J65, 60J15, 60K05.

1. Introduction

We consider six stochastic processes, namely, the Brownian motion, the reflecting Brownian motion, the Brownian meander, the Brownian bridge, the reflecting Brownian bridge and the Brownian excursion. For each process we determine explicitly the distribution function and the moments of the local time. This paper is a sequel of the author's papers [37], [41] and [42] in which more elaborate methods were used to find the distribution and the moments of the local time for the Brownian excursion, the Brownian meander, and the reflecting Brownian motion. In this paper we shall show that we can determine all the above mentioned distributions and moments in a very simple way. We approximate each process by a suitably chosen random walk and determine the moments of the local time of the approximating random walk by making use of a conveniently chosen sequence of recurrent events. By letting the number of steps in the random walks tend to infinity we obtain the moments of local times of the processes considered. In each case the sequence of moments uniquely determines the distribution of the corresponding local time and the distribution function can be determined explicitly. It is very surprising that for each process the moments of the local time can be expressed simply by the moments of the local time of the Brownian motion.

In this section we introduce the notations used and describe briefly the results obtained. The proofs are given in subsequent sections.

Throughout this paper we use the following notations:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \tag{1}$$

the normal density function,

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$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du, \tag{2}$$

the normal distribution function, and

$$H_n(x) = n! \sum_{j=0}^{[n/2]} \frac{(-1)^j x^{n-2j}}{2^j j!(n-2j)!}, \tag{3}$$

the nth Hermite polynomial ($n = 0, 1, 2, \ldots$). We have

$$H_n(x) = xH_{n-1}(x) - (n-1)H_{n-2}(x) \tag{4}$$

for $n \geq 2$ where $H_0(x) = 1$ and $H_1(x) = x$. The jth derivative of $\varphi(x)$ is equal to

$$\varphi^{(j)}(x) = (-1)^j \varphi(x) H_j(x). \tag{5}$$

We define

$$J_r(\alpha) = \alpha^{r+1} \int_1^{\infty} e^{-\alpha^2 x^2/2} (x-1)^r dx \tag{6}$$

for $\alpha > 0$ and $r = 0, 1, 2, \ldots$. In particular,

$$J_0(\alpha) = \sqrt{2\pi}[-\Phi(\alpha)], \tag{7}$$

and

$$J_1(\alpha) = e^{-\alpha^2/2} - \alpha\sqrt{2\pi}[1 - \Phi(\alpha)]. \tag{8}$$

We have

$$J_{r+1}(\alpha) = rJ_{r-1}(\alpha) - \alpha J_r(\alpha) \tag{9}$$

for $r = 1, 2, \ldots$.

The Brownian motion: Let $\{\xi(t), 0 \leq t \leq 1\}$ be a standard Brownian motion process. We have $\mathbf{P}\{\xi(t) \leq x\} = \Phi(x/\sqrt{t})$ for $0 < t \leq 1$. Let us define

$$\tau(\alpha) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \text{ measure } \{t: \alpha \leq \xi(t) < \alpha + \epsilon, \ 0 \leq t \leq 1\} \tag{10}$$

for any real α. The limit (10) exists with probability one and $\tau(\alpha)$ is called the *local time at level* α. The concept of local time was introduced by P. Lévy [26], [27]. See also H. Trotter [43], K. Itô and H.P. McKean, Jr. [19], and A.N. Borodin [6].

Our approach is based on a symmetric random walk $\{\zeta_r, r \geq 0\}$ where $\zeta_r = \xi_1 + \xi_2 + \ldots + \xi_r$ for $r \geq 1$, $\zeta_0 = 0$ and $\{\xi_r, r \geq 1\}$ is a sequence of independent and identically distributed random variables for which

$$\mathbf{P}\{\xi_r = 1\} = \mathbf{P}\{\xi_r = -1\} = 1/2. \tag{11}$$

Let us define $\tau_n(a)$ $(a = 1, 2, \ldots)$ as the number of subscripts $r = 1, 2, \ldots, n$ for which $\zeta_{r-1} = a - 1$ and $\zeta_r = a$. If $a \geq 1$, $\zeta_{r-1} = a - 1$ and $\zeta_r = a$, then we say that in the symmetric random walk a transition $a - 1 \to a$ takes places at the rth step and $\tau_n(a)$ is the number of transitions $a - 1 \to a$ in the first n steps. In a similar way we define $\tau_n(-a)$ $(a = 1, 2, \ldots, n)$ as the number of subscripts $r = 1, 2, \ldots, n$ for which $\zeta_{r-1} = -a + 1$ and $\zeta_r = -a$, that is, $\tau_n(-a)$ is the number of transitions $-a+1 \to -a$ in the first n steps.

By the results of M. Donsker [11], if $n \to \infty$, the process $\{\zeta_{[nt]}/\sqrt{n}, 0 \le t \le 1\}$ converges weakly to the Brownian motion $\{\xi(t), 0 \le t \le 1\}$. See also I.I. Gikhman and A.V. Skorokhod [16], pp. 490-495. By using an argument of F. Knight [24], we can prove that

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{2\tau_n([\alpha\sqrt{n}])}{\sqrt{n}} \le x\right\} = \mathbf{P}\{\tau(\alpha) \le x\} \tag{12}$$

for any α and $x > 0$. By calculating the left-hand side of (12) and letting $n \to \infty$ we obtain that

$$\mathbf{P}\{\tau(\alpha) \le x\} = 2\Phi(|\alpha| + x) - 1 \tag{13}$$

for $x \ge 0$. Hence it follows that

$$\mathbf{E}\{[\tau(\alpha)]^r\} = m_r(\alpha) = 2J_r(\alpha)/\sqrt{2\pi} \tag{14}$$

for $\alpha > 0$ and $r \ge 1$.

We shall prove also that

$$\lim_{n\to\infty} \mathbf{E}\left\{\left(\frac{2\tau_n([\alpha\sqrt{n}])}{\sqrt{n}}\right)^r\right\} = m_r(\alpha) \tag{15}$$

for $r = 1, 2, \ldots$.

The reflecting Brownian motion: We use the same notations as we used for the Brownian motion. The reflecting Brownian motion process is defined as $\{|\xi(t)|, 0 \le t \le 1\}$. Its local time at level $\alpha > 0$ is $\tau(\alpha) + \tau(-\alpha)$. In the same way as (12) we can prove that

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{2\tau_n([\alpha\sqrt{n}]) + 2\tau_n(-[\alpha\sqrt{n}])}{\sqrt{n}} \le x\right\} = \mathbf{P}\{\tau(\alpha) + \tau(-\alpha) \le x\} \tag{16}$$

for $\alpha > 0$ and $x > 0$. By calculating the moments of $\tau_n(a) + \tau_n(-a)$ and determining their limit behavior as $n \to \infty$ we obtain that

$$\mathbf{E}\{[\tau(\alpha) + \tau(-\alpha)]^r\} = 2\sum_{\ell=1}^{r} \binom{r-1}{\ell-1} m_r((2\ell-1)\alpha) \tag{17}$$

for $\alpha > 0$ and $r \ge 1$ where $m_r(\alpha)$ is defined by (14). By using (17) we shall prove that

$$\begin{aligned} \mathbf{P}\{\tau(\alpha) + \tau(-\alpha) \le x\} &= 1 - 4\sum_{\ell=1}^{\infty} (-1)^{\ell-1}[1 - \Phi((2\ell-1)\alpha + x)] \\ &\quad + 4\sum_{\ell=2}^{\infty} \sum_{j=1}^{\ell-1} \binom{\ell-1}{j} \frac{(-1)^{\ell-1}x^j}{j!} \varphi^{(j-1)}((2\ell-1)\alpha + x) \end{aligned} \tag{18}$$

if $x \ge 0$ and $\alpha > 0$.

The Brownian meander: Let $\{\xi^+(t), 0 \le t \le 1\}$ be a standard Brownian meander. The *Brownian meander* is a Markov process for which $\mathbf{P}\{\xi^+(0) = 0\} = 1$ and $\mathbf{P}\{\xi^+(t) \ge 0\} = 1$ for $0 \le t \le 1$. If $0 < t \le 1$, then $\xi^+(t)$ has a density function $f(t,x)$. Obviously, $f(t,x) = 0$ if $x < 0$. If $0 < t < 1$ and $x > 0$, we have

$$f(t,x) = \sqrt{\frac{2\pi}{t}}\varphi\left(\frac{x}{\sqrt{t}}\right)\frac{x}{t}\left[2\Phi\left(\frac{x}{\sqrt{1-t}}\right) - 1\right]. \tag{19}$$

The Brownian meander is the subject of the papers by B. Belkin [2], [3], E. Bolthausen

[5], D.L. Iglehart [18], R.T. Durrett and D.L. Iglehart [12], R.T. Durrett, D.L. Iglehart and D.R. Miller [13], W.D. Kaigh [20], [21], [22], K.L. Chung [7] and E. Csáki and S.G. Mohanty [9], [10]. Let us define the local time $\tau^+(\alpha)$ for $\alpha \geq 0$ by

$$\tau^+(\alpha) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \text{ measure } \{t: \alpha \leq \xi^+(t) < \alpha + \epsilon, 0 \leq t \leq 1\}. \tag{20}$$

The limit (20) exists with probability one and $\tau^+(\alpha)$ is a nonnegative random variable.

Let us define a random walk in the following way: Denote by S_n the set of sequences consisting of n elements such that each element may be either $+1$ or -1 and the sum of the first i elements is ≥ 0 for every $i = 1, 2, \ldots, n$. The number of such sequences is

$$|S_n| = \binom{n}{[n/2]} \tag{21}$$

for $n = 1, 2, \ldots$. Let us choose a sequence at random in S_n, assuming that all the possible choices are equally probable. Denote by ζ_i^+ $(i = 1, 2, \ldots, n)$ the sum of the first i elements in a random sequence and set $\zeta_0^+ = 0$. The sequence $\{\zeta_i^+,\ 0 \leq i \leq n\}$ describes a random walk which is usually called a *Bernoulli meander*. We are concerned with the random variable $\tau_n^+(a)$ defined as follows: $\tau_n^+(a) =$ the number of subscripts $i = 1, 2, \ldots, n$ for which $\zeta_{i-1}^+ = a - 1$ and $\zeta_i^+ = a$ where $n \geq 1$ and $a \geq 1$.

