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A STOCHASTIC MODEL FOR THE FINANCIAL MARKET WITH DISCONTINUOUS PRICES[1]

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ABSTRACT

This paper models some situations occurring in the financial market. The asset prices evolve according to a stochastic integral equation driven by a Gaussian martingale. A portfolio process is constrained in such a way that the wealth process covers some obligation. A solution to a linear stochastic integral equation is obtained in a class of cadlag stochastic processes.

Key words: Contingent Claim Valuation, Representation of Martingales, Stochastic Integral Equation, Option Pricing, Portfolio Processes.

AMS (MOS) subject classifications: 60H20, 60H30.

1. Introduction

In the present paper we model investments of an economic agent whose decisions cannot affect market prices (a "small investor").

Karatzas and Shreve in [7] considered a market model in which prices evolve according to a stochastic differential equation, driven by Brownian motion. Aase [1] and M. Picqué and M. Pontier [9] studied a more general model in which the evolution of asset prices is a combination of a continuous process based on Brownian motion (a semimartingale) and a Poisson point process.

The security price model that we use is a linear stochastic equation driven by a Gaussian martingale. This is a natural generalization, because the market is not continuous and the Brownian motion cannot model jump processes. Moreover, the instants of jumps of a Gaussian martingale are nonrandom.

The techniques we use include the martingale representation theorem and the Girsanov's type theorem. We also find a solution to a linear stochastic integral equation.

2. The Model

We consider a model of a security market where an economic agent is allowed to trade continuously up to some fixed planning horizon $0 \leq T < \infty$. We shall denote by X_t the wealth of this

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agent at time t. Let the process $\mathbf{M} = (M_t, F_t, 0 \le t \le T)$ be a Gaussian martingale on a fixed probability space $(\Omega, F, \mathbf{P})$ and the filtration $\mathbf{F} = \{F_t, 0 \le t \le T\}$ be the augmentation under $\mathbf{P}$ of a natural filtration $F_t^M = \sigma(M_s, 0 \le s \le t)$, $0 \le t < \infty$. F_0 contains the null sets of $\mathbf{P}$ and $\mathbf{F}$ is right continuous. $\langle M \rangle_t = EM_t^2$, $t \in \mathbb{R}_+ = [0, \infty)$ is the square characteristic of $\mathbf{M}$.

Let us suppose that the agent invests in two assets (or "securities"). One of the assets, called *bond*, has a finite variation on $[0,T]$, and its price model is

$$P_0(t) = \int_{(0,t]} P_0(s-)r(s-)d\langle M \rangle_s, \ \ P_0(0) = p_0, \ 0 \le t \le T.$$

The other one, called *stock*, is "risky". Its price is modeled by the linear stochastic equation

$$P(t) = \int_{(0,t]} P(s-)A(s-)d\langle M \rangle_s + \int_{(0,t]} P(s-)\sigma(s-)dM_s, \ \ P(0) = p.$$

Here the interest rate process $r(t) > 0$, $0 \le t < \infty$ of the bond, the appreciation rate process $A(t)$ of the stock, and volatility process $\sigma(t) > 0$, $0 \le t < \infty$ will all be nonrandom, F-predictable processes such that

$$\int_{(0,\infty)} r^2(s-)d\langle M \rangle_s < \infty, \quad \int_{(0,\infty)} A^2(s-)d\langle M \rangle_s < \infty, \qquad (1)$$
$$\int_{(0,\infty)} \sigma^2(s-)d\langle M \rangle_s < \infty, \ \ \mathbf{P}\text{-a.s.}$$

In addition, $A(t-)\Delta\langle M \rangle_t + \sigma(t-)\Delta M_t > -1$, $t \in (0,T]$, to ensure a limited liability of the stock.

Let $m(t)$ denote the number of stocks held at time t. Then the amount invested in the stocks is

$$\Pi(t) = m(t)P(t).$$

The process $(\Pi(t), F_t)$, $0 \le t \le T$ describes the investment policy and will be called a *portfolio process*. It is assumed to be measurable, F_t-predictable and

$$\int_{(0,T]} \Pi^2(s-)d\langle M \rangle_s < \infty, \ \ \mathbf{P}\text{-a.s.} \qquad (2)$$

for every finite number $T > 0$. Note that $\Pi(t)$ can be negative, which amounts to selling the stock short.

