J. of Inequal. & Appl., 1997, Vol. 1, pp. 139–147
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Embedding Variational Inequalities and their Generalizations into a Separation Scheme

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(Received 2 August 1996)

When the domain of a variational inequality is given as intersection of level sets of functionals, then it is useful to introduce a separation scheme. Starting from this it is possible to carry out, in a uniform way, several topics, as penalization, regularization, gap functions, duality.

Keywords: Variational and quasi-variational inequalities; separation of sets; penalization; regularization; gap functions; duality.

AMS Classification: 65K, 90C.

1 INTRODUCTION

Let Ξ be a real Hilbert space, $\varphi : \Xi \to \mathbb{R}$, $\mathcal{F} : \Xi \rightrightarrows \Xi$ and $K : \Xi \rightrightarrows \Xi$ point-to-set maps. Consider the following Generalized Quasi-Variational Inequality (in short, GQVI): find $y \in K(y)$ and $F \in \mathcal{F}(y)$ such that

$$\langle F, x - y\rangle + \varphi(x) - \varphi(y) \geq 0 \ , \ \forall x \in K(y). \tag{1.1}$$

We suppose that $K(y) := \{x \in X(y) : g(y; x) \in \mathcal{C}\}$, where $X : \Xi \rightrightarrows \Xi$ is a point-to-set map, $g : X(y) \times X(y) \to \mathbb{R}^m$, $m \in \mathbb{N}\backslash\{0\}$, and $\mathcal{C} \subset \mathbb{R}^n$ is a closed and convex cone with apex at the origin. Of course, when $g(y; x) \in \mathcal{C}$ is identically true, then the equality $K(y) = X(y)$ holds and shows that the present format is not less general than the usual ones. When $X(y)$ and $g(y; x)$ are independent of y and $\mathcal{F}(y)$ is a singleton $\forall y \in \Xi$

(in which case they will be denoted by X, $g(x)$ and $F(y)$, respectively), then (1.1) collapses to a Variational Inequality (in short, VI). (1.1) recovers a Generalized Quasi-Complementarity System (in a Hilbert space), when, $\forall y \in \Xi$, $K(y)$ is a closed and convex cone with apex at the origin and $\varphi \equiv 0$:

$$y \in K(y)\ ,\ \ F \in \mathcal{F}(y) \cap K^*(y)\ ,\ \ \langle F, y \rangle = 0, \tag{1.2}$$

where $K^*(y)$ denotes the (positive) polar of $K(y)$. If K is independent of y and $\mathcal{F}(y)$ is a singleton, then (1.2) collapses to the usual (nonlinear) Complementarity System [12]. (1.2) is motivated by the following:

PROPOSITION 1.1 *If, $\forall y \in \Xi$, $K(y)$ is a closed and convex cone with apex at the origin, and φ is identically zero, then y is a solution of (1.1) iff it is a solution of (1.2).*

Proof **"If"**. $y \in K(y)$ and $F \in \mathcal{F}(y)$ are obvious. $F \in K^*(y)$ implies $\langle F, x \rangle \geq 0\ \forall x \in K(y)$; subtracting side by side from this inequality the equality in (1.2) yields (1.1) at $\varphi \equiv 0$.
"Only if". Again $y \in K(y)$ and $F \in \mathcal{F}(y)$ are obvious. Ab absurdo, assume that $F \notin K^*(y)$. Then $\exists x(y) \in K(y)$ such that $\langle F, x(y) \rangle < 0$, and hence $\langle F, w - y \rangle < 0$ where $w := y + x(y) \in K(y)$ since $K(y)$ is a convex cone. The latest inequality contradicts the assumption that y be the solution of (1.1). Hence $F \in K^*(y)$. This relation and $y \in K(y)$ imply

$$\langle F, y \rangle \geq 0.$$

Now, suppose that $\langle F, y \rangle > 0$. This implies $y \notin 0$. Since $\tilde{x} := \frac{1}{2}y \in K(y)$, we find

$$\langle F, \tilde{x} - y \rangle = -\frac{1}{2}\langle F, y \rangle < 0,$$

which contradicts the assumption that y be solution of (1.1). Hence $\langle F, y \rangle = 0$, and y is solution of (1.2). This completes the proof. □

When K is independent of y and convex, φ convex, and $\mathcal{F}$ is the subdifferential of a convex functional, say $f : \Xi \to \mathbb{R}$, then (1.1) gives a 1st order necessary condition for the problem

$$\min[f(x) + \varphi(x)]\ ,\quad \text{s.t.}\ \ x \in K,$$

as it is easy to show. If, in addition, f is differentiable, then the above format shows an interesting splitting of the objective function into a differentiable part and a nondifferentiable one.