If $n \to \infty$, the process $\{\zeta_{[nt]}^+/\sqrt{n}, 0 \leq t \leq 1\}$ converges weakly to the Brownian meander $\{\xi^+(t), 0 \leq t \leq 1\}$ and in the same way as (12) we can prove that

$$\lim_{n \to \infty} \mathbf{P}\left\{\frac{2\tau_n^+([\alpha\sqrt{n}])}{\sqrt{n}} \leq x\right\} = \mathbf{P}\{\tau^+(\alpha) \leq x\} \tag{22}$$

for any $\alpha \geq 0$ and $x > 0$. By calculating the moments of $\tau_n^+(a)$ and determining their limit behavior as $n \to \infty$ we obtain that

$$\mathbf{E}\{\tau^+(\alpha)\} = 2\sqrt{2\pi}[\Phi(2\alpha) - \Phi(\alpha)] \tag{23}$$

and

$$\mathbf{E}\{[\tau^+(\alpha)]^r\} = r\sqrt{2\pi} \sum_{\ell=1}^{r} (-1)^{\ell-1}\binom{r-1}{\ell-1}[m_{r-1}((2\ell-1)\alpha) - m_{r-1}(2\ell\alpha)] \tag{24}$$

for $r \geq 2$ where $m_r(\alpha)$ is defined by (14). By (23) and (24) we obtain that

$$\mathbf{P}\{\tau^+(\alpha) \leq x\} = 1 + 2\sqrt{2\pi} \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{x^j}{j!}[\varphi^{(j)}(2k\alpha + x) - \varphi^{(j)}((2k-1)\alpha + x)] \tag{25}$$

for $x \geq 0$.

The Brownian bridge: Let $\{\eta(t), 0 \leq t \leq 1\}$ be a *standard Brownian bridge*. We have $\mathbf{P}\{\eta(t) \leq x\} = \Phi(x/\sqrt{t(1-t)})$ for $0 < t < 1$. Let us define

$$\omega(\alpha) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \text{ measure } \{t: \alpha \leq \eta(t) < \alpha + \epsilon, 0 \leq t \leq 1\} \tag{26}$$

for any real α. The limit (26) exists with probability one and $\omega(\alpha)$ is a nonnegative random variable which is called the *local time at level* α.

We consider a tied-down random walk defined in the following way: Let us denote by T_{2n} the set of sequences consisting of $2n$ elements such that n elements are equal to $+1$ and n elements are equal to -1. The number of such sequences is $|T_{2n}| = \binom{2n}{n}$. Let us choose a sequence at random, assuming that all the possible $\binom{2n}{n}$ choices are equally probable. Denote by η_r $(r = 1, 2, \ldots, 2n)$ the sum of the first r elements in a random sequence and set $\eta_0 = 0$. The sequence $\{\eta_0, \eta_1, \ldots, \eta_{2n}\}$ describes a random walk which is called a *tied-down random walk.* In this random walk $\eta_{2n} = \eta_0 = 0$. The stochastic law of the tied-down random walk $\{\eta_0, \eta_1, \ldots, \eta_{2n}\}$ is identical to the stochastic law of the symmetric random walk $\{\zeta_0, \zeta_1, \ldots, \zeta_{2n}\}$ under the condition that $\zeta_{2n} = 0$.

Let us define $\omega_{2n}(a)$ $(a = 1, 2, \ldots)$ as the number of subscripts $r = 1, 2, \ldots, 2n$ for which $\eta_{r-1} = a - 1$ and $\eta_r = a$. If $a \geq 1$, $\eta_{r-1} = a - 1$ and $\eta_r = a$, then we say that in the tied-down random walk a transition $a - 1 \to a$ occurs at the rth step and $\omega_{2n}(a)$ is the number of transitions $a - 1 \to a$ in the $2n$ steps. In a similar way we define $\omega_{2n}(-a)$ $(a = 1, 2, \ldots, 2n)$ as the number of subscripts $r = 1, 2, \ldots, 2n$ for which $\eta_{r-1} = -a + 1$ and $\eta_r = -a$, that is, $\omega_{2n}(-a)$ is the number of transitions $-a + 1 \to -a$ in the $2n$ steps.

By the results of M. Donsker [11], if $n \to \infty$, the process $\{\eta_{[2nt]}/\sqrt{2n}, 0 \leq t \leq 1\}$ converges weakly to the Brownian bridge $\{\eta(t), 0 \leq t \leq 1\}$. See also I.I. Gikhman and A.V. Skorokhod [16], pp. 490-495. By using an argument similar to the one F. Knight [24] used for the symmetric random walk, we can prove that

$$\lim_{n \to \infty} \mathbf{P}\left\{\frac{2\omega_{2n}([\alpha\sqrt{2n}])}{\sqrt{2n}} \leq x\right\} = \mathbf{P}\{\omega(\alpha) \leq x\} \tag{27}$$

for any $\alpha \geq 1$ and $x > 0$. By calculating the left-hand side of (27) and letting $n \to \infty$ we obtain that

$$\mathbf{P}\{\omega(\alpha) \leq x\} = 1 - e^{-(2|\alpha| + x)^2/2} \tag{28}$$

for $x \geq 0$. Hence it follows that

$$\mathbf{E}\{[\omega(\alpha)]^r\} = \mu_r(\alpha) = rJ_{r-1}(2\alpha) \tag{29}$$

for $\alpha > 0$ and $r \geq 1$. We shall prove also that

$$\lim_{n \to \infty} \mathbf{E}\left\{\left(\frac{2\omega_{2n}([\alpha\sqrt{2n}])}{\sqrt{2n}}\right)^r\right\} = \mu_r(\alpha) \tag{30}$$

for $\alpha > 0$ and $r = 1, 2, \ldots$.

The reflecting Brownian bridge: We use the same notations as we used for the Brownian bridge. The reflecting Brownian bridge is defined as $\{|\eta(t)|, 0 \leq t \leq 1\}$. Its local time at level $\alpha > 0$ is $\omega(\alpha) + \omega(-\alpha)$. In the same way as (27) we can prove that

$$\lim_{n \to \infty} \mathbf{P}\left\{\frac{2\omega_{2n}([\alpha\sqrt{2n}]) + 2\omega_{2n}(-[\alpha\sqrt{2n}])}{\sqrt{2n}} \leq x\right\} = \mathbf{P}\{\omega(\alpha) + \omega(-\alpha) \leq x\} \tag{31}$$

for $\alpha > 0$ and $x > 0$. By calculating the moments of $\omega_{2n}(a) + \omega_{2n}(-a)$ and determining their limit behavior as $n \to \infty$ we obtain that

$$\mathbf{E}\{[\omega(\alpha) + \omega(-\alpha)]^r\} = 2\sum_{\ell=1}^{r} \binom{r-1}{\ell-1} \mu_r(\ell\alpha) \tag{32}$$

where $\mu_r(\alpha)$ is defined by (29). By using (32) we shall prove that

$$\mathbf{P}\{\omega(\alpha) + \omega(-\alpha) \leq x\} = 1 - 2\sqrt{2\pi} \sum_{\ell=1}^{\infty} \sum_{j=0}^{\ell-1} \binom{\ell-1}{j} \frac{(-1)^{\ell-1} x^j}{j!} \varphi^{(j)}(2\ell\alpha + x) \tag{33}$$

if $x \geq 0$ and $\alpha > 0$.

The Brownian excursion: Let $\{\eta^+(t), 0 \leq t \leq 1\}$ be a standard Brownian excursion. The *Brownian excursion* is a Markov process for which $\mathbf{P}\{\eta^+(0) = 0\} = 1$, $\mathbf{P}\{\eta^+(1) = 0\} = 1$ and $\mathbf{P}\{\eta^+(t) \geq 0\} = 1$ for $0 \leq t \leq 1$. If $0 < t < 1$, then $\eta^+(t)$ has a density function $g(t, x)$. Obviously, $g(t,x)=0$ if $x<0$. If $0<t<1$ and $x>0$, we have

$$g(t,x)=\frac{2x^2}{\sqrt{2\pi t^3(1-t)^3}}e^{-x^2/(2t(1-t))}. \tag{34}$$

Let us define the local time $\omega^+(\alpha)$ for $\alpha\geq 0$ by

$$\omega^+(\alpha)=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\text{ measure }\{t:\alpha\leq\eta^+(t)<\alpha+\epsilon, 0\leq t\leq 1\}. \tag{35}$$

The limit (35) exists with probability one and $\omega^+(\alpha)$ is a nonnegative random variable.

Let us define a random walk in the following way: Let us consider a set of sequences each consisting of $2n$ elements such that n elements are equal to $+1$, n elements are equal to -1, and the sum of the first i elements is ≥ 0 for every $i=1,2,\ldots,2n$. The number of such sequences is given by the nth Catalan number,

$$C_n=\binom{2n}{n}\frac{1}{n+1}. \tag{36}$$

We have $C_0=1$, $C_1=1$, $C_2=2$, $C_3=5$, $C_4=14$, $C_5=42$, ... Let us choose a sequence at a random, assuming that all the possible C_n sequences are equally probable. Denote by η_i^+ ($i=1, 2,\ldots,2n$) the sum of the first i elements in a random sequence and set $\eta_0^+=0$. The sequence $\{\eta_0^+, \eta_1^+,\ldots,\eta_{2n}^+\}$ describes a random walk which is usually called a Bernoulli excursion. We have $\eta_{2n}^+=\eta_0^+=0$. If $n\to\infty$, then the finite-dimensional distributions of the process $\{\eta_{[2nt]}^+/\sqrt{2n}, 0\leq t\leq 1\}$ converge to the corresponding finite-dimensional distributions of the Brownian excursion process $\{\eta^+(t), 0\leq t\leq 1\}$, and we have weak convergence too.