On the other hand, $C(t)$, $0 \le t \le T$ is a non-negative consumption process, assumed to be nondecreasing and F_t-predictable, such that

$$\int_{(0,T]} C(s-)d\langle M \rangle_s < \infty, \ \ \mathbf{P}\text{-a.s.} \qquad (3)$$

for every finite number $T > 0$.

The quantity

$$\Pi_0(t) = X_t - \Pi(t),$$

is invested in the bond at any particular time and may also become negative. This is to be interpreted as borrowing at the interest rate $r(t)$.

We assume now that the investor starts with some initial wealth $x \geq 0$, and the wealth at time t satisfies the linear stochastic equation

$$X_t = \int_{(0,t]} \Pi(s-)\sigma(s-)dM_s + \int_{(0,t]} \Pi(s-)[A(s-)-r(s-)]d\langle M\rangle_s + \int_{(0,t]} [X_{s-}r(s-)-C(s-)]d\langle M\rangle_s, \quad 0 < t \leq T; \qquad X(0) = x. \tag{4}$$

Conditions (1), (2), and (3) ensure that the stochastic equation (4) has a unique solution in the class of cadlag adapted processes (see Section 5 and Theorem 3).

3. Characterization of the Portfolio Process

If $A(t) = r(t)$ for every $t \in [0,\infty)$, the drift

$$\int_{(0,t]} \Pi(s-)[A(s-)-r(s-)]d\langle M\rangle_s$$

vanishes from the right-hand side of (4). When $A(t) \not\equiv r(t)$ we introduce a new probability measure $\bar{\mathbf{P}}$ which removes this drift.

Let us denote by ϕ_t the solution of the equation

$$\phi_t = 1 - \int_{(0,t]} \phi_{s-}\theta(s-)dM_s, \quad 0 \leq t \leq T,$$

where

$$\theta(t) = \frac{A(t)-r(t)}{\sigma(t)}.$$

From our assumptions on A, r, and σ, it follows that $\theta(t)$ is bounded, measurable and adapted to $\{F_t-\}$. Then the exponential supermartingale

$$\phi_t = \exp\left[-\int_{(0,t]} \theta(s-)dM_s^c - \frac{1}{2}\int_{(0,t]} \theta^2(s-)d\langle M^c\rangle_s\right] \cdot \prod_{s\leq t}[1-\theta(s-)\Delta M_s],$$

is actually a martingale, where

$$\Delta M_t = M_t - M_{t-} \not\equiv \frac{\sigma(t-)}{A(t-)-r(t-)} \quad \text{for } 0 < t \leq T.$$

Here M_t^c and $\langle M^c\rangle_t$ are the continuous parts of the processes M_t and $\langle M\rangle_t$, respectively, for $t \in \mathbb{R}_+$.

We define the new probability measure $\bar{\mathbf{P}}$:

$$\bar{\mathbf{P}}(A) = \mathbf{E}(\phi_T \mathbf{I}_A), \quad A \in F_T \text{ on } (\Omega, F).$$

The probability measures $\mathbf{P}$ and $\bar{\mathbf{P}}$ are mutually absolutely continuous on F_T.

The process

$$\bar{M}_t = M_t + \int_{(0,t]} \theta(s-)d\langle M\rangle_s, \quad 0 \leq t \leq T, \tag{6}$$

is a $\bar{\mathbf{P}}$-Gaussian martingale [8], and

$$(\langle \bar{M}\rangle_t, \bar{\mathbf{P}}) \equiv (\langle M\rangle_t, \mathbf{P}), \quad 0 \leq t \leq T.$$

