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ON WEAK SOLUTIONS OF RANDOM DIFFERENTIAL INCLUSIONS

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ABSTRACT

In the paper we study the existence of solutions of the random differential inclusion

$$\begin{aligned} &\dot{x}_t \in G(t, x_t) \quad P.1, t \in [0,T]\text{-a.e.} \\ &x_0 \overset{d}{=} \mu, \end{aligned} \tag{I}$$

where G is a given set-valued mapping value in the space K^n of all nonempty, compact and convex subsets of the space $\mathbb{R}^n$, and μ is some probability measure on the Borel σ-algebra in $\mathbb{R}^n$. Under certain restrictions imposed on F and μ, we obtain weak solutions of problem (I), where the initial condition requires that the solution of (I) has a given distribution at time $t = 0$.

Key words: Set-Valued Mappings, Hukuchara's Derivative, Aumann's Integral, Tightness and Weak Convergence of Probability Measures.

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1. Preliminaries

Problems of existence of solutions of differential inclusions were studied by many. In particular, random cases were considered in [3], [5], [7]. This work deals with the inclusion with a purely stochastic initial condition. First, we recall several notions and results needed in the sequel. Let $K_c(S)$ be the space of all nonempty compact and convex subsets of a metric space S equipped with the Hausdorff metric H (see e.g., [1], [4]): $H(A,B) = \max(\bar{H}(A,B), \bar{H}(B,A)); A, B \in K_c(S)$, where $\bar{H}(A,B) = \sup_{a \in A} \inf_{b \in B} \rho(a,b)$. By $\| A \|$ we denote the distance $H(A, 0)$. For S being a separable Banach space, $(K_c(S), H)$ is a polish metric space.

Let $I = [0,T]$, $T > 0$. For a given multifunction $G: I \rightarrow K_c(S)$ by $D_H G(t_0)$, we denote its Hukuchara derivative at the point $t_0 \in I$ (see e.g., [2], [9]) by the limits (if they exist in $K_c(S)$)

$$\lim_{h \rightarrow 0^+} \frac{F(t_0 + h) - F(t_0)}{h}, \lim_{h \rightarrow 0^+} \frac{F(t_0) - F(t-h)}{h},$$

both equal to the same set $D_H F(t_0) \in K_c(S)$.

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For $S = \mathbb{R}^n$ and $K^n = K_c(\mathbb{R}^n)$, we denote by $C_I = C(I, K^n)$ the space of all H-continuous set-valued mappings. In C_I we consider a metric ρ of uniform convergence

$$\rho(F,G):=\sup_{0 \leq t \leq T} H(X(t),Y(t)), \text{ for } X,Y \in C_T.$$

Then C_I is a polish metric space.

Let $(\Omega, \mathfrak{F}, P)$ be a given complete probability space. We recall now the notion of a multivalued stochastic process. The family of set-valued mappings $X = (X_t)_{t \geq 0}$ is said to be a *multivalued stochastic process* if for every $t \geq 0$, the mapping $X_t: \Omega \to K^n$ is measurable, i.e., $\bar{X}_t(U):=\{\omega: X_t(\omega) \cap U \neq \emptyset\} \in \mathfrak{F}$, for every open set $U \subseteq \mathbb{R}^n$ (see e.g., [1, 4]). It can be noted that U can be also chosen as closed or Borel subset. We restrict our interest to the case when $0 \leq t \leq T$, $T > 0$. If the mapping $t \to X_t(\omega)$ is continuous (H-continuous) with probability on $(P.1)$, then we say that the process X has *continuous "paths."*

Let us notice that the set-valued stochastic process X can be though as a random element X: $\Omega \to C_I$. Indeed, it follows immediately from [3] and from the fact that the topology of the uniform convergence and the compact-open topology in C_I are the same.

Definition 1: A probability measure μ (on C_I) is a *distribution of the set-valued process* $X = (X_t)_{0 \leq t \leq T}$ if one has $\mu(A) = P(\bar{X}(A))$ for every Borel subset A from C_I.

A distribution of X will be denoted by P^X.