In the same way as (27) we can prove that

$$\lim_{n\to\infty}\mathbf{P}\left\{\frac{2\omega_{2n}^+([\alpha\sqrt{2n}])}{\sqrt{2n}}\leq x\right\}=\mathbf{P}\{\omega^+(\alpha)\leq x\} \tag{37}$$

for any $\alpha\geq 0$ and $x>0$. By calculating the moments of $\omega_{2n}^+(a)$ and determining their limit behavior as $n\to\infty$ we obtain that

$$\mathbf{E}\{[\omega^+(\alpha)]^r\}=2r(r-1)\sum_{\ell=1}^{r}(-1)^{\ell-1}\binom{r-1}{\ell-1}\mu_{r-2}(\ell\alpha) \tag{38}$$

if $r\geq 3$ where $\mu_r(\alpha)$ is defined by (29). If $r=1$, we have

$$\mathbf{E}\{\omega^+(\alpha)\}=4\alpha e^{-2\alpha^2} \tag{39}$$

and if $r=2$,

$$\mathbf{E}\{[\omega^+(\alpha)]^2\}=4[e^{-2\alpha^2}-e^{-8\alpha^2}]. \tag{40}$$

By (38), (39) and (40) we obtain that

$$\mathbf{P}\{\omega^+(\alpha)\leq x\}=1-2\sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{x^j}{j!}\varphi^{(j+2)}(2k\alpha+x) \tag{41}$$

for $x\geq 0$.

2. Recurrent Events

In proving the various results mentioned in the Introduction, we shall make frequent use of the notion of recurrent events. This section contains the basic material needed.

Let us suppose that in the time interval $(0,\infty)$ events occur at random at times $\theta_1 + \theta_2 + \ldots + \theta_k$ $(k = 1,2,\ldots)$ where $\{\theta_k\}$ is a sequence of independent random variables which take on positive integers only. The variables θ_k $(k = 2,3,\ldots)$ are identically distributed, but θ_1 may have a different distribution. Let

$$\sum_{n=1}^{\infty} \mathbf{P}\{\theta_1 = n\} z^n = \psi^*(z) \tag{42}$$

and

$$\sum_{n=1}^{\infty} \mathbf{P}\{\theta_k = n\} z^n = \psi(z) \tag{43}$$

for $k = 2,3,\ldots$ and $|z| \leq 1$. Denote by ν_n the number of events occurring in the time interval $(0,n]$, and let $\nu_0 = 0$. Clearly,

$$\{\nu_n \geq k\} \equiv \{\theta_1 + \ldots + \theta_k \leq n\} \tag{44}$$

for $k = 1,2,\ldots$ and $n = 1,2,\ldots$. The expectation

$$\mathbf{E}\left\{\binom{\nu_n}{r}\right\} = \sum_{k=r}^{\infty} \binom{k}{r} \mathbf{P}\{\nu_n = k\} \tag{45}$$

is called the rth binomial moment of ν_n and by (44) we have

$$(1-z)\sum_{n=1}^{\infty} \mathbf{E}\left\{\binom{\nu_n}{r}\right\} z^n = \frac{\psi^*(z)[\psi(z)]^{r-1}}{[1-\psi(z)]^r} \tag{46}$$

for $r = 1,2,\ldots$ and $|z| < 1$. By (44)

$$(1-z)\sum_{n=1}^{\infty} \mathbf{P}\{\nu_n \geq k\} z^n = \psi^*(z)[\psi(z)]^{k-1} \tag{47}$$

if $k \geq 1$ and $|z|$, or

$$(1-z)\sum_{n=1}^{\infty} \mathbf{P}\{\nu_n = k\} z^n = \psi^*(z)[1-\psi(z)][\psi(z)]^{k-1} \tag{48}$$

if $k \geq 1$ and $|z| < 1$. If we multiply (48) by $\binom{k}{r}$ and add the product for $k = 1,2,\ldots$, we obtain (46).

We note that if A_n denotes the event that a random event occurs at time n, then $\mathbf{P}\{A_n\} = \mathbf{E}\{\nu_n\} - \mathbf{E}\{\nu_{n-1}\}$ for $n = 1,2,\ldots$ and if $r = 1$ in (46), we obtain that

$$\sum_{n=1}^{\infty} \mathbf{P}\{A_n\} z^n = \frac{\psi^*(z)}{1-\psi(z)} \tag{49}$$

for $|z| < 1$. We say that $\{A_n\}$ is a sequence of recurrent events.

3. A Symmetric Random Walk

Let us recall some results for the symmetric random walk $\{\zeta_r, r \geq 0\}$ which we need in this paper. See Takács [34], [35]. We have

$$\mathbf{P}\{\zeta_n = 2j - n\} = \binom{n}{j}\frac{1}{2^n} \tag{50}$$

for $j = 0, 1, \ldots, n$ and by the central limit theorem

$$\lim_{n\to\infty} \mathbf{P}\{\frac{\zeta_n}{\sqrt{n}} \leq x\} = \Phi(x) \tag{51}$$

where $\Phi(x)$ is defined by (2). We have also

$$\lim_{n\to\infty} \mathbf{E}\{(\frac{\zeta_n}{\sqrt{n}})^r\} = \int_{-\infty}^{\infty} x^r \varphi(x)dx \tag{52}$$

for $r = 0, 1, 2, \ldots$ where $\varphi(x)$ is defined by (1).

Let us define $\rho(a)$ as the first passage time through a, that is,

$$\rho(a) = \inf\{r: \zeta_r = a \text{ and } r \geq 0\}. \tag{53}$$

We have

$$\mathbf{P}\{\rho(a) = a + 2j\} = \frac{a}{a+2j}\binom{a+2j}{j}\frac{1}{2^{a+2j}} = \left[\binom{a+2j-1}{j} - \binom{a+2j-1}{j-1}\right]\frac{1}{2^{a+2j}} \tag{54}$$

for $a \geq 1$ and $j \geq 0$. By the reflection principle it is evident that

$$\mathbf{P}\{\rho(a) = n\} = \tfrac{1}{2}[\mathbf{P}\{\zeta_{n-1} = a-1\} - \mathbf{P}\{\zeta_{n-1} = a+1\}] \tag{55}$$

for $a \geq 1$ and $n \geq 1$.

By (54)

$$\sum_{n=a}^{\infty} \mathbf{P}\{\rho(a) = n\}z^n = [\gamma(z)]^a \tag{56}$$

for $a \geq 1$ and $|z| \leq 1$ where

$$\gamma(z) = \frac{1 - \sqrt{1-z^2}}{z}. \tag{57}$$

We note that the identity

$$\sum_{j=0}^{n} \mathbf{P}\{\rho(a) = j\}\mathbf{P}\{\rho(b) = n-j\} = \mathbf{P}\{\rho(a+b) = n\} \tag{58}$$

is valid for any $a \geq 1$, $b \geq 1$ and $n \geq 1$.

Furthermore, we have

$$\sum_{n=r}^{\infty}\binom{n}{r}z^n = \frac{z^r}{(1-z)^{r+1}} \tag{59}$$

for $r = 0, 1, 2, \ldots$ and $|z| < 1$.

The symmetric random walk $\{\zeta_n, n \geq 0\}$ provides a simple example for recurrent events. Let us define $A_n = \{\zeta_n = a\}$ for $n \geq 1$ and $a \geq 1$. Then $\{A_n\}$ is a sequence of recurrent events. In this sequence θ_1 has the same distribution as $\rho(a)$ and θ_k $(k = 2, 3, \ldots)$ has the same distribution as $\rho(1) + 1$. Consequently, by (49)

$$\sum_{n=a}^{\infty} \mathbf{P}\{\zeta_n = a\}z^n = \frac{[\gamma(z)]^a}{1 - z\gamma(z)} = \frac{2[\gamma(z)]^{a+1}}{(1 - [\gamma(z)]^2)z} \tag{60}$$

for $|z| < 1$.

4. The Brownian Motion

Let us consider the symmetric random walk $\{\zeta_r, r \geq 0\}$ and denote by A_n the event that $\zeta_{n-1} = a-1$ and $\zeta_n = a$ where $n \geq 1$ and $a \geq 1$. Then $\{A_n\}$ is a sequence of recurrent events for which θ_1 has the same distribution as $\rho(a)$ and θ_k $(k \geq 2)$ has the same distribution as $\rho(2)$. Now $\tau_n(a)$ is the number of events occurring among $A_1, A_2, \ldots, A_n$. Thus

$$\begin{aligned} \mathbf{P}\{\tau_n(a) > k\} &= \mathbf{P}\{\theta_1 + \ldots + \theta_{k+1} \leq n\} = \mathbf{P}\{\rho(a+2k) \leq n\} \\ &= \mathbf{P}\{\zeta_n \geq a+2k\} + \mathbf{P}\{\zeta_n > a+2k\} \end{aligned} \tag{61}$$

for $k \geq 1$ and $a \geq 1$ and

$$\mathbf{P}\{\tau_n(a) = 0\} = \mathbf{P}\{\rho(a) > n\} = \mathbf{P}\{-a \leq \zeta_n < a\}. \tag{62}$$

By symmetry, $\tau_n(-a)$ has the same distribution as $\tau_n(a)$. By (61) we obtain that

$$\mathbf{E}\{[2\tau_n(a)]^r\} = 2\mathbf{E}\{([\zeta_n - a]^+)^r\} \tag{63}$$

if $n+a$ is odd and $r \geq 1$, and

$$\mathbf{E}\{[2\tau_n(a)]^r\} = \mathbf{E}\{([\zeta_n - a]^+)^r\} + \mathbf{E}\{([\zeta_n - a + 2]^+)^r\} \tag{64}$$

if $n+a$ is even and $r \geq 1$. Here $[x]^+ = x$ for $x \geq 0$ and $[x]^+ = 0$ for $x < 0$.

Theorem 1: *If $\alpha > 0$ and $x > 0$, then*

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{2\tau_n([\alpha\sqrt{n}])}{\sqrt{n}} \geq x\right\} = 2[1 - \Phi(\alpha + x)] \tag{65}$$

and

$$\lim_{n\to\infty} \mathbf{E}\left\{\left[\frac{2\tau_n([\alpha\sqrt{n}])}{\sqrt{n}}\right]^r\right\} = m_r(\alpha) \tag{66}$$

for $r \geq 1$ where $m_r(\alpha)$ is given by (14).