With respect to a new probability measure, equation (4) can be rewritten as

$$X_t = \int_{(0,t]} \Pi(s-)\sigma(s-)d\bar{M}_s + \int_{(0,t]} [X_{s-}r(s-) - C(s-)]d\langle \bar{M}\rangle_s, 0 < t \leq T, \tag{7}$$

$$X(0) = x$$

and the solution (see Section 5) for $0 \leq t \leq T$, leads to

$$\frac{X_t}{\Phi(t)} + \int_{(0,t]} \frac{C(s-)}{\Phi(s-)[1 + r(s-)\Delta\langle M\rangle_s]} d\langle \bar{M}\rangle_s = x + \int_{(0,t]} \frac{\Pi(s-)\sigma(s-)}{\Phi(s-)[1 + r(s-)\Delta\langle M\rangle_s]} d\bar{M}_s, \tag{8}$$

where

$$\Phi(t) = \exp\left[\int_{(0,t]} r(s-)d\langle M^c\rangle_s\right] \cdot \prod_{s \leq t} [1 + r(s-)\Delta\langle M\rangle_s]$$

is a unique strong solution of the homogeneous equation corresponding to (7):

$$\Phi(t) = 1 + \int_{(0,t]} \Phi(s-)r(s-)d\langle M\rangle_s.$$

If we suppose that $1 + r(s-)\Delta\langle M\rangle_s \leq 0$ for some $s \in \mathbb{S}$, then $\Delta\langle M\rangle_s \leq -\frac{1}{r(s-)}$. But this is impossible if $r(s)$ is nonnegative. Consequently, $1 + r(s-)\Delta\langle M\rangle_s > 0$ for every $s \in \mathbb{R}_+$.

Let us notice also that

$$\inf_{t \in \mathbb{R}_+} |\Phi(t)| > 0.$$

The right-hand side of (8) is a $\bar{\mathbf{P}}$-local martingale. If (Π, C) is an admissible pair (i.e., $X_t \geq 0$, $0 \leq t \leq T$ a.s.), the left-hand side is nonnegative, consequently it is a nonnegative supermartingale under $\bar{\mathbf{P}}$. From the supermartingale property we obtain that

$$\bar{\mathbf{E}}\left[\frac{X_T}{\Phi(T)} + \int_{(0,T]} \frac{C(s-)}{\Phi(s-)[1 + r(s-)\Delta\langle M\rangle_s]} d\langle \bar{M}\rangle_s\right] \leq x, \tag{10}$$

where $\bar{\mathbf{E}}$ denotes the expectation operator under measure $\bar{\mathbf{P}}$.

This condition is also sufficient for the admissibility in the sense of the following theorem.

Theorem 1: *Suppose that $x \geq 0$ and B_T is a nonnegative F_T-measurable random variable, such that*

$$\bar{\mathbf{E}}\left[\frac{B_T}{\Phi(T)} + \int_{(0,T]} \frac{C(s-)}{\Phi(s-)[1 + r(s-)\Delta\langle M\rangle_s]} d\langle \bar{M}\rangle_s\right] \leq x. \tag{11}$$

Then there exists a portfolio process Π such that the pair (Π, C) is admissible for the initial endowment x and the terminal wealth X_T is at least B_T.

Proof: It is obvious that we can assume equality to hold in (11).

Let us define the nonnegative process

$$\bar{\mu}_t = \bar{\mathbf{E}}\left[\frac{B_T}{\Phi(T)} + \int_{(0,T]} \frac{C(s-)}{\Phi(s-)[1 + r(s-)\Delta\langle M\rangle_s]} d\langle \bar{M}\rangle_s \,\middle|\, F_t\right],$$

$$\bar{\mu}_0 = x, \tag{12}$$

which is a $\bar{\mathbf{P}}$-martingale and has "cadlag" paths.

Define the process μ_t, $0 \le t \le T$ by

$$\mu_t = \bar{\mu}_t + \int_{(0,t]} \phi_{s-}^{-1} d\langle \phi, \bar{\mu} \rangle_s, \tag{13}$$

where ϕ_t is the density (5). It is well known that the process μ_t is a $\mathbf{P}$-martingale [5], $\mu_0 = \bar{\mu}_0$, and $\langle \mu \rangle = \langle \bar{\mu} \rangle$.