2 A SEPARATION SCHEME

When the domain is explicitly given in terms of (generalized) level sets, as in (1.1), then a Variational Inequality or its generalizations can be associated with a separation scheme; this can be considered as a root for developing several topics. Such an approach starts from the obvious remark that $y \in K(y)$ is a solution of (1.1) iff $\exists F \in \mathcal{F}(y)$, such that the system (in the unknown x):

$$u := \langle F, y - x\rangle + \varphi(y) - \varphi(x) > 0, \ v := g(y; x) \in \mathcal{C}, \ x \in X(y) \quad (2.1)$$

is impossible. The space where (u, v) runs is the *image space* associated with (1.1), and the set

$$\mathcal{K}(y, F) := \{(u, v) \in \mathbb{R} \times \mathbb{R}^m : u = \langle F, y - x\rangle + \varphi(y) - \varphi(x),$$
$$v = g(y; x), \ x \in X(y)\}, \ \ y \in \Xi, \ F \in \mathcal{F}(y),$$

is the image of (1.1). System (2.1) is associated with the set:

$$\mathcal{H} := \{(u, v) \in \mathbb{R} \times \mathbb{R}^m : u > 0 \, , \ v \in \mathcal{C}\}.$$

Another obvious remark is that the impossibility of (2.1) holds iff $\mathcal{H} \cap \mathcal{K}(y, F) = \emptyset$. Separation arguments appear now as a useful tool to show disjunction between the above sets. To this end let us introduce a family of functions $\gamma : \Xi \times \mathbb{R}^m \times \Omega \to \mathbb{R}$ defined in the following way. The set of parameters ω, i.e. Ω, is given and must verify, $\forall y \in \Xi$, the conditions:

$$\text{lev}_{>0}[u + \gamma(y; v; \omega)] \supseteq \mathcal{H}, \quad (2.2)$$

$$\bigcap_{\omega \in \Omega} \text{lev}_{>0}[u + \gamma(y; v; \omega)] = \text{cl}\ \mathcal{H}, \quad (2.3)$$

where lev and cl denote level set and closure, respectively; the level sets are considered with respect to (u, v) only. The function $w(y; u, v; \omega) := u + \gamma(y; v; \omega)$ is called *weak separation function* [7,14].

PROPOSITION 2.1 *$y \in K(y)$ is a solution of (1.1) if $\exists \omega \in \Omega$ and $\exists F \in \mathcal{F}(y)$ such that*

$$w(y; \langle F, y - x\rangle + \varphi(y) - \varphi(x), \ g(y; x); \omega) \leq 0, \ \ \forall x \in X(y). \quad (2.4)$$

Proof Taking into account the definition of $\mathcal{K}(y, F)$, it is easy to see that (2.4) implies

$$\mathcal{K}(y, F) \subseteq \text{lev}_{\leq 0} w(y; u, v; \omega),$$

where the level set is considered with respect to (u, v) only. Because of (2.2) the above inclusion implies $\mathcal{H} \cap \mathcal{K}(y, F) = \emptyset$, which shows that y is a solution of (1.1). This completes the proof. □

The sufficient condition (2.4) can be considered as a starting point for deriving several theories. See [6–10, 14] for some details.

Instead of (1.1) consider the following inequality: find $y \in K(y)$ such that

$$\langle F^*, y - x \rangle + \varphi(y) - \varphi(x) \leq 0\,, \ \forall F^* \in \mathcal{F}(x)\,, \ \forall x \in K(y). \qquad (2.5)$$

When K is independent of y and convex, φ is convex, $\mathcal{F}$ is monotone and upper semicontinuous, then y is a solution of (1.1) iff it is a solution of (2.5); see [9]. If, moreover, $\mathcal{F}$ is single-valued ($\mathcal{F}(x)$ is a singleton), then such an equivalence is the classic Minty Lemma [3, 13]. When equivalence between (1.1) and (2.5) holds, then the image set — which can in any case be defined for (2.5) as well as $\mathcal{K}$ has been defined for (1.1) — of (2.5) is another interesting set to be associated to (1.1) [9].