Proof: If in (61) we put $a = [\alpha\sqrt{n}]$ where $\alpha > 0$ and $k = [x\sqrt{n}/2]$ where $x > 0$, then by (51) we obtain (65). By (12), (65) proves (13) also. If $a = [\alpha\sqrt{n}]$ where $\alpha > 0$, then by (63) and (64) we obtain that

$$\mathbf{E}\left\{\left[\frac{2\tau_n([\alpha\sqrt{n})]}{\sqrt{n}}\right]^r\right\} \sim 2\mathbf{E}\left\{\left(\left[\frac{\zeta_n}{\sqrt{n}} - a\right]^+\right)^r\right\} \tag{67}$$

as $n \to \infty$. By (52)

$$\lim_{n\to\infty} \mathbf{E}\left\{\left[\frac{2\tau_n([\alpha\sqrt{n}])}{\sqrt{n}}\right]^r\right\} = 2\mathbf{E}\{([\xi - \alpha]^+)^r\} \tag{68}$$

for $r \geq 1$ where $\mathbf{P}\{\xi \leq x\} = \Phi(x)$. This proves (15).

In the sequence of recurrent events $\{A_n\}$ we have $\psi^*(z) = [\gamma(z)]^a$ and $\psi(z) = [\gamma(z)]^2$ where $\gamma(z)$ is defined by (57). Thus be (46) we obtain that

$$(1-z)\sum_{n=a}^{\infty} \mathbf{E}\left\{\binom{\tau_n(a)}{r}\right\} z^n = \frac{[\gamma(z)]^{a+2(r-1)}}{(1-[\gamma(z)]^2)^r} \tag{69}$$

for $r \geq 1$, $a \geq 1$ and $|z| < 1$.

5. The Reflecting Brownian Motion

The distribution of $\tau_n(a) + \tau_n(-a)$ is determined by its binomial moments. We have

$$\mathbf{P}\{\tau_n(a) + \tau_n(-a) = k\} = \sum_{r=k}^{n} (-1)^{r-k}\binom{r}{k}\mathbf{E}\left\{\binom{\tau_n(a) + \tau_n(-a)}{r}\right\} \tag{70}$$

for $k = 0, 1, 2, \ldots, n$.

Theorem 2: *If $r = 1, 2, \ldots$ and $a = 1, 2, \ldots$, we have*

$$\mathbf{E}\left\{\binom{\tau_n(a) + \tau_n(-a)}{r}\right\} = 2\sum_{\ell=1}^{r}\binom{r-1}{\ell-1}\mathbf{E}\left\{\binom{\tau_n((2\ell-1)a - 2\ell + 2)}{r}\right\} \tag{71}$$

where the right-hand side is determined by (69).

Proof: Let us consider the symmetric random walk $\{\zeta_r, r \geq 0\}$ and denote by A_n the event that $|\zeta_{n-1}| = a-1$ and $|\zeta_n| = a$ where $n \geq 1$ and $a \geq 1$. Then $\{A_n\}$ is a sequence of recurrent events and $\tau_n(a) + \tau_n(-a)$ is the number of events occurring among $A_1, A_2, \ldots, A_n$. The generating function of θ_1 is denoted by $\psi^*(z)$ and the generating function of θ_k $(k \geq 2)$ by $\psi(z)$. Now we shall determine these generating functions. By (46)

$$(1-z)\sum_{n=a}^{\infty}\mathbf{E}\left\{\binom{\tau_n(a) + \tau_n(-a)}{r}\right\}z^n = \frac{\psi^*(z)[\psi(z)]^{r-1}}{[1-\psi(z)]^r} \tag{72}$$

for $r = 1, 2, \ldots$ and $|z| < 1$. By symmetry, $\mathbf{E}\{\tau_n(-a)\} = \mathbf{E}\{\tau_n(a)\}$ and if $r = 1$, then by (69), (72) reduces to

$$2(1-z)\sum_{n=a}^{\infty}\mathbf{E}\{\tau_n(a)\}z^n = \frac{\psi^*(z)}{1-\psi(z)} = \frac{2[\gamma(z)]^{2a}}{1-[\gamma(z)]^2}. \tag{73}$$

In order to determine $\psi(z)$ let us consider again the symmetric random walk $\{\zeta_r, r \geq 0\}$ and now define an event B_n in such a way that it occurs if either $\zeta_{n-1} = 1$ and $\zeta_n = 0$ or $\zeta_{n-1} = 2a-1$ and $\zeta_n = 2a$. Then $\{B_n\}$ is a sequence of recurrent events for which $\psi^*(z) = \psi(z)$ where $\psi(z)$ has the same meaning as above. Now by (49)

$$\sum_{n=1}^{\infty}\mathbf{P}\{B_n\}z^n = \frac{1}{2}\sum_{n=1}^{\infty}[\mathbf{P}\{\zeta_{n-1} = 1\} + \mathbf{P}\{\zeta_{n-1} = 2a-1\}]z^n = \frac{\psi(z)}{1-\psi(z)}. \tag{74}$$

Thus by (60) we obtain that

$$\frac{\psi(z)}{1-\psi(z)} = \frac{z\gamma(z) + z[\gamma(z)]^{2a-1}}{2[1-z\gamma(z)]} = \frac{[\gamma(z)]^2 + [\gamma(z)]^{2a}}{1+[\gamma(z)]^2}. \tag{75}$$

The two equations (73) and (75) determine $\psi(z)$ and $\psi^*(z)$. By (75) we have

$$\psi(z) = \frac{[\gamma(z)]^{2a} + [\gamma(z)]^2}{1+[\gamma(z)]^{2a}} \tag{76}$$

and by (73)

$$\psi^*(z) = \frac{2[\gamma(z)]^a}{1+[\gamma(z)]^{2a}} \tag{77}$$

for $|z| < 1$. Thus by (72)

$$(1-z)\sum_{n=a}^{\infty}\mathbf{E}\left\{\binom{\tau_n(a) + \tau_n(-a)}{r}\right\}z^n = \frac{2[\gamma(z)]^a([\gamma(z)]^{2a} + [\gamma(z)]^2)^{r-1}}{(1-[\gamma(z)]^2)^r}$$

$$= 2\sum_{\ell=1}^{r} \binom{r-1}{\ell-1} \frac{[\gamma(z)]^{(2\ell-1)a+2r-2\ell}}{(1-[\gamma(z)]^2)^r}. \tag{78}$$

In the above sum, each term can be expressed in the form of (69) and thus we get (71).

Now we shall prove some limit theorems:

Theorem 3: *If* $\alpha > 0$ *and* $r \geq 1$, *then the limit*

$$\lim_{n\to\infty} \mathbf{E}\left\{\left[\frac{2\tau_n([\alpha\sqrt{n}]) + 2\tau_n(-[\alpha\sqrt{n}])}{\sqrt{n}}\right]^r\right\} = M_r(\alpha) \tag{79}$$

exists and

$$M_r(\alpha) = 2\sum_{\ell=1}^{r} \binom{r-1}{\ell-1} m_r((2\ell-1)\alpha) \tag{80}$$

where $m_r(\alpha)$ *is defined by* (14).

Proof: If in (71) we put $a = [\alpha\sqrt{n}]$ where $\alpha > 0$ and let $n\to\infty$, then by (66) we get (80).

Theorem 4: *IF* $\alpha > 0$, *then there exists a distribution function* $L_\alpha(x)$ *of a nonnegative random variable such that*

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{2\tau_n([\alpha\sqrt{n}]) + 2\tau_n(-[\alpha\sqrt{n}])}{\sqrt{n}} \leq x\right\} = L_\alpha(x) \tag{81}$$

in every continuity point of $L_\alpha(x)$. *The distribution function* $L_\alpha(x)$ *is uniquely determined by its moments*

$$\int_{-0}^{\infty} x^r dL_\alpha(x) = M_r(\alpha) \tag{82}$$

for $r \geq 0$ *where* $M_0(\alpha) = 1$ *and* $M_r(\alpha)$ *for* $r \geq 1$ *is given by* (80).

Proof: Since

$$M_r(\alpha) \leq 2^{r+1}\mathbf{E}\{\mid \xi \mid^r\} \tag{83}$$

for $r \geq 1$ where $\mathbf{P}\{\xi \leq x\} = \Phi(x)$, the sequence of moments $\{M_r(\alpha)\}$ uniquely determines $L_\alpha(x)$, and $L_\alpha(x) = 0$ for $x < 0$. By the moment convergence theorem of M. Fréchet and J. Shohat [14] we can conclude that (82) implies (81).

By (16) it follows from (81) that

$$\mathbf{P}\{\tau(\alpha) + \tau(-\alpha) \leq x\} = L_\alpha(x) \tag{84}$$

where $L_\alpha(x)$ is determined by (82). From (82) it follows that (17) is true for all $r \geq 1$. Formula (17) is a surprisingly simple expression for the rth moment of $\tau(\alpha) + \tau(-\alpha)$. If we know the rth moment of $\tau(\alpha)$ for $\alpha > 0$, then by (17) the rth moment of $\tau(\alpha) + \tau(-\alpha)$ can immediately be determined for $\alpha > 0$. Moreover, formula (17) makes it possible to determine $L_\alpha(x)$ explicitly.