Now by the martingale representation theorem [8], if $(\mathbf{M}, \mu)$ is a Gaussian process, there exists an F_t-predictable measurable process $h(s)$, such that

$$\int_{(0,T]} h^2(s-) d\langle M \rangle_s < \infty, \quad \mathbf{P}\text{-a.s.}$$

for every finite $T > 0$ and

$$\mu_t = \mu_0 + \int_{(0,t]} h(s-) dM_s, \quad 0 \le t \le T. \tag{14}$$

The process (13) can be represented as

$$\mu_t = \bar{\mu}_t - \int_{(0,t]} \theta(s-) h(s-) d\langle M \rangle_s, \quad 0 \le t \le T. \tag{15}$$

From equalities (6), (14), and (15) it follows that

$$\begin{aligned} \bar{\mu}_t &= \mu_t + \int_{(0,t]} \theta(s-) h(s-) d\langle M \rangle_s \\ &= \mu_0 + \int_{(0,t]} h(s-)[d\bar{M}_s - \theta(s-) d\langle M \rangle_s] + \int_{(0,t]} \theta(s-) h(s-) d\langle M \rangle_s \\ &= \mu_0 + \int_{(0,t]} h(s-) d\bar{M}_s. \end{aligned} \tag{16}$$

Now,

$$\Pi(t-) = \frac{h(t-)\Phi(t-)[1 + r(t-)\Delta\langle M \rangle_t]}{\sigma(t-)}, \quad 0 < t \le T \tag{17}$$

is a well-defined portfolio process.

From (12), (16), and (17), we get

$$\begin{aligned} \bar{\mu}_t &= \bar{\mathbf{E}} \left[\frac{B_T}{\Phi(T)} + \int_{(0,T]} \frac{C(s-)}{\Phi(s-)[1 + r(s-)\Delta\langle M \rangle_s]} d\langle \bar{M} \rangle_s \middle| F_t \right] \\ &= x + \int_{(0,t]} h(s-) d\bar{M}_s. \end{aligned} \tag{18}$$

By using (18) and (8), we obtain

$$\bar{\mu}_t = \frac{X_t^{\Pi, C, x}}{\Phi(t)} + \int_{(0,t]} \frac{C(s-)}{\Phi(s-)[1 + r(s-)\Delta\langle M \rangle_s]} d\langle \bar{M} \rangle_s, \tag{19}$$

where $X_t^{\Pi, C, x}$ is a solution of equation (7) for the pair (Π, C) and the initial capital $x \geq 0$.

Now, from (18) and (19), it follows that

$$\frac{X_t^{\Pi, C, x}}{\Phi(t)} = \bar{\mathbf{E}}\left[\frac{B_T}{\Phi(T)} + \int_{(t,T]} \frac{C(s-)}{\Phi(s-)[1 + r(s-)\Delta\langle M\rangle_s} d\langle \bar{M}\rangle_s \middle| F_t\right]. \tag{20}$$

Consequently, $X_t^{\Pi, C, x}$ is nonnegative and (Π, C) is an admissible strategy. □

4. Valuation of Contingent Claim

Definition: A contingent claim is a nonnegative F_T-measurable random variable B that satisfies

$$0 < \bar{\mathbf{E}}\left[\frac{B}{\Phi(T)}\right] \leq x.$$

The hedging price of this contingent claim is defined by

$$U \stackrel{\text{def}}{=} \inf\left\{x > 0, \exists\, (\Pi, C) - \text{admissible, such that } X_T^{\Pi, C, x} \geq B \ \mathbf{P}\text{-a.s.}\right\}.$$

Theorem 2: *The value of the contingent claim is attained and*

$$U = \bar{\mathbf{E}}\left[\frac{B}{\Phi(T)}\right].$$

Proof: Let us suppose that $X_T^{\Pi, C, x} \geq B$ a.s. for some value of $x > 0$ and a suitable pair (Π, C). Then from (10) it follows that

$$\bar{\mathbf{E}}\left[\frac{B}{\Phi(T)}\right] \leq \bar{\mathbf{E}}\left[\frac{X_T^{\Pi, C, x}}{\Phi(T)}\right] \leq x.$$

Consequently, $z = \bar{\mathbf{E}}\left[\frac{B}{\Phi(T)}\right] \leq U$.

Let us define the nonnegative random process

$$X_0(t) = \Phi(t) \cdot \bar{\mathbf{E}}\left[\frac{B}{\Phi(T)} \middle| F_t\right], \quad 0 \leq t \leq T,$$

where $\bar{m}_t = \bar{\mathbf{E}}\left[\frac{B}{\Phi(T)} \middle| F_t\right]$ is a $\bar{\mathbf{P}}$-Gaussian martingale, such that $\bar{m}_0 = \bar{\mathbf{E}}\left[\frac{B}{\Phi(T)}\right]$.

