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Weighted L^2 Inequalities for Classical and Semiclassical Weights

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Dedicated to the memory of Professor A.K. Varma

We give weighted L^2 Markov-Bernstein or Landau type inequalities for classical and semiclassical weights.

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1 INTRODUCTION

By an orthogonal polynomial system (OPS), we always mean a sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$, where $\deg(P_n) = n$, $n \geq 0$, and there is an increasing function $\mu(x)$ on an interval I such that

$$\int_I P_m(x)P_n(x)d\mu(x) = K_n\delta_{mn}, \quad m \quad \text{and} \quad n \geq 0,$$

where K_n are positive constants. In this case, we say that $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS relative to a positive measure $d\mu(x)$ (or a positive weight $w(x)$ if $d\mu(x) = w(x)dx$) on I.

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It is well known ([2, 8, 10]) that there are essentially (i.e., up to a linear change of variable) only three distinct OPS's (called classical OPS's) that arise as eigenfunctions of second order differential equation of hypergeometric type:

$$A(x)y''(x) + B(x)y'(x) + \lambda_n y(x) = 0, \tag{1.1}$$

where $A(x) = ax^2+bx+c \not\equiv 0$, $B(x) = dx+e$, and $\lambda_n = -n[a(n-1)+d]$, $n = 0, 1, 2, \cdots$. They are Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ (α, $\beta > -1$), Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ ($\alpha > -1$), and Hermite polynomials $\{H_n(x)\}_{n=0}^{\infty}$ satisfying

$$\begin{cases} (1-x^2)y''(x) + [(\beta-\alpha)-(\alpha+\beta+2)x]y'(x) & \\ +n(n+\alpha+\beta+1)]y(x) = 0 & \text{for } \{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty} \\ xy''(x) + (1+\alpha-x)y'(x) + ny(x) = 0 & \text{for } \{L_n^{(\alpha)}(x)\}_{n=0}^{\infty} \\ y''(x) - 2xy'(x) + 2ny(x) = 0 & \text{for } \{H_n(x)\}_{n=0}^{\infty} \end{cases} \tag{1.2}$$

and are orthogonal relative to

$$w(x) = \begin{cases} (1-x)^{\alpha}(1+x)^{\beta} & \text{on } [-1,1] & \text{for } \{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty} \\ x^{\alpha}e^{-x} & \text{on } [0,\infty) & \text{for } \{L_n^{(\alpha)}(x)\}_{n=0}^{\infty} \\ e^{-x^2} & \text{on } (-\infty,\infty) & \text{for } \{H_n(x)\}_{n=0}^{\infty} \end{cases} . \tag{1.3}$$

For these three classical weights in (1.3), Guessab and Milovanović [5] and Guessab [4] obtained weighted L^2-Markov or Bernstein type inequalities for polynomials. In 1987, Varma [13] obtained weighted L^2-Landau type inequalities for $w(x) = e^{-x^2}$ and later, Agarwal and Milovanović [1] extended Varma's result to all three classical weights in (1.3).

Although these inequalities must remain valid under any linear change of variable, it is not clear then what kinds of weights $w(x)$ are allowed to ensure such inequalities. In section two, we give weighted L^2-Markov or Bernstein type inequalities for classical weights in such a way that does not depend on specific form of $w(x)$ as in (1.3). In section three, we extend L^2-Landau type inequalities of Agarwal and Milovanović for classical weights to any semiclassical and positive-semidefinite weights. Finally we illustrate this extension by two examples, one for a classical weight e^{-x^2} and another for a nonclassical weight $|x|^{2\mu}e^{-x^2}$ ($\mu > -\frac{1}{2}$). For similar Markov-type inequalities for discrete classical weights we refer to [6, 7].

All polynomials are assumed to be real polynomials unless stated otherwise. We use the notation $\deg(P)$ to denote the degree of a polynomial $P(x)$ with the convention that $\deg(0) = -1$.

2 MARKOV-BERNSTEIN TYPE INEQUALITIES

The results in this section are not really new but some modifications of results obtained by Guessab and Milovanović [5] and Guessab [4], which is based on the following result (see [8, Theorem 2.9]) and [11, Theorem 2]).