Theorem 5: *If* $x \geq 0$ *and* $\alpha > 0$, *we have*

$$L_\alpha(x) = 1 - 4\sum_{\ell=1}^{\infty} (-1)^{\ell-1}[1 - \Phi((2\ell-1)\alpha + x)] \tag{85}$$

$$+4\sum_{\ell=2}^{\infty}\sum_{j=1}^{\ell-1}\binom{\ell-1}{j}\frac{(-1)^{\ell-1}x^j}{j!}\varphi^{(j-1)}((2\ell-1)\alpha+x).$$

Proof: For $\alpha>0$ the Laplace-Stieltjes transform

$$\Psi_\alpha(s)=\int_{-0}^{\infty}e^{-sx}dL_\alpha(x) \tag{86}$$

can be expressed as

$$\Psi_\alpha(s)=\sum_{r=0}^{\infty}(-1)^r M_r(\alpha)s^r/r! \tag{87}$$

and the series is convergent on the whole complex plane. In (87) $M_r(\alpha)$ is given by (80). If we put (80) into (87), express $m_r(\alpha)$ by (14) and (6) and interchange summations with respect to r and ℓ, we obtain that

$$\Psi_\alpha(s)=1+4\sum_{\ell=1}^{\infty}\frac{1}{(\ell-1)!}\int_0^{\infty}\left(\frac{d^\ell e^{-sx}}{dx^\ell}\right)[1-\Phi((2\ell-1)\alpha+x)]x^{\ell-1}dx$$

$$=1+4\sum_{\ell=1}^{\infty}(-1)^\ell[1-\Phi((2\ell-1)\alpha)] \tag{88}$$

$$+4\sum_{\ell=1}^{\infty}\frac{(-1)^\ell}{(\ell-1)!}\int_0^{\infty}e^{-sx}\left(\frac{d^\ell[1-\Phi((2\ell-1)\alpha+x)]x^{\ell-1}}{dx^\ell}\right)dx.$$

Hence we can conclude that

$$L_\alpha(x)=1+4\sum_{\ell=1}^{\infty}\frac{(-1)^\ell}{(\ell-1)!}\left(\frac{d^{\ell-1}[1-\Phi((2\ell-1)\alpha+x)]x^{\ell-1}}{dx^{\ell-1}}\right) \tag{89}$$

for $x\geq 0$. This proves (18) and (85).

6. The Brownian Meander

Let us consider again the symmetric random walk $\{\zeta_r, r\geq 0\}$ and define the following event:

$$A_n\equiv\{\zeta_{n-1}=a-1, \zeta_n=a \text{ and } \zeta_r\geq 0 \text{ for } 0\leq r\leq n\} \tag{90}$$

for $n\geq 1$ and $a\geq 1$. Then $\{A_n\}$ is a sequence of recurrent events. By the reflection principle we obtain that

$$\mathbf{P}\{A_n\}=\tfrac{1}{2}[\mathbf{P}\{\zeta_{n-1}=a-1\}-\mathbf{P}\{\zeta_{n-1}=a+1\}]=\mathbf{P}\{\rho(a)=n\}. \tag{91}$$

As before let us denote the generating function of θ_1 by $\psi^*(z)$ and the generating function of θ_k $(k\geq 2)$ by $\psi(z)$. Our next aim is to determine these generating functions. By (49) and (56) we have

$$\sum_{n=1}^{\infty}\mathbf{P}\{A_n\}z^n=[\gamma(z)]^a=\frac{\psi^*(z)}{1-\psi(z)}. \tag{92}$$

In order to determine $\psi(z)$ let us consider again the symmetric random walk $\{\zeta_r, r \geq 0\}$ but now define an event B_n in the following way:

$$B_n \equiv \{\zeta_{n-1} = 1, \zeta_n = 0 \text{ and } \zeta_r \leq a+1 \text{ for } 0 \leq r \leq n\}. \tag{93}$$

Then $\{B_n\}$ is again a sequence of recurrent events for which $\psi^*(z) = \psi(z)$ where $\psi(z)$ has the same meaning as in (92). By the reflection principle

$$\mathbf{P}\{B_n\} = \tfrac{1}{2}[\mathbf{P}\{\zeta_{n-1} = 1\} - \mathbf{P}\{\zeta_{n-1} = 2a+1\}] \tag{94}$$

and by (60)

$$\begin{aligned}\sum_{n=1}^{\infty} \mathbf{P}\{B_n\} z^n &= \frac{z(\gamma(z) - [\gamma(z)]^{2a+1})}{2[1 - z\gamma(z)]} \\ &= \frac{[\gamma(z)]^2(1 - [\gamma(z)]^{2a})}{1 - [\gamma(z)]^2} = \frac{\psi(z)}{1 - \psi(z)}.\end{aligned} \tag{95}$$

The two equations (92) and (95) determine $\psi(z)$ and $\psi^*(z)$. By (95) we have

$$\psi(z) = \frac{[\gamma(z)]^2(1 - [\gamma(z)]^{2a})}{1 - [\gamma(z)]^{2a+2}} \tag{96}$$

and by (92)

$$\psi^*(z) = \frac{[\gamma(z)]^a(1 - [\gamma(z)]^2)}{1 - [\gamma(z)]^{2a+2}}. \tag{97}$$

Let us consider the sequence of events $\{A_n\}$ defined by (90). We can interpret $\tau_n^+(a)$ as the number of events occurring among $A_1, A_2, \ldots, A_n$ given that $D_n = \{\zeta_r \geq 0 \text{ for } 0 \leq r \leq n\}$ occurs. Since

$$\mathbf{P}\{D_n\} = \binom{n}{[n/2]} \frac{1}{2^n}, \tag{98}$$

we obtain that

$$\mathbf{P}\{\tau_n^+(a) \geq k\} \binom{n}{[n/2]} \frac{1}{2^n} = \sum_{j=1}^{n} \mathbf{P}\{\theta_1 + \ldots + \theta_k = j\} \mathbf{P}\{\rho(a+1) > n - j\}. \tag{99}$$

Since by (56)

$$(1 - z) \sum_{j=0}^{\infty} \mathbf{P}\{\rho(a+1) > j\} z^j = 1 - [\gamma(z)]^{a+1}, \tag{100}$$

by (99) we obtain that

$$(1 - z) \sum_{n=1}^{\infty} \mathbf{P}\{\tau_n^+(a) \geq k\} \binom{n}{[n/2]} \frac{z^n}{2^n} = (1 - [\gamma(z)]^{a+1}) \psi^*(z) [\psi(z)]^{k-1}. \tag{101}$$

Since

$$\mathbf{E}\left\{\binom{\tau_n^+(a)}{r}\right\} = \sum_{k=r}^{n} \binom{k}{r} \mathbf{P}\{\tau_n^+(a) = k\} = \sum_{k=r}^{n} \binom{k-1}{r-1} \mathbf{P}\{\tau_n^+(a) \geq k\}, \tag{102}$$

by (101)

$$(1 - z) \sum_{n=1}^{\infty} \mathbf{E}\left\{\binom{\tau_n^+(a)}{r}\right\} \binom{n}{[n/2]} \frac{z^n}{2^n} = (1 - [\gamma(z)]^{a+1}) \psi^*(z) \frac{[\psi(z)]^{r-1}}{(1 - [\psi(z)])^r}$$

$$= \frac{[\gamma(z)]^{a+2r-2}(1-[\gamma(z)]^{a+1})(1-[\gamma(z)]^{2a})^{r-1}}{(1-[\gamma(z)]^2)^{r-1}} \tag{103}$$

$$= \sum_{\ell=1}^{r}(-1)^{\ell-1}\binom{r-1}{\ell-1}\frac{[\gamma(z)]^{(2\ell-1)a+2r-2}-[\gamma(z)]^{2\ell a+2r-1}}{(1-[\gamma(z)]^2)^{r-1}}$$

for $|z| < 1$. Comparing (69) and (103) we can conclude that

$$\mathbf{E}\{\tau_n^+(a)\}\binom{n}{[n/2]}\frac{1}{2^n} = \mathbf{P}\{\rho(2a+1) > n\} - \mathbf{P}\{\rho(a) > n\} \tag{104}$$

and

$$\mathbf{E}\left\{\binom{\tau_n^+(a)}{r}\right\}\binom{n}{[n/2]}\frac{1}{2^n}$$
$$= \sum_{\ell=1}^{r}(-1)^{\ell-1}\binom{r-1}{\ell-1}\left[\mathbf{E}\left\{\binom{\tau_n((2\ell-1)a+2)}{r-1}\right\} - \mathbf{E}\left\{\binom{\tau_n(2\ell a+3)}{r-1}\right\}\right] \tag{105}$$

for $r \geq 2$ where the right-hand side can be obtained by (69). The distribution of $\tau_n^+(a)$ is determined by its binomial moments. We have

$$\mathbf{P}\{\tau_n^+(a) = k\} = \sum_{r=k}^{n}(-1)^{r-k}\binom{r}{k}\,\mathbf{E}\left\{\binom{\tau_n^+(a)}{r}\right\} \tag{106}$$

for $k = 0, 1, 2, \ldots, n$. Now we shall prove the following limit theorems:

Theorem 6: *If $\alpha > 0$ and $r \geq 1$, then the limit*

$$\lim_{n\to\infty}\mathbf{E}\left\{\left[\frac{2\tau_n^+([\alpha\sqrt{n}])}{\sqrt{n}}\right]^r\right\} = M_r^+(\alpha) \tag{107}$$

exists where

$$M_1^+(\alpha) = 2\sqrt{2\pi}[\Phi(2\alpha) - \Phi(\alpha)] \tag{108}$$

and

$$M_r^+(\alpha) = r\sqrt{2\pi}\sum_{\ell=1}^{r}(-1)^{\ell-1}\binom{r-1}{\ell-1}[m_{r-1}((2\ell-1)\alpha) - m_{r-1}(2\ell\alpha)] \tag{109}$$

for $r \geq 2$. In (109) $m_r(\alpha)$ is defined by (14).

Proof: If in (104) and (105) we put $a = [\alpha\sqrt{n}]$ where $\alpha > 0$ and let $n\to\infty$, then by (66) we get (108) and (109). Here we used that

$$\binom{n}{[n/2]}\frac{1}{2^n} \sim \sqrt{\frac{2}{n\pi}} \tag{110}$$

as $n\to\infty$.