Analogously to the proof of the Theorem 1, we can apply the generalized Girsanov's theorem and the martingale representation theorem.

By comparing the processes

$$\frac{X_0(t)}{\Phi(t)} = z + \int_{(0,t]} h(s-)d\bar{M}_s$$

and $\frac{X_t^{\Pi, 0, x}}{\Phi(t)}$ we obtain that

$$X_0(t) \equiv X_t^{\Pi, 0, z}, \quad 0 \leq t \leq T. \tag{21}$$

Consequently, $z \geq U$. □

Remark 1: Let us note that (21) yields

$$X_0(T) \equiv X_T^{\Pi, 0, z} = B, \text{ a.s.,}$$

i.e., the contingent claim is attained with the initial capital U, portfolio Π, and zero consumption. This fact could be used as a starting point for solving appropriate optimal problems.

Remark 2: If $\langle M\rangle_t \equiv t$, we have $(\mathbf{M},\mathbf{P})$ and $(\bar{\mathbf{M}},\bar{\mathbf{P}})$ (standard) Wiener processes, and $\mathbb{S}$ empty. Then Theorem 1 and Theorem 2 reduce to the results of Karatzas and Shreve [7] and Cvitanić and Karatzas [2].

Corollary: *Let $C(t)\equiv 0$ and let the agent invest in one stock asset. Then, the following representations hold:*

$$X_t = \Phi(t)\cdot\bar{\mathbf{E}}\left[\frac{P(T)}{\Phi(T)}\middle| F_t\right],\quad 0\le t\le T; \qquad (i)$$

$$X_t = p\cdot\exp\left[\int_{(0,t]}\sigma(s-)d\bar{M}^c_s - \frac{1}{2}\int_{(0,t]}\sigma^2(s-)d\langle\bar{M}^c\rangle_s + \int_{(0,t]} r(s-)d\langle\bar{M}^c\rangle_s\right] \qquad (ii)$$

$$\cdot\prod_{s\le t}[1+r(s-)\Delta\langle\bar{M}\rangle_s + \sigma(s-)\Delta\bar{M}_s],\quad 0\le t\le T.$$

Proof: Representation (i) follows from (20) when $C(t)\equiv 0$.

By Ito's rule it can be proved that $\frac{P(T)}{\Phi(T)}$ is a $\bar{\mathbf{P}}$-Gaussian martingale and is a unique solution of the following stochastic equation:

$$\frac{P(T)}{\Phi(T)} = p + \int_{(0,T]}\frac{P(s-)}{\Phi(s-)}\cdot\frac{\sigma(s-)}{[1+r(s-)\Delta\langle M\rangle_s]}d\bar{M}_s,\quad T>0.$$

Consequently, from representation (i) it follows that $\frac{X_t}{\Phi(t)}$, $0\le t\le T$ is a $\bar{\mathbf{P}}$-Gaussian martingale and it yields representation (ii). □

5. A Linear Stochastic Integral Equation

In this section we will obtain a solution of the stochastic equation

$$X_t = X_0 + \int_{(0,t]}[S(s)X_{s-}+\sigma(s)]dM_s + \int_{(0,t]}[A(s)X_{s-}+a(s)]db(s), \qquad (22)$$

$0\le t<\infty$, which is a more general than we anticipate.

Let $\mathbf{M}=(M_t,F_t)$, $M_0=0$, $F_t=\sigma(M_s, s\le t)$, $t\in\mathbb{R}_+=[0,\infty)$, be a cadlag Gaussian martingale, $b=b(t)$, $t\in\mathbb{R}_+$-nonrandom, right-continuous function with finite variation on each finite interval. Suppose the function $b(t)$ be a real-valued deterministic function, absolutely continuous with respect to $\langle M\rangle_t = \mathbf{E}M_t^2$ and

$$b(t) = \int_{(0,t]}\gamma_s d\langle M\rangle_s,\quad t\in\mathbb{R}_+,$$

where $\gamma=(\gamma_t, F_{t-})$, $F_{t-}=\sigma(M_s, 0<s<t)$ is a F-predictable function, $X_0=(X_0,F_0)$ be a Gaussian random variable, independent of $\mathbf{M}$. $A(t)$, $\sigma(t)$, $a(t)$, and $S(t)$ are nonrandom F-predictable functions, such that

$$\int_{(0,\infty)}\sigma^2(s)d\langle M\rangle_s<\infty,\qquad \int_{(0,\infty)}A^2(s)db(s)<\infty,$$

$$\int_{(0,\infty)}a^2(s)db(s)<\infty\quad\text{and}\quad\int_{(0,\infty)}S^2(s)db(s)<\infty,\ \ \mathbf{P}\text{-a.s.}$$