THEOREM 2.1 *The differential equation* (1.1) *has an OPS* $\{P_n(x)\}_{n=0}^{\infty}$ *as solutions if and only if*

$$s_n := an + d \neq 0 \quad \text{and} \quad \frac{s_{n-1}}{s_{2n-1}s_{2n+1}} A\left(\frac{-(bn+e)}{s_{2n}}\right) > 0, \quad n \geq 0. \tag{2.1}$$

Moreover, $\{P_n(x)\}_{n=0}^{\infty}$ *is orthogonal relative to a weight* $w(x)$ *on* I, *where* $w(x)$ *is any nonnegative solution of Pearson differential equation*

$$(A(x)w(x))' - B(x)w(x) = 0 \quad \text{on} \quad Int(I) \tag{2.1}$$

and

$$I = \begin{cases} [m, M] & \text{if } A(x) \text{ has 2 real zeros } m \text{ and } M \\ [m, \infty) & \text{if } \deg(A) = 1 \text{ and } A(m) = 0 \\ (-\infty, \infty) & \text{if } \deg(A) = 0. \end{cases}$$

The first condition in (2.1) is the necessary and sufficient condition for the differential equation (1.1) to have a unique monic polynomial solution $P_n(x)$ of degree n for each $n \geq 0$ and the second condition in (2.1) is the necessary and sufficient condition for $\{P_n(x)\}_{n=0}^{\infty}$ to be an OPS. Theorem 2.1 is used in [8] to classify all classical OPS's, up to a real linear change of variable, including OPS's orthogonal relative to signed measures.

In Theorem 2.1, we may take

$$w(x) = \frac{\epsilon}{A(x)} \exp \int \frac{B(x)}{A(x)} dx \quad \text{on} \quad Int(I), \tag{2.3}$$

where $\epsilon = \pm 1$ depending on $A(x) \geq 0$ or $A(x) \leq 0$ on I respectively.

We also note that when the equation (1.1) has an OPS as solutions, $\{\lambda_n\}_{n=0}^{\infty}$ must be strictly monotone. More precisely, $\{\lambda_n\}_{n=0}^{\infty}$ is strictly increasing or strictly decreasing depending on $A(x) \geq 0$ or $A(x) \leq 0$ on I respectively. It can be easily seen because $\{\lambda_n\}_{n=0}^{\infty}$ remain unchanged under any linear change of variable and $\{\lambda_n\}_{n=0}^{\infty}$ is strictly increasing in all three cases in (1.2).

We set

$$||P||^2 := \int_I P^2(x)w(x)\,dx.$$

THEOREM 2.2 [5] *Let $w(x)$ be any nonnegative solution of the equation* (2.2), *where $A(x)$ and $B(x)$ satisfy the condition* (2.1). *Then for any integers m and n with $1 \leq m \leq n$*

$$||\,|A(x)|^{m/2}P^{(m)}|| \leq \left(\prod_{k=0}^{m-1}\sqrt{|\lambda_{n,k}|}\right)||P|| \tag{2.4}$$

for any complex polynomial $P(x)$ of degree $\leq n$, where

$$\lambda_{n,k} := -(n-k)[(n+k-1)a+d], \quad n \geq k \geq 0.$$

Moreover, the equality holds in (2.4) *if and only if $P(x) = CP_n(x)$ for some constant C, where $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS relative to $w(x)$ on I.*

Now, Theorem 2.2 can be proved essentially in the same way as the one in [5, Theorem 2.1] or [4, Lemma 3.1] even though they proved it only for three classical weights in (1.3) and for real polynomials $P(x)$ since the condition (2.1) guarantees the existence of an OPS $\{P_n(x)\}_{n=0}^{\infty}$ satisfying the equation (1.1) by Theorem 2.1.

Using the inequality (2.4), Guessab [4] obtained another weighted Markov-Bernstein type inequality for three classical weights in (1.3) and for real polynomials. In much the same way as before, we can reformulate his inequality [4, Theorem 2.1] as

THEOREM 2.3 [4] *Let $w(x)$ be the same as in Theorem 2.2 and $w_m(x) := |A(x)|^m w(x)$, $m \geq 0$ an integer. Then*

$$||(\sqrt{|A|}/w_m)(w_mP^{(m)})'||_m \leq \sqrt{|\beta_{n,m}\prod_{k=0}^{m-1}\lambda_{n,k}|}\;||P||_0 \qquad (\lambda_{n,-1}=0) \tag{2.5}$$

for any complex polynomial $P(x)$ of degree n $(\geq m)$, where

$$||P||_m^2 = \int_I |P(x)|^2 w_m(x)dx$$

and

$$\beta_{n,m} := \lambda_{n,m} - [2(m-1)a + d].$$

Moreover, the equality holds in (2.5) if and only if $P(x) = CP_n(x)$ for some constant C.

Theorem 2.2 and Theorem 2.3 give conditions (2.1) on $A(x)$ and $B(x)$ under which any nonnegative solution $w(x)$ of the equation (2.2) give rise to a corresponding weighted Markov-Bernstein type inequality for polynomials in $L^2(I : w(x)dx)$.