Theorem 7: *If $\alpha > 0$, then there exists a distribution function $L_\alpha^+(x)$ of a nonnegative random variable such that*

$$\lim_{n\to\infty}\mathbf{P}\left\{\frac{2\tau_n^+([\alpha\sqrt{n}])}{\sqrt{n}} \leq x\right\} = L_\alpha^+(x) \tag{111}$$

in every continuity point of $L_\alpha^+(x)$. The distribution function $L_\alpha^+(x)$ is uniquely determined by its moments

$$\int_{-0}^{\infty} x^r dL_\alpha^+(x) = M_r^+(\alpha) \tag{112}$$

for $r \geq 0$ where $M_0^+(\alpha) = 1$, $M_1^+(\alpha)$ is given by (108) and $M_r^+(\alpha)$ for $r \geq 2$ is given by (109).

Proof: Since

$$M_r^+(\alpha) \leq r\sqrt{2\pi}2^r \mathbf{E}\{|\xi|^{r-1}\} \tag{113}$$

for $r \geq 2$ where $\mathbf{P}\{\xi \leq x\} = \Phi(x)$, the sequence of moments $\{M_r^+(\alpha)\}$ uniquely determines $L_\alpha^+(x)$, and $L_\alpha^+(x) = 0$ for $x < 0$. By the moment convergence theorem of M. Fréchet and J. Shohat [14] we can conclude that (112) implies (111).

By (22) it follows from (111) that

$$\mathbf{P}\{\tau^+(\alpha) \leq x\} = L_\alpha^+(x) \tag{114}$$

where $L_\alpha^+(x)$ is determined by (112). This proves (23), and (24) for all $r \geq 2$. Formula (24) is a very simple expression for the rth moment of $\tau^+(\alpha)$. If we know the rth moment of $\tau(\alpha)$ for $\alpha > 0$, then by (24) the rth moment of $\tau^+(\alpha)$ can immediately be determined for $\alpha > 0$. Moreover, formulas (23) and (24) make it possible to determine $L_\alpha^+(x)$ explicitly.

Theorem 8: *If $x \geq 0$ and $\alpha > 0$, we have*

$$L_\alpha^+(x) = 1 + 2\sqrt{2\pi}\sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{x^j}{j!}[\varphi^{(j)}(2k\alpha + x) - \varphi^{(j)}((2k-1)\alpha + x)] \tag{115}$$

for $x \geq 0$.

Proof: For $\alpha > 0$, the Laplace-Stieltjes transform

$$\Psi_\alpha^+(s) = \int_{-0}^{\infty} e^{-sx} dL_\alpha^+(x) \tag{116}$$

can be expressed as

$$\Psi_\alpha^+(s) = \sum_{r=0}^{\infty} (-1)^r M_r^+(\alpha) s^r / r! \tag{117}$$

and the series is convergent on the whole complex plane. Here $M_r^+(\alpha)$ is given by (108) and (109). If we put (108) and (109) into (117), express $m_r(\alpha)$ by (14) and interchange summations with respect to r and ℓ, we obtain that

$$\Psi_\alpha^+(s) = 1 + \sum_{k=1}^{\infty} h(2k\alpha) + \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \int_0^{\infty} e^{-sx} \frac{d^k x^{k-1} h(2k\alpha + x)}{dx^k} dx \tag{118}$$

where

$$h(x) = 2[e^{-x^2/2} - e^{-(x-\alpha)^2/2}]. \tag{119}$$

From (118) it follows that

$$L_\alpha^+(x) = 1 + \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \frac{d^{k-1} x^{k-1} h(2k\alpha + x)}{dx^{k-1}} \tag{120}$$

for $x \geq 0$. This proves (115) and (25).

In a different setting, the distribution function $L_\alpha^+(x)$ was found by E. Csáki and S.G. Mohanty [10]. They expressed $L_\alpha^+(x)$ in the form of a complicated complex integral.

7. The Brownian Bridge

We use the same notations as in Section 4. We consider the symmetric random walk $\{\zeta_r, r \geq 0\}$ and denote by A_n the event that $\zeta_{n-1} = a-1$ and $\zeta_n = a$ where $n \geq 1$ and $a \geq 1$. Then $\{A_n\}$ is a sequence of recurrent events for which θ_1 has the same distribution as $\rho(a)$ and θ_k $(k \geq 2)$ has the same distribution as $\rho(2)$. We can interpret $\omega_{2n}(a)$ for $a \geq 1$ as the number of events occurring among $A_1, A_2, \ldots, A_{2n}$ given that $\{\zeta_{2n} = 0\}$ occurs. We have

$$\mathbf{P}\{\zeta_{2n} = 0\} = \binom{2n}{n}\frac{1}{2^{2n}} \sim \frac{1}{\sqrt{n\pi}} \tag{121}$$

as $n \to \infty$. Now

$$\mathbf{P}\{\zeta_{2n} = 0\}\mathbf{P}\{\omega_{2n}(a) > k\} = \mathbf{P}\{\zeta_{2n} = 2a + 2k\}, \tag{122}$$

or

$$\binom{2n}{n}\mathbf{P}\{\omega_{2n}(a) > k\} = \binom{2n}{n+a+k} \tag{123}$$

for $k \geq 0$, $a \geq 1$ and $n \geq 1$. For

$$\begin{aligned} &\mathbf{P}\{\zeta_{2n} = 0\}\mathbf{P}\{\omega_{2n}(a) > k\} \\ &= \sum_{j=0}^{n-a}\mathbf{P}\{\theta_1 + \ldots + \theta_{k+1} = a + 2j\}\mathbf{P}\{\zeta_{2n-a-2j} = a\} \end{aligned} \tag{124}$$

$$= \sum_{j=0}^{n-a}\mathbf{P}\{\rho(a+2k) = a + 2j\}\mathbf{P}\{\zeta_{2n-a-2j} = a\} = \mathbf{P}\{\zeta_{2n} = 2a + 2k\}.$$

We obtain the last equality if we take into consideration that the event $\zeta_{2n} = 2a + 2k$ can occur in such a way that the smallest $r = 1, 2, \ldots, 2n$ for which $\zeta_r = a + 2k$ may be $r = a + 2j$ where $j = 0, 1, \ldots, n-a$.

If we use (55), then by (122) we have

$$\mathbf{P}\{\zeta_{2n} = 0\}\mathbf{P}\{\omega_{2n}(a) = k\} = 2\mathbf{P}\{\rho(2a + 2k - 1) = 2n + 1\} \tag{125}$$

for $k \geq 1$.

By (123) we obtain that

$$\binom{2n}{n}\mathbf{E}\{[\omega_{2n}(a)]^r\} = \sum_{j=0}^{n-a}[(j+1)^r - j^r]\binom{2n}{n+a+j} \tag{126}$$

if $r \geq 1$.

For each $r = 1, 2, \ldots$ we can express (126) in a compact form. If $r = 1$, then by (126)

$$\binom{2n}{n}\mathbf{E}\{\omega_{2n}(a)\} = \sum_{j=0}^{n-a}\binom{2n}{n+a+j} = 2^{2n}\mathbf{P}\{\zeta_{2n} \geq 2a\} \tag{127}$$

or

$$\mathbf{P}\{\zeta_{2n}=0\}\mathbf{E}\{\omega_{2n}(a)\}=\mathbf{P}\{\zeta_{2n}\geq 2a\}. \tag{128}$$

For $r=2$ and $r=3$ we obtain that

$$\mathbf{P}\{\zeta_{2n}=0\}\mathbf{E}\{[\omega_{2n}(a)]^2\}=(n+a)\mathbf{P}\{\zeta_{2n}=2a\}-(2a-1)\mathbf{P}\{\zeta_{2n}\geq 2a\} \tag{129}$$

and

$$\mathbf{P}\{\zeta_{2n}=0\}\mathbf{E}\{[\omega_{2n}(a)]^3\}=(\tfrac{3n}{2}+3a^2-3a+1)\mathbf{P}\{\zeta_{2n}\geq 2a\}-\tfrac{3}{4}(n+a)(2a-1)\mathbf{P}\{\zeta_{2n}=2a\}. \tag{130}$$

In a similar way we can express (126) for all $r>3$ as a combination of the two probabilities $\mathbf{P}\{\zeta_{2n}=2a\}$ and $\mathbf{P}\{\zeta_{2n}\geq 2a\}$.

Theorem 9: *If $\alpha>0$ and $x>0$, then*

$$\lim_{n\to\infty}\mathbf{P}\left\{\frac{2\omega_{2n}([\alpha\sqrt{2n}])}{\sqrt{2n}}\leq x\right\}=1-e^{-2(2\alpha+x)^2} \tag{131}$$

and

$$\lim_{n\to\infty}\mathbf{E}\left\{\left[\frac{2\omega_{2n}([\alpha\sqrt{2n}])}{\sqrt{2n}}\right]^r\right\}=\mu_r(\alpha) \tag{132}$$

for $r\geq 1$ where $\mu_r(\alpha)$ is given by (29).

Proof: If in (123) we put $a=[\alpha\sqrt{2n}]$ where $\alpha>0$, and $k=[x\sqrt{2n}/2]$ where $x>0$, then by letting $n\to\infty$ we obtain (131). By (27), (131) proves (28). If in (128) we put $a=[\alpha\sqrt{2n}]$ where $\alpha>0$, then by letting $n\to\infty$ we obtain (132) for $r=1$. If $r\geq 2$, then we can express (126) in the following form:

$$\mathbf{P}\{\zeta_{2n}=0\}\mathbf{E}\{[\omega_{2n}(a)]^r\}=\mathbf{P}\{\zeta_{2n}\geq 2a\}+\sum_{s=1}^{r-1}\binom{r}{s}\mathbf{E}\{([\zeta_{2n}-2a]^+)^s\}2^{-s}, \tag{133}$$

and by (121), we obtain that

$$\mathbf{E}\left\{\left[\frac{2\omega_{2n}([\alpha\sqrt{2n}])}{\sqrt{2n}}\right]^r\right\}\sim r\sqrt{2\pi}\mathbf{E}\left\{\left(\left[\frac{\zeta_{2n}}{\sqrt{2n}}-2\alpha\right]^+\right)^{r-1}\right\} \tag{134}$$

as $n\to\infty$. Accordingly, for $r\geq 2$

$$\lim_{n\to\infty}\mathbf{E}\left\{\left[\frac{2\omega_{2n}([\alpha\sqrt{2n}])}{\sqrt{2n}}\right]^r\right\}=r\sqrt{2\pi}\mathbf{E}\{([\xi-2\alpha]^+)^{r-1}\} \tag{135}$$

where $\mathbf{P}\{\xi\leq x\}=\Phi(x)$. This proves (132) for $r\geq 2$ and completes the proof of (29) for $r\geq 1$ and $a>0$.