3 GAP FUNCTIONS

Consider the set $K^0 := \{y \in \Xi : y \in K(y)\}$. A function $\psi : K^0 \to \mathbb{R}$ is said to be a *gap function* iff $\psi(y) \geq 0 \ \forall y \in K^0$ and $\psi(y) = 0$ iff y is a solution of (1.1); K^0 is the set of fixed-points of the point-to-set map K.

Gap functions can be obtained as a by-product of the separation scheme introduced in Section 2. To this end we consider a particular case where:

$$\mathcal{I} := \{1, \ldots, m\}, \ g(y; x) = (g_i(y; x), i \in I),$$
$$O_p := (0, \ldots, 0) \in \mathbb{R}^p, \ \mathcal{C} = O_p \times \mathbb{R}^{m-p}_+,$$
$$p, m \in \mathbb{Z}_+, 0 \leq p \leq m, \ \mathcal{C} = \mathbb{R}^m_+ \text{ if } p = 0 \text{ and } \mathcal{C} = O_m \text{ if } p = m.$$

Then, we consider, as separation function*,

$$w(y; u, v; \lambda, \omega) := u + \langle \lambda, G(y; v; \omega) \rangle, \ u \in \mathbb{R}, \ v \in \mathbb{R}^m, \ \lambda \in \mathcal{C}^*, \ \omega \in \Omega,$$

where $\mathcal{C}^* := \{\lambda \in \mathbb{R}^m : \lambda_i \geq 0, \ i = p+1, \ldots, m\}$ is the (positive) polar of $\mathcal{C}$, and

$$G(y; v; \omega) := (G_i(y; v_i; \omega_i), \ i \in \mathcal{I}),$$

$$G_i : \Xi \times \mathbb{R} \times \Omega_i \to \mathbb{R}, \ \omega = (\omega_i, i \in \mathcal{I}), \ \omega_i \in \Omega_i, \ \Omega = \bigtimes_{i=1}^{m} \Omega_i.$$

Each G can be considered as a transformation of g_i; the particular cases of G_i exponential have been shown to be useful at least when (1.1) is a VI and $p = 0$ (so that $\mathcal{C} = \mathbb{R}^m_+$): or

$$G_i(y; v_i; \omega_i) = v_i \exp(-\omega_i v_i),$$

$$G_i(y; v_i; \omega_i) = 1 - \exp(-\omega_i v_i), \qquad (\omega_i \in \Omega_i := \mathbb{R}_+).$$

The inequality (2.4) now becomes:

$$\langle F, y - x \rangle + \varphi(y) - \varphi(x) + \langle \lambda, G(y; g(y; x); \omega) \rangle \leq 0, \ \forall x \in X(y). \quad (3.1)$$

The study of condition (2.4), namely (3.1), leads to the introduction of the following function:

$$\psi(y, F) := \min_{\lambda \in \mathcal{C}^*} \max_{x \in X(y)} [\langle F, y - x \rangle + \varphi(y) - \varphi(x) + \langle \lambda, G(y; g(y; x); \omega) \rangle]. \quad (3.2)$$

Under suitable assumptions it is possible to show that ψ is a gap function for (1.1); see [7]. In such cases a solution can be found by solving the problem:

$$\min \psi(y, F), \ \text{ s.t. } \ y \in K(y), \ F \in \mathcal{F}(y). \quad (3.3)$$

A particular case of special interest is met when a GQVI collapses to (1.2), $\mathcal{F}$ is single-valued, and $\varphi \equiv 0$; in this case (3.3) becomes ($\mathcal{F}$ is now denoted by F itself):

$$\min \langle F(y), y \rangle, \ \text{ s.t. } \ y \in K(y), \ F(y) \in K^*(y), \quad (3.4)$$

where K^* is the (positive) polar of K. (3.4) shows a gap function for a Quasi-Complementarity System.

*Without any fear of confusion, we use the same symbol w as in Section 2, even if the function is different.