We will find a solution of equation (22) in the class of cadlag adapted processes (i.e., processes with right-continuous paths and finite left limits) and it provides a fairly explicit representation. According to [4], such a solution exists and it is unique in the sense of **P**-indistinguishability.

Recall [6] that the random process **M** and the deterministic nondecreasing function $\langle M\rangle$ have their jumps at the same nonrandom moments of time which form a countable set $\mathbb{S}\subset\mathbb{R}_+\backslash\{0\}$. Let us notice now that the function $b(t)$ and the process X_t have their jumps at the same moments.

Let us suppose that the function $b=b(t)$, $t\in\mathbb{R}_+$ has no more than a countable subset of jumps $\{0\le s_0<s_1<\ldots<s_k<\ldots<\infty\}\subseteq\mathbb{S}$ with

$$\Delta b(s_k)=-\frac{1+S(s_k)\Delta M_{s_k}}{A(s_k)},\ k\ge 1.$$

It is obvious that

$$\begin{aligned}\Delta X_{s_k}&=[A(s_k)X_{s_k-}+a(s_k)]\Delta b(s_k)+[S(s_k)X_{s_k-}+\sigma(s_k)]\Delta M_{s_k}\\ &=-X_{s_k-}-\frac{a(s_k)}{A(s_k)}[1+S(s_k)\Delta M_{s_k}]+\sigma(s_k)\Delta M(s_k).\end{aligned}$$

Consequently,

$$X_{s_k}=\sigma(s_k)\Delta M_{s_k}-\frac{a(s_k)}{A(s_k)}[1+S(s_k)\Delta M_{s_k}]. \tag{23}$$

We will find a solution of equation (22) on the interval $[s_k,s_{k+1})$, $k\ge 0$, with an initial condition X_{s_k}, independent of increments $M_t-M_{s_k}$, $s_k\le t<s_{k+1}$, $k\ge 0$, according to (23) and the conditions imposed on X_0.

The homogeneous equation corresponding to (22) is

$$\Phi(t,s_k)=1+\int\limits_{(s_k,t]}\Phi(s-,s_k)A(s)db(s)+\int\limits_{(s_k,t]}\Phi(s-,_k)S(s)dM_s,$$

$$\Phi(s_k,s_k)=1,\ \ k\ge 0$$

and has a unique solution [3]:

$$\Phi(t,s_k)=\exp\left\{\int\limits_{(s_k,t]}A(s)db(s)+\int\limits_{(s_k,t]}S(s)dM_s-\frac{1}{2}\int\limits_{(s_k,t]}S^2(s)d\langle M^c\rangle_s\right\}$$
$$\cdot\prod_{s_k<s\le t}\{1+A(s)\Delta b(s)+S(s)\Delta M_s\}\cdot\ \exp\{-A(s)\Delta b(s)-S(s)\Delta M_s\};$$

$$\Phi(t,s_k)=\exp\left\{\int\limits_{(s_k,t]}A(s)db^c(s)+\int\limits_{(s_k,t]}S(s)dM_s^c-\frac{1}{2}\int\limits_{(s_k,t]}S^2(s)d\langle M^c\rangle_s\right\}$$
$$\cdot\prod_{s_k<s\le t}\{1+A(s)\Delta b(s)+S(s)\Delta M_s\}, \tag{24}$$

where $b^c(t)$ is the continuous path of the function $b(t)$, $s_k\le t<s_{k+1}$, $k\ge 0$.

Let us notice that if

$$A(t)\Delta b(t)+S(t)\Delta M_t\not\equiv -1$$

on (s_k, s_{k+1}), from the solution of (24) it follows

$$\inf_{s_k \leq t < s_{k+1} \wedge T} |\Phi(t, s_k)| > 0 \tag{25}$$

with some $T \in [s_k, \infty)$, $k \geq 0$.