If we use (56), then by (125) we obtain that

$$\sum_{n=1}^{\infty}\mathbf{P}\{\zeta_{2n}=0\}\mathbf{P}\{\omega_{2n}(a)=k\}z^{2n+1}=\frac{2[\gamma(z)]^{2a+2k-1}}{1-[\gamma(z)]^2} \tag{136}$$

for $|z|<1$ where $\gamma(z)$ is given by (57). If we multiply both sides of (136) by $\binom{k}{r}$ and form the sum for $k\geq r$, then we obtain that

$$\sum_{n=a}^{\infty}\mathbf{E}\left\{\binom{\omega_{2n}(a)}{r}\right\}\binom{2n}{n}\frac{z^{2n+1}}{2^{2n+1}}=\frac{[\gamma(z)]^{2a+2r-1}}{(1-[\gamma(z)]^2)^{r+1}} \tag{137}$$

for $|z| < 1$, $r \geq 1$ and $a \geq 1$.

We note that

$$\mathbf{E}\{[\omega(\alpha)]^r\} = r\sqrt{2\pi}\mathbf{E}\{[\tau(2\alpha)]^{r-1}\}/2 \tag{138}$$

for $r \geq 2$ where $\tau(\alpha)$ is defined by (10).

The distribution and the asymptotic distribution of $\omega_{2n}(a)$ have already been determined by V.S. Mihalovič [29] and N.V. Smirnov [32] in the context of order statistics.

8. The Reflecting Brownian Bridge

We use the same notation as in Section 5. We consider the symmetric random walk $\{\zeta_r, r \geq 0\}$ and denote by A_n the event that $|\zeta_{n-1}| = a-1$ and $|\zeta_n| = a$ where $n \geq 1$ and $a \geq 1$. Then $\{A_n\}$ is a sequence of recurrent events for which θ_1 has the generating function $\psi^*(z)$ given by (77) and θ_k ($k \geq 2$) has the generating function $\psi(z)$ given by (76).

The distribution of $\omega_{2n}(a) + \omega_{2n}(-a)$ is determined by its binomial moments. We have

$$\mathbf{P}\{\omega_{2n}(a) + \omega_{2n}(-a) = k\} = \sum_{r=k}^{n}(-1)^{r-k}\binom{r}{k}\mathbf{E}\left\{\binom{\omega_{2n}(a) + \omega_{2n}(-a)}{r}\right\} \tag{139}$$

for $k = 0, 1, 2, \ldots, n$.

Theorem 10: *If $r = 1, 2, \ldots$ and $a = 1, 2, \ldots$, we have*

$$\mathbf{E}\left\{\binom{\omega_{2n}(a) + \omega_{2n}(-a)}{r}\right\} = 2\sum_{\ell=1}^{r}\binom{r-1}{\ell-1}\mathbf{E}\left\{\binom{\omega_{2n}(\ell a - \ell + 1)}{r}\right\} \tag{140}$$

where the right-hand side is determined by (137).

Proof: We can interpret $\omega_{2n}(a) + \omega_{2n}(-a)$ for $a \geq 1$ as the number of events occurring among $A_1, A_2, \ldots, A_{2n}$ given that $\{\zeta_{2n} = 0\}$ occurs. Thus we can write that

$$\mathbf{P}\{\omega_{2n}(a) + \omega_{2n}(-a) \geq k\}\binom{2n}{n}\frac{1}{2^{2n}} = \sum_{j=1}^{2n}\mathbf{P}\{\theta_1 + \ldots + \theta_k = j\}\mathbf{P}\{\zeta_{2n-j} = a\}. \tag{141}$$

Since by (60)

$$\sum_{n=1}^{\infty}\mathbf{P}\{\zeta_n = a\}z^{n+1} = \frac{z[\gamma(z)]^a}{1 - z\gamma(z)} = \frac{2[\gamma(z)]^{a+1}}{1 - [\gamma(z)]^2} \tag{142}$$

for $a \geq 1$ and $|z| < 1$, by (141) we obtain that

$$\sum_{n=1}^{\infty}\mathbf{P}\{\omega_{2n}(a) + \omega_{2n}(-a) \geq k\}\binom{2n}{n}\frac{z^{2n+1}}{2^{2n+1}} = \frac{[\gamma(z)]^{a+1}\psi^*(z)[\psi(z)]^{k-1}}{1 - [\gamma(z)]^2} \tag{143}$$

where $\psi(z)$ is given by (76) and $\psi^*(z)$ by (77). If we multiply (143) by $\binom{k-1}{r-1}$ and add for $k \geq r$, then we obtain that

$$\begin{aligned}\sum_{n=1}^{\infty}\mathbf{E}\left\{\binom{\omega_{2n}(a) + \omega_{2n}(-a)}{r}\right\}\binom{2n}{n}\frac{z^{2n+1}}{2^{2n+1}} &= \frac{[\gamma(z)]^{a+1}\psi^*(z)[\psi(z)]^{r-1}}{(1 - [\gamma(z)]^2)[1 - \psi(z)]^r} \\ &= \frac{2[\gamma(z)]^{2a+1}([\gamma(z)]^{2a} + [\gamma(z)]^2)^{r-1}}{(1 - [\gamma(z)]^2)^{r+1}} = 2\sum_{\ell=1}^{r}\binom{r-1}{\ell-1}\frac{[\gamma(z)]^{2\ell a + 2r - 2\ell + 1}}{(1 - [\gamma(z)]^2)^{r+1}}\end{aligned} \tag{144}$$

for $|z| < 1$. In the above sum each term can be expressed in the form of (137) and thus we get (140).

Now we shall prove some limit theorems:

Theorem 11: *If $\alpha > 0$ and $r \geq 1$, then the limit*

$$\lim_{n\to\infty} \mathbf{E}\left\{\left[\frac{2\omega_{2n}([\alpha\sqrt{2n}]) + 2\omega_{2n}(-[\alpha\sqrt{2n}])}{\sqrt{2n}}\right]^r\right\} = Q_r(\alpha) \tag{145}$$

exists and

$$Q_r(\alpha) = 2\sum_{\ell=1}^{r}\binom{r-1}{\ell-1}\mu_r(\ell\alpha) \tag{146}$$

where $\mu_r(\alpha)$ is defined by (29).

Proof: If in (140) we put $a = [\alpha\sqrt{2n}]$ where $\alpha > 0$ and let $n\to\infty$, we get (145).

Theorem 12: *If $\alpha > 0$, then there exists a distribution function $T_\alpha(x)$ of a nonnegative random variable such that*

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{2\omega_{2n}([\alpha\sqrt{2n}]) + 2\omega_{2n}(-[\alpha\sqrt{2n}])}{\sqrt{2n}} \leq x\right\} = T_\alpha(x) \tag{147}$$

in every continuity point of $T_\alpha(x)$. The distribution function $T_\alpha(x)$ is uniquely determined by its moments

$$\int_{-0}^{\infty} x^r dT_\alpha(x) = Q_r(\alpha) \tag{148}$$

for $r \geq 0$ where $Q_0(\alpha) = 1$ and $Q_r(\alpha)$ for $r \geq 1$ is given by (146).

Proof: Since

$$Q_r(\alpha) \leq 2^r\mu_r(\alpha) \leq 2^r r\int_0^\infty x^{r-1}e^{-x^2/2}dx \tag{149}$$

for $r \geq 1$, the sequence of moments $\{Q_r(\alpha)\}$ uniquely determines $T_\alpha(x)$, and $T_\alpha(x) = 0$ for $x < 0$. By the moment convergence theorem of M. Fréchet and J. Shohat [14] we can conclude that (148) implies (147). By (31) it follows from (147) that

$$\mathbf{P}\{\omega(\alpha) + \omega(-\alpha) \leq x\} = T_\alpha(x) \tag{150}$$

where $T_\alpha(x)$ is determined by (148). From (148) it follows that (32) is valid for all $r \geq 1$. If we know the rth moment of $\omega(\alpha)$ for $\alpha > 0$, then by (32) the rth moment of $\omega(\alpha) + \omega(-\alpha)$ can immediately be determined for $\alpha > 0$.

The moments (146) uniquely determine the distribution of $\omega(\alpha) + \omega(-\alpha)$ for $\alpha > 0$ and we have the following result.

Theorem 13: *If $x \geq 0$ and $\alpha > 0$, we have*

$$T_\alpha(x) = 1 - 2\sqrt{2\pi}\sum_{\ell=1}^{\infty}\sum_{j=0}^{\ell-1}\binom{\ell-1}{j}\frac{(-1)^{\ell-1}x^j}{j!}\varphi^{(j)}(2\ell\alpha + x). \tag{151}$$

Proof: For $\alpha > 0$ let us define the Laplace-Stieltjes transform of $T_\alpha(x)$ by

$$\Omega_\alpha(s) = \int_{-0}^{\infty} e^{-sx}dT_\alpha(x). \tag{152}$$

We have

$$\Omega_\alpha(s) = \sum_{r=0}^{\infty}(-1)^r Q_r(\alpha)s^r/r! \tag{153}$$

and the series is convergent on the whole complex plane. Here $Q_r(\alpha)$ is given by (146) and $\mu_r(\alpha)$ by (29). If we put (146) into (153), express $\mu_r(\alpha)$ by (29) and interchange summations with respect to r and ℓ, we obtain that

$$\begin{aligned}\Omega_\alpha(s) &= 1 + 2\sqrt{2\pi}\sum_{\ell=1}^{\infty}\frac{1}{(\ell-1)!}\int_0^{\infty}\left(\frac{d^\ell e^{-sx}}{dx^\ell}\right)\varphi(2\ell\alpha+x)x^{\ell-1}dx \\ &= 1 + 2\sqrt{2\pi}\sum_{\ell=1}^{\infty}(-1)^\ell\varphi(2\ell\alpha) + 2\sqrt{2\pi}\sum_{\ell=1}^{\infty}\frac{(-1)^\ell}{(\ell-1)!}\int_0^{\infty}e^{-sx}\left(\frac{d^\ell\varphi(2\ell\alpha+x)x^{\ell-1}}{dx^\ell}\right)dx.\end{aligned} \tag{154}$$

Hence we can conclude that

$$T_\alpha(x) = 1 + 2\sqrt{2\pi}\sum_{\ell=1}^{\infty}\frac{(-1)^\ell}{(\ell-1)!}\left(\frac{d^{\ell-1}\varphi(2\ell\alpha+x)x^{\ell-1}}{dx^{\ell-1}}\right) \tag{155}$$

for $x \geq 0$. This proves (151).