3 SEMICLASSICAL WEIGHTS

All polynomials in section three are assumed to be real polynomials. Agarwal and Milovanović [1] proved a Landau type inequality [9] for three classical weights in (1.3): Let $w(x)$ be one of the classical weights in (1.3). Then for any integer $n \geq 0$,

$$(2\lambda_n + B'(x)) \, ||\sqrt{A}P'||^2 \leq \lambda_n{}^2 ||P||^2 + ||AP''||^2 \tag{3.1}$$

for any real polynomial $P(x)$ of degree $\leq n$. Moreover, equality holds in (3.1) if and only if $P(x) = CP_n(x)$ for some real constant C. When $w(x) = e^{-x^2}$, the inequality (3.1) was found first by Varma [13]. As in section two, we can reformulate and extend (3.1) as:

THEOREM 3.1 *Let $w(x)$ be the same as in Theorem 2.2. Then*

$$(2\lambda + B'(x)) \, ||\sqrt{|A|}P'||^2 \leq \lambda^2 ||P||^2 + ||AP''||^2 \tag{3.2}$$

for any polynomial $P(x)$ and any real constant λ. Moreover, equality holds if and only if $\lambda = \lambda_n$ and $P(x) = CP_n(x)$ for some real constant C, where $n := \deg(P)$.

Note that we claim the inequality (3.2) holds for any λ and any polynomial $P(x)$ regardless of $\deg(P)$. We can further extend Theorem 3.1 into a more general situation like: Let σ be any moment functional on the space of polynomials ([3]). We call σ to be the positive-semidefinite if $\langle\sigma, P^2\rangle \geq 0$ for any polynomial $P(x)$. We call σ to be semiclassical if there is a pair of polynomials $(A(x), B(x)) \neq (0, 0)$ such that

$$(A(x)\sigma)' = B(x)\sigma, \tag{3.3}$$

where $\langle\sigma', \phi\rangle := -\langle\sigma, \phi'\rangle$ and $\langle\psi\sigma, \phi\rangle := \langle\sigma, \psi\phi\rangle$ for any polynomials $\phi(x)$ and $\psi(x)$. Note that here, we do not assume σ to be regular contrary to the usual definition of semiclassical moment functionals (see [12]). Any classical weight $w(x)$ satisfying the condition (2.1) and (2.2) defines a positive-definite semiclassical moment functional σ by

$$\langle\sigma, P\rangle := \int_I P(x)w(x)dx. \tag{3.4}$$

THEOREM 3.2 *Let σ be a positive-semidefinite and semiclassical moment functional satisfying*

$$(A(x)\sigma)' = (B(x) + D(x))\sigma \quad \text{and} \quad (D(x)\sigma)' = E(x)\sigma \tag{3.5}$$

for some polynomial A, B, D, E with $A^2(x) + B^2(x) \not\equiv 0$. Then for any polynomial $C(x)$ (which may depend on some parameters), we have

$$\begin{aligned}\langle\sigma, (AB' + 2AC + BD)(P')^2\rangle \\ \leq \langle\sigma, (AC'' + BC' + C^2 + 2C'D + CE)P^2\rangle + \langle\sigma, (AP'')^2\rangle\end{aligned} \tag{3.6}$$

for any polynomial $P(x)$. Moreover, if σ is positive-definite, then equality holds in (3.6) if and only if

$$L[P](x) := A(x)P''(x) + B(x)P'(x) + C(x)P(x) \equiv 0. \tag{3.7}$$

Proof We have

$$\langle\sigma, L[P]^2\rangle = \langle\sigma, (AP'')^2 + (BP')^2 + (CP)^2\rangle + 2\langle\sigma, ABP'P''\rangle + 2\langle\sigma, ACPP''\rangle + 2\langle\sigma, BCPP'\rangle. \tag{3.8}$$

We also have by (3.5)

$$\begin{aligned} 2\langle\sigma, ABP'P''\rangle &= \langle BA\sigma, [(P')^2]'\rangle = -\langle (BA\sigma)', (P')^2\rangle \\ &= -\langle\sigma, (AB' + B^2 + BD)(P')^2\rangle \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} 2\langle\sigma, ACPP''\rangle + 2\langle\sigma, BCPP'\rangle &= -2\langle (CPA\sigma)', P'\rangle + 2\langle\sigma, BCPP'\rangle \\ &= -2\langle (AC' + CD)P\sigma, P'\rangle - 2\langle\sigma, AC(P')^2\rangle \\ &= -\langle (AC' + CD)\sigma, (P^2)'\rangle - 2\langle\sigma, AC(P')^2\rangle \\ &= \langle ((AC' + CD)\sigma)', P^2\rangle - 2\langle\sigma, AC(P')^2\rangle \\ &= \langle\sigma, (AC'' + BC' + 2C'D + CE)P^2\rangle - 2\langle\sigma, AC(P')^2\rangle. \end{aligned} \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.8), we obtain

$$\begin{aligned} \langle\sigma, L[P]^2\rangle = &\langle\sigma, (AP'')^2 + (BP')^2 + (CP)^2\rangle \\ &- \langle\sigma, (AB' + B^2 + BD)(P')^2\rangle \\ &+ \langle\sigma, (AC'' + BC' + 2C'D + CE)P^2\rangle - 2\langle\sigma, AC(P')^2\rangle \end{aligned}$$

from which (3.6) follows since $\langle\sigma, L[P]^2\rangle \geq 0$. When σ is positive-definite, $\langle\sigma, L[P]^2\rangle = 0$ if and only if $L[P] = 0$ so that equality holds in (3.6) if and only if $L[P] = 0$.