Definition 2: A set-valued mapping $F: I \times K^n \to K^n$ is said to be an *integrably bounded of the Caratheodory type* if:

1) there exists a measurable function $m: I \to \mathbb{R}_+$ such that $\int_0^T m(t)dt < \infty$ and $\| F(t,A) \| \leq m(t)$ t-a.e., $A \in K^n$.
2) $F(t, \cdot)$ is H-continuous t-a.e.
3) $F(\cdot, A)$ is a measurable multifunction for every $A \in K^n$.

Let us consider now the multivalued random differential equation:

$$D_H X_t = F(t, X_t) \ P.1, t \in [0,T]\text{-a.e.} \qquad (II)$$

$$X_0 \stackrel{d}{=} \mu$$

where the initial condition requires that the set-valued solution process $X = (X_t)_{t \in I}$ has a given distribution μ at the time $t = 0$. By a *weak solution* of (II) we understand a system $(\Omega, \mathfrak{F}, P(X_t)_{t \in I})$ where $(X_t)_{t \in I}$ is a set-valued process on some probability space $(\Omega, \mathfrak{F}, P)$ such that (II) is met.

We state the following theorem (see e.g. [6]).

Theorem 1: *Let $F: I \times K^n \to K^n$ be an integrably bounded set-valued function of the Caratheodory type and let μ be an arbitrary probability measure on the space K^n. Then there exists a weak solution of (II).*

2. Weak Solutions of Random Differential Inclusions

As an application of Theorem 1, we show the existence of a weak solution of the random differential inclusion

$$\dot{x}_t = G(t, x_t) \quad P.1, t \in [0,T]\text{-a.e.}$$

$$x_0 \stackrel{d}{=} \mu. \qquad (I)$$

The weak solution of (I) is understood similarly as above, where μ is now a given probability measure on $\mathbb{R}^n$.

Let $\mathcal{T}_0$ denote the family of nonempty open subsets of $\mathbb{R}^n$, and let $C = \{C_V; V \in \mathcal{T}_0\}$, where $C_V = \{K \in K^n : K \cap V \neq \emptyset\}$. Then we have that $\mathcal{B}^n = \sigma(C)$ (see e.g. Proposition 3.1 [4]), where $\mathcal{B}^n$ is a Borel σ-field induced by the metric space (K^n, H).

Lemma 1: *The following hold true:*

i) $K^n \in C$,

ii) *if* $A_1, A_2, \ldots \in C$ *then* $\bigcup_{n=1}^{\infty} A_n \in C$,

iii) *if* $C_{V_1} \subseteq C_{V_2} \subseteq \ldots$ *then* $V_1 \subseteq V_2 \subseteq \ldots$.

Proof: The property *i)* is obvious. Let $V_1, V_2, \ldots, \in \mathcal{T}_0$ be such that $A_n = C_{V_n}$ for $n = 1, 2, \ldots$. To establish *ii)*, let us observe that $\bigcup_{n=1}^{\infty} A_n = C_{\bigcup_{n=1}^{\infty} V_n}$.

Let us suppose that *iii)* does not hold. Then for some $k \geq 1$, $V_k \nsubseteq V_{k+1}$. Hence there exists a point $x \in V_k$ such that $x \notin V_{k+1}$. But then $\{x\} \in C_{V_k}$ and $\{x\} \notin C_{V_{k+1}}$ contradicts to $C_{V_k} \subseteq C_{V_{k+1}}$.

To obtain our main result we need the following lemma:

Lemma 2: *If μ is a probability measure on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$, then there exists a probability measure $\widehat{\mu}$ on the space K^n such that $\widehat{\mu}(C_V) = \mu(V)$, $V \in \mathcal{T}_0$.*

Proof: Let C be the family generating Borel σ-field $\mathcal{B}^n$. We define a set-function ν on C by $\nu(C_V) = \mu(V)$. Let us observe that ν is well-defined. Indeed, if $C_{V_1} = C_{V_2}$ and $\mu(V_1) \neq \mu(V_2)$ then $V_1 \neq V_2$. Hence $V_1 \backslash V_2 \neq \emptyset$ or $V_2 \backslash V_1 \neq \emptyset$. Without loss of generality we may assume the first case. Then there exists $x \in V_2$ such that $x \notin V_1$. But then $\{x\} \in C_{V_2}$ and $\{x\} \notin C_{V_1}$ which contradicts with an equality $C_{V_1} = C_{V_2}$. Similarly, it can be shown that if the sets C_{V_1} and C_{V_2} are disjoint, then the sets V_1, V_2 have the same property too. Hence we get $\nu(C_{V_1} \cup C_{V_2}) = \nu(C_{V_1}) + \nu(C_{V_2})$ for disjoint C_{V_1} and C_{V_2}. From Lemma 1 we conclude that, if $C_{V_1} \subseteq C_{V_2} \subseteq \ldots$, then

$$\bigcup_{n=1}^{\infty} C_{V_n} \in C \text{ and } \nu\Big(\bigcup_{n=1}^{\infty} C_{V_n}\Big) = \lim \nu(C_{V_n}).$$