Let now define the function $\Phi(t)$, $t \in \mathbb{R}_+$, where

$$\Phi(t) = \Phi(t, s_k), \quad s_k \leq t < s_{k+1}, \; k \geq 0.$$

It follows from(25) that the function $\Phi^{-1}(t)$, $t \in \mathbb{R}_+$ is correct defined and bounded on every finite interval $[0, T]$, $T \in \mathbb{R}_+$. Consequently, for every $t \in \mathbb{R}_+$, it holds true that

$$\int_{(0,t]} \Phi^{-2}(s) d\langle M \rangle_s < \infty. \tag{26}$$

Theorem 3: *The unique solution of the equation* (22) *is given by*

$$\begin{aligned} X_t^k = \Phi(t, s_k) \Bigg[X_{s_k} &+ \int_{(s_k, t]} \frac{\sigma(s)}{\Phi(s)} dM_s + \int_{(s_k, t]} \frac{a(s)}{\Phi(s)} db(s) \\ &- \int_{(s_k, t]} \frac{S(s)\sigma(s)}{\Phi(s)} d\langle M^c \rangle_s \Bigg], \quad s_k \leq t < s_{k+1}, \; k \geq 0. \end{aligned} \tag{27}$$

Proof: Observe that (25) ensures that the process X_t^k is well defined. We will show that the process X_t^k from (27) is a solution of equation (22) over the interval $[s_k, s_{k+1})$, $k \geq 0$.

We apply Ito's rule to (27) on the interval (s_k, s_{k+1}):

$$X_t^k = X_{s_k} + \int_{(s_k, t]} X_{s-} A(s) db(s) + \int_{(s_k, t]} X_{s-} S(s) dM_s$$

$$+ \int_{(s_k, t]} \frac{1}{1 + A(s)\Delta b(s) + S(s)\Delta M_s} \left[\mathbf{I}_{\{\Delta b(s) = 0\}} + \mathbf{I}_{\{\Delta b(s) \neq 0\}} \right]$$

$$\cdot \left[\sigma(s) dM_s + a(s) db(s) - S(s)\sigma(s) \mathbf{I}_{\{\Delta b(s) = 0\}} d\langle M^c \rangle_s \right] + \int_{(s_k, t]} S(s)\sigma(s) d\langle M^c \rangle_s$$

$$+ \sum_{s_k < s \leq t} \left[1 - \frac{1}{1 + A(s)\Delta b(s) + S(s)\Delta M_s} \right] \left[\sigma(s)\Delta M_s + a(s)\Delta b(s) \right];$$

$$X_t^k = X_{s_k} + \int_{(s_k, t]} X_{s-} A(s) db(s) + \int_{(s_k, t]} X_{s-} S(s) dM_s$$

$$+ \int_{(s_k, t]} \sigma(s) dM_s + \int_{(s_k, t]} a(s) db(s).$$

If $s_k < t < s_{k+1}$, then

$$X_t = X_{s_k} + \int_{(s_k, t]} [X_s - A(s) + a(s)]db(s) + \int_{(s_k, t]} [X_s - S(s) + \sigma(s)]dM_s.$$

The last representation and (23) lead to (22). □

Remark 3: The coefficients $A(t)$, $\sigma(t)$, $S(t)$, and $a(t)$ of equation (22) are f-predictable functions. It will be convenient for applications to represent solution (27) in the form

$$X_t^k = \Phi(t, s_k)\left[X_{s_k} + \int_{(s_k, t]} \frac{\sigma(s)}{\Phi(s-)[1 + A(s)\Delta b(s) + S(s)\Delta M_s]}dM_s\right.$$

$$+ \int_{(s_k, t]} \frac{a(s)}{\Phi(s-)[1 + A(s)\Delta b(s) + S(s)\Delta M_s]}db(s)$$

$$\left. - \int_{(s_k, t]} \frac{S(s)\sigma(s)}{\Phi(s-)[1 + A(s)\Delta b(s) + S(s)\Delta M_s]}d\langle M^c\rangle_s\right], \quad t \in [s_k, s_{k+1}), \quad k \geq 0.$$

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