The distribution function $T_\alpha(x)$ was determined by N.V. Smirnov [32], [33] in 1939 in the context of order statistics. In 1973 in the context of random mappings G.V. Proskurin [30] also found the distribution function $T_\alpha(x)$. Recently D. Aldous and J. Pitman [1] proved the result of G.V. Proskurin [30] by using a Brownian bridge approach.

9. The Brownian Excursion

We use the same notation as in Section 6. The distribution of $\omega_{2n}^+(a)$ is determined by its binomial moments for which we have the following formulas:

Theorem 14: *If $a \geq 1$, then*

$$C_n\mathbf{E}\{\omega_{2n}^+(a)\} = 2^{2n+1}\mathbf{P}\{\rho(2a+1) = 2n+1\} \tag{156}$$

and

$$C_n\mathbf{E}\left\{\binom{\omega_{2n}^+(a)}{2}\right\} = 2^{2n}[\mathbf{P}\{\zeta_{2n} = 2a+2\} - \mathbf{P}\{\zeta_{2n} = 4a+2\}] \tag{157}$$

where C_n is defined by (36). If $r \geq 3$ and $a = 1, 2, \ldots$, we have

$$\mathbf{E}\left\{\binom{\omega_{2n}^+(a)}{r}\right\} = (n+1)\sum_{\ell=1}^{r}(-1)^{\ell-1}\binom{r-1}{\ell-1}\mathbf{E}\left\{\binom{\omega_{2n}(\ell a+2)}{r-2}\right\} \tag{158}$$

where the right-hand side is determined by (137).

Proof: Now $\omega_{2n}^+(a)$ for $a \geq 1$ is the number of events occurring among $A_1, A_2, \ldots, A_{2n}$ given that $\{\zeta_{2n} = 0\}$ occurs. Thus we can write that

$$\mathbf{P}\{\omega_{2n}^+(a) \geq k\}\frac{C_n}{2^{2n}} = 2\sum_{j=1}^{2n}\mathbf{P}\{\theta_1 + \ldots + \theta_k = j\}\mathbf{P}\{\rho(a+1) = 2n+1-j\}. \tag{159}$$

Since by (56)

$$\sum_{n=1}^{\infty} \mathbf{P}\{\rho(a+1)=n\}z^n = [\gamma(z)]^{a+1} \tag{160}$$

for $a \geq 1$ and $|z| < 1$, by (159) we obtain that

$$\sum_{n=1}^{\infty} \mathbf{P}\{\omega_{2n}^{+}(a) \geq k\}\frac{C_n z^{2n+1}}{2^{2n+1}} = [\gamma(z)]^{a+1}\psi^*(z)[\psi(z)]^{k-1} \tag{161}$$

where $\psi(z)$ is given by (96) and $\psi^*(z)$ by (97). If we multiply (161) by $\binom{k-1}{r-1}$ and add for $k \geq r$, then we obtain that

$$\sum_{n=1}^{\infty} \mathbf{E}\left\{\binom{\omega_{2n}^{+}(a)}{r}\right\}\frac{C_n z^{2n+1}}{2^{2n+1}} = \frac{[\gamma(z)]^{a+1}\psi^*(z)[\psi(z)]^{r-1}}{[1-\psi(z)]^r} \tag{162}$$

$$= \frac{[\gamma(z)]^{2a+2r-1}(1-[\gamma(z)]^{2a})^{r-1}}{(1-[\gamma(z)]^2)^{r-1}} = \sum_{\ell=1}^{r}(-1)^{\ell-1}\binom{r-1}{\ell-1}\frac{[\gamma(z)]^{2\ell a+2r-1}}{(1-[\gamma(z)]^2)^{r-1}}$$

for $|z| < 1$. If $r=1$ and $r=2$ in (162), by (56) and (60) we obtain (156) and (157). Comparing (162) and (137) we obtain (158) for $r \geq 3$.

Now we shall prove some limit theorems.

Theorem 15: *If $\alpha > 0$ and $r \geq 1$, then the limit*

$$\lim_{n\to\infty}\mathbf{E}\left\{\left[\frac{2\omega_{2n}^{+}([\alpha\sqrt{2n}])}{\sqrt{2n}}\right]^r\right\} = Q_r^{+}(\alpha) \tag{163}$$

exists and

$$Q_1^{+}(\alpha) = 4\alpha e^{-2\alpha^2}, \tag{164}$$

$$Q_2^{+}(\alpha) = 4[e^{-2\alpha^2} - e^{-8\alpha^2}] \tag{165}$$

and

$$Q_r^{+}(\alpha) = 2r(r-1)\sum_{\ell=1}^{r}(-1)^{\ell-1}\binom{r-1}{\ell-1}\mu_{r-2}(\ell\alpha) \tag{166}$$

where $\mu_r(\alpha)$ is defined by (29).

Proof: If in (156), (157) and (158) we put $a = [\alpha\sqrt{2n}]$ where $\alpha > 0$ and let $n\to\infty$, we get (164), (165) and (166).

Theorem 16: *If $\alpha > 0$, then there exists a distribution function $T_\alpha^{+}(x)$ of a nonnegative random variable such that*

$$\lim_{n\to\infty}\mathbf{P}\left\{\frac{2\omega_{2n}^{+}([\alpha\sqrt{2n}])}{\sqrt{2n}} \leq x\right\} = T_\alpha^{+}(x) \tag{167}$$

in every continuity point of $T_\alpha^{+}(x)$. The distribution function $T_\alpha^{+}(x)$ is uniquely determined by its moments

$$\int_{-0}^{\infty} x^r dT_\alpha(x) = Q_r^{+}(\alpha) \tag{168}$$

for $r \geq 0$ where $Q_0^{+}(\alpha) = 1$ and $Q_r^{+}(\alpha)$ for $r \geq 1$ is given by (164), (165) and (166).

Proof: Since

$$Q_r^+(\alpha) \le 2^r \mu_r(\alpha) \le 2^r r \int_0^\infty x^{r-1} e^{-x^2/2} dx \tag{169}$$

for $r \ge 3$, the sequence of moments $\{Q_r^+(\alpha)\}$ uniquely determines $T_\alpha^+(x)$, and $T_\alpha^+(x) = 0$ for $x < 0$. By the moment convergence theorem of M. Fréchet and J. Shohat [14] we can conclude that (168) implies (167). By (37) it follows from (167) that

$$\mathbf{P}\{\omega^+(\alpha) \le x\} = T_\alpha^+(x) \tag{170}$$

where $T_\alpha^+(x)$ is determined by (168). Now (168) proves (39), (40), and (38) for all $r \ge 3$. If we know the rth moment of $\omega(\alpha)$ for $\alpha > 0$, then by (38), (39), and (40) the rth moment of $\omega^+(\alpha)$ can immediately be determined for $\alpha > 0$ and $r \ge 1$.

The moments (168) uniquely determine the distribution of $\omega^+(\alpha)$ for $\alpha > 0$ and we have the following result.

Theorem 17: *If $x \ge 0$ and $\alpha > 0$, we have*

$$T_\alpha^+(x) = 1 - 2\sum_{k=1}^\infty \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{x^j}{j!} \varphi^{(j+2)}(2k\alpha + x) \tag{171}$$

for $x \ge 0$.

Proof: For $\alpha > 0$ let us define the Laplace-Stieltjes transform of $T_\alpha^+(x)$ by

$$\Omega_\alpha^+(s) = \int_{-0}^\infty e^{-sx} dT_\alpha^+(x). \tag{172}$$

We have

$$\Omega_\alpha^+(s) = \sum_{r=0}^\infty (-1)^r Q_r^+(\alpha) s^r / r! \tag{173}$$

and the series is convergent on the whole complex plane. Here $Q_r^+(\alpha)$ is given by (164), (165) and (166) and $\mu_r(\alpha)$ by (29). If we put these formulas into (173), express $\mu_r(\alpha)$ by (29) and interchange summations with respect to r and ℓ, we obtain that

$$\Omega_\alpha^+(s) = 1 + \sum_{k=1}^\infty h(2k\alpha) + \sum_{k=1}^\infty \frac{1}{(k-1)!} \int_0^\infty e^{-sx} \frac{d^k x^{k-1} h(2k\alpha + x)}{dx^k} dx \tag{174}$$

where

$$h(x) = 2e^{-x^2/2}(1 - x^2). \tag{175}$$

From (174) it follows that

$$T_\alpha^+(x) = 1 + \sum_{k=1}^\infty \frac{1}{(k-1)!} \frac{d^{k-1} x^{k-1} h(2k\alpha + x)}{dx^{k-1}} \tag{176}$$

for $x \ge 0$. This proves (171) and (41).

In various forms, the distribution function $T_\alpha^+(x)$ was found by V.E. Stepanov [31] in 1969 in the context random trees, D.P. Kennedy [23] in 1975 in the context of branching processes and J.W. Cohen and G. Hooghiemstra [8] in 1981 in the context of queueing processes. See also K.L. Chung [7], R.K. Getoor and N.J. Sharpe [15], G. Louchard [28], E. Csáki and S.G. Mohanty [10], Ph. Biane and M. Yor [4], F.B. Knight [25], G. Hooghiemstra [17] and L. Takács [36], [37], [38], [39], [40].

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