4 PENALIZATION

Let us go back to the separation function w of Section 2. By specializing γ in a suitable way we can achieve penalization for (1.1). For the sake of simplicity, in this section we assume that $\mathcal{F}$ be single-valued and replace $\mathcal{F}(y)$ with the notation $F(y)$. Assume that $K(y)$ be closed $\forall y \in \Xi$, and γ continuous in $\Xi \times \mathbb{R}^m \times \Omega$ and such that, $\forall \omega \in \Omega$,

$$\gamma(y; v; \omega) \begin{cases} = 0\,,\ \forall v \in \mathbb{R}^m, & \text{if } y \in \ \text{int } K(y), \quad \text{or} \\ \quad\ ,\ \forall v \in \mathcal{C}, & \text{if } y \in \ \text{frt } K(y), \\ < 0\,,\ \forall v \in \mathbb{R}^m \backslash \mathcal{C}, & \text{if } y \in \ \text{frt } K(y), \\ > 0\,,\ \forall v \in \mathcal{C}\backslash\{0\}, & \text{if } y \in X(y)\backslash K(y), \end{cases} \tag{4.1}$$

where $\{0\}$ is cut off to embed the linear case; int and frt denote interior and frontier, respectively. Moreover, we assume that $\exists \omega_0 \in \Omega$ such that

$$\lim_{\omega \to \omega_0} \gamma(y; v; \omega) \begin{cases} -\infty\,,\ \forall v \in \mathbb{R}^m \backslash \mathcal{C}, & \forall y \in \ \text{frt } K(y), \\ +\infty\,,\ \forall v \in \mathcal{C}, & \forall y \in X(y)\backslash K(y). \end{cases} \tag{4.2}$$

Thus, we are considering (weak) separation functions of type:

$$w(y; u, v; \omega) = u + \gamma(y; v; \omega)\ ,\ \omega \in \Omega,$$

where γ satisfies (4.1)–(4.2). The inequality (2.4) becomes:

$$\langle F(y), y - x \rangle + \varphi(y) - \varphi(x) + \gamma(y; g(y; x); \omega) \leq 0,\ \forall x \in X(y). \tag{4.3}$$

Of course, (4.3) is fulfilled iff the inequality (in the unknown x):

$$\langle F(y), y - x \rangle + \varphi(y) - \varphi(x) + \gamma(y; g(y; x); \omega) > 0,\ \ \forall x \in X(y) \tag{4.4}$$

is impossible. This suggests a penalization. To this end consider a neighbourhood of ω_0 as

$$\Omega(\epsilon) := \begin{cases} \{\omega \in \Omega : \|\omega - \omega_0\| < \epsilon\}\,, & \text{if } \|\omega_0\| < +\infty, \\ \{\omega \in \Omega : \frac{1}{\|\omega\|} < \epsilon\}\,, & \text{if } \|\omega_0\| = +\infty, \end{cases} \qquad (\epsilon \in \mathbb{R}_+ \backslash \{0\}).$$

It is now intuitive that, if ω is close enough to ω_0, then γ acts as a penalization and forces y to fulfil (4.3). Indeed, the following proposition holds [8].

PROPOSITION 4.1 *Let γ be the function defined by (4.1)–(4.2) and assume that $\forall \omega \in \Omega(\bar{\epsilon})$ $\exists y_\omega \in X(y_\omega)$ such that (4.4) is impossible. Then, $\exists \tilde{\epsilon} \in]0, \bar{\epsilon}]$ such that, $\forall \omega \in \Omega(\tilde{\epsilon})$, y_ω belongs to $K(y_\omega)$ and is a solution of (1.1).*

As a consequence of Proposition 4.1, we have that a solution of the inequality: find $y \in X(y)$ such that

$$\langle F(y), x - y\rangle + \varphi(x) - \varphi(y) - \gamma(y; g(y; x); \omega) \geq 0, \quad \forall x \in X(y), \quad (4.5)$$

with ω close enough to ω_0 is a solution of (1.1) too. Hence, we have achieved exact penalization.