Moreover, $\nu(K^n) = 1$. Finally let us observe that ν is σ-subadditive. Next we define another set function $\widehat{\nu}$ as follows:

$$\widehat{\nu}(A) := \inf\{\nu(D) : A \subset D, D \in C\}, \quad A \subset K^n.$$

Standard calculations show that $\widehat{\nu}$ is an outer measure on K^n. Thus from the Caratheodory Theorem, $\widehat{\nu}$ is a probability measure on the σ-field of $\widehat{\nu}$-measurable subsets in K^n. Setting $\widehat{\mu} = \widehat{\nu} \mid_{\mathcal{B}^n}$, we obtain a desired probability measure.

We now present the following existence theorem.

Theorem 2: *Let us suppose that $G: I \times \mathbb{R}^n \rightarrow K^n$ is an integrably bounded multifunction of the Caratheodory type. Then for any probability measure μ on $\mathbb{R}^n$, there exists a weak solution of problem (I).*

Proof: Lemma 2 yields the existence of a probability measure $\widehat{\mu}$ on the metric space (K^n, H) with the property: $\widehat{\mu}(C_V) = \mu(V)$, $V \in \mathcal{T}_0$. Let $F: I \times K^n \rightarrow K^n$ be a multifunction defined by $F(t, A) = \overline{\text{co}}\, G(t, A)$, for $A \in K^n$. Hence from Lemma 1.1 [9], the set-valued mapping F is integrably bounded of the Caratheodory type too. Consequently, by Theorem 1, there exists a probability space $(\Omega, \mathcal{F}, P)$ and the set-valued stochastic process $X = (X_t)_{0 \leq t \leq T}$ (on it) with continuous "paths" and with values in K^n which is a weak solution of the equation

$$D_H X_t = F(t, X_t) \quad P.1, t \in [0,T]\text{-a.e.}$$

$$X_0 \stackrel{d}{=} \widehat{\mu}.$$

From Kuratowski and Ryll-Nardzewski Selection Theorem [4] we can choose $\xi: \Omega \to \mathbb{R}^n$ as a measurable selection of X_0. Then by Theorem 4 [5] (see also [3]), there exists a stochastic process $x = (x_t)_{0 \leq t \leq T}$ as a selection of X that is a solution (in strong sense) of the random differential inclusion:

$$\dot{x}_t \in G(t, x_t) \quad P.1, \ t \in [0,T]\text{-a.e.}$$

$$x_0 \in U \quad P.1,$$

where $U(\omega) = \{\xi(\omega)\}$ for $\omega \in \Omega$.

To complete the proof, it is sufficient to show that $x_0 \stackrel{d}{=} \mu$. Let us notice that $\{\omega: x_0(\omega) \in V\} = \{\omega: \xi(\omega) \in V\} \subset \{\omega: X_0 \cap V \neq \emptyset\}$, $V \in \mathcal{T}_0$. Because of $X_0 \stackrel{d}{=} \widehat{\mu}$ and $\widehat{\mu}(C_V) = \mu(V)$ we have

$$P^{x_0}(V) \leq \mu(V). \tag{$*$}$$

Using regularity properties of probability measures (on a separable metric space) (see e.g., Th. 1.2 [8]), we have that

$$P^{x_0}(B) = \inf\{P^{x_0}(V): B \subset V, V \in \mathcal{T}_0\}$$

and $\mu(B) = \inf\{\mu(V): B \subset V, V \in \mathcal{T}_0\}$ for every Borel subset B of $\mathbb{R}^n$. Hence from inequality ($*$) we get $P^{x_0}(B) \leq \mu(B)$. But P^{x_0} and μ are probability measures. Therefore they have to be equal.

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