Inequality (4.5) may receive special forms. For instance, if in (1.1) we set $m = 1, \mathcal{C} = \mathbb{R}_+, \Omega = \mathbb{R}_+, \omega_0 = +\infty$, and

$$g(y; x) = \Psi'(y; x-y)\,, \; \gamma(y; g(y; x); \omega) = -\omega g(y; x) = -\omega\Psi'(y; x-y),$$

where $\Psi : \Xi \to \mathbb{R}$ is a directionally derivable function whose directional derivative fulfils (4.1)–(4.2), then (4.5) becomes:

$$\langle F(y), x - y\rangle + \varphi(x) - \varphi(y) + \omega\Psi'(y; x - y) \geq 0, \quad \forall x \in X(y), \quad (4.6)$$

and has a more "variational aspect" than (4.5).

One might desire to handle a Ψ which is differentiable on $X(y)$. This can be obtained by weakening (4.1), in the sense that we renounce to distinguish between interior and frontier of $K(y)$ and to control v outside $\mathcal{C}$; (4.1) is replaced with

$$\gamma(y; v; \omega) \begin{cases} = 0\,, \ \forall v \in \mathcal{C}\,, & \text{if } y \in K(y), \\ > 0\,, \ \forall v \in \mathcal{C}\,, & \text{if } y \notin K(y). \end{cases} \quad (4.7)$$

Now, the thesis of Proposition 4.1 is no longer guaranteed; however we obtain penalization even if not necessarily exact. In the particular case where $\varphi \equiv 0$ and (4.7) is of type:

$$\gamma(y; v; \omega) = -\omega\langle G(y), y - x\rangle,$$

G being an operator which is not necessarily a gradient of any functional, then (4.6) becomes:

$$\langle F(y) + \omega G(y), x - y\rangle \geq 0\,, \; \forall x \in X(y). \quad (4.8)$$

Here we are faced with a Quasi-Variational Inequality having an operator, which is a pencil of the given operator and the penalization one. Since in case (4.7) we cannot replace (1.1) with only one inequality of type (4.8), it is natural to set up a sequence of (4.8), such that a corresponding sequence of their solutions converge in some sense to a solution of (1.1). More precisely, we can construct a sequence $\{y_{\omega_r}\}_{r=1}^{\infty}$ such that y_{ω_r} be a solution of (4.8) at $\omega = \omega_r$ and $\lim_{r \to +\infty} \omega_r = \omega_0$. If $X(y)$ is open, then in (4.8) we must have equality, so that we are led to find y_{ω_r} as a solution of the equation

$$F(y) + \omega_r G(y) = 0.$$

This result embeds the classic one for VI (K independent of y); see [2].

5 FURTHER DEVELOPMENTS

In the preceding two sections we have seen that, starting from a separation scheme (Section 2), we can carry out two theories, which appear different because they require different "technical tools", but they substantially differ in the language only. This does not mean that the two theories should be joined; it means that it is suitable to carry them out within the common framework of separation of sets.

The topics outlined in Sections 3 and 4 are not the only ones which can be embedded in a separation scheme. The classic question of regularize a VI [3,13] is strictly related to penalization and then can be developed by exploiting the approach of Section 2; see [9]. A topic, which has not yet met a full treatment, is that of defining a dual VI. Starting from (2.4), instead of operating as in Section 3, we can merely look for the $\inf_\omega$ of the $\sup_x$ of function w; this leads to a dual problem of (1.1), which deserves further investigation [1].

The generalizations of Variational Inequalities and of Complementarity Systems have been numerous; often they have been proposed independently of each other. Hence, a format which embeds at least most of them might be useful. One possible format might be the following: find $y \in H(y, K(y))$ and $F \in \mathcal{F}(y)$ such that

$$\langle F, x - y \rangle + \varphi(x) - \varphi(y) \geq 0 \ , \ \forall x \in K(y), \tag{5.1}$$

where $H : \Xi \times 2^{\Xi} \rightrightarrows \Xi$ is a given point-to-set map. The classic formats of VI and Quasi-VI are obviously embraced by (5.1), as well as Complementarity and Quasi-Complementarity Systems. Beside this, some new formats might be embraced. For instance, by means of a suitable transformation, it is possible to reduce to (5.1) the new VI introduced and studied in [11]; see also [12, p. 169]. The remarkable aspect of (5.1) would be that the beginning of a theory for it would already have existed for several years, even if not explicitly developed for any kind of problem as (5.1). More precisely, Theorem 2.1 of [15] should give an existence result for (5.1) at least when K is independent of y.

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