When $\sigma = w(x)dx$ is a classical moment functional given by (3.4), $C(x) = \lambda$ is a constant, and $D(x) = E(x) \equiv 0$, Theorem 3.2 reduces to Theorem 3.1.

Remark 3.1 Conversely, if σ satisfies the inequality (3.6) with $C(x) = \lambda$, an arbitrary constant, then σ must be positive-semidefinite since if we divide (3.6) by λ^2 and let λ tend to ∞, then we obtain $\langle\sigma, P^2\rangle \geq 0$.

COROLLARY 3.3 *Let σ be the same as in Theorem 3.2. Then for any polynomial $P(x)$*

$$\langle\sigma, (AB' + BD)(p')^2\rangle \leq \langle\sigma, (AP'')^2\rangle. \tag{3.11}$$

Furthermore, if $\langle\sigma, P^2\rangle = 0$, then

$$\langle\sigma, EP^2\rangle - 2\langle\sigma, A(P')^2\rangle = 0. \tag{3.12}$$

If $\langle\sigma, P^2\rangle > 0$, then

$$\langle\sigma, EP^2 - 2A(P')^2\rangle^2 \leq 4\langle\sigma, P^2\rangle\langle\sigma, (AP'')^2 - (AB' + BD)(P')^2\rangle. \tag{3.13}$$

Proof Take $C(x) = \lambda$, an arbitrary constant, in (3.6). Then we obtain

$$\langle\sigma, P^2\rangle\lambda^2 + \langle\sigma, EP^2 - 2A(P')^2\rangle\lambda + \langle\sigma, (AP'')^2 - (AB' + BD)(P')^2\rangle \geq 0. \tag{3.14}$$

When $\lambda = 0$ in (3.14), we obtain (3.11). If $\langle\sigma, P^2\rangle = 0$, then (3.14) becomes

$$\langle\sigma, EP^2 - 2A(P')^2\rangle\lambda + \langle\sigma, (AP'')^2 - (AB' + BD)(P')^2\rangle \geq 0$$

so that (3.12) follows since λ is arbitrary. If $\langle\sigma, P^2\rangle > 0$, then (3.14) implies (3.13). □

When $\sigma = w(x)dx$ on I is a classical moment functional, we can have the following interesting Landau-type inequality:

COROLLARY 3.4 *Let $w(x)$ be any classical weight as in Theorem 2.2. Then*

$$2||\sqrt{|A|}\, P'||^2 \leq |d|\, ||P||^2 + ||P||\sqrt{d^2||P||^2 + 4||AP''||^2} \tag{3.15}$$

for any polynomial $P(x)$.

Proof Any classical weight $w(x)$ satisfies the condition (3.5) with $D(x) \equiv E(x) \equiv 0$ and $A(x)B'(x) \leq 0$ (see (1.2)). Hence, (3.13) becomes

$$\langle\sigma, A(P')^2\rangle^2 \leq \langle\sigma, P^2\rangle\{\langle\sigma, (AP'')^2\rangle + \langle\sigma, -AB'(P')^2\rangle\},$$

that is,

$$||\sqrt{|A|}P'||^4 \leq ||P||^2(||AP''||^2 + |d|\, ||\sqrt{|A|}P'||^2),$$

from which (3.15) follows immediately. □

Remark 3.2 When σ is a moment functional as in Theorem 3.2 satisfying (3.5) with $D(x) \equiv E(x) \equiv 0$ and $\deg(B) = 1$, we can obtain a similar inequality as (3.15) for $\langle \sigma, A(P')^2 \rangle$.

Finally, we give two examples illustrating Theorem 3.2.

EXAMPLE 3.1 Varma [13] proved the inequality (3.1) for $w(x) = e^{-x^2}$:

$$||P'||^2 \leq \frac{1}{2(2n-1)}||P''||^2 + \frac{2n^2}{2n-1}||P^2||, \qquad \deg(P) \leq n. \tag{3.16}$$

Equality holds in (3.16) if and only if $P(x) = CH_n(x)$. Applying Theorem 3.2 to $\sigma = e^{-x^2}dx$ with $A(x) = 1$, $B(x) = -2x$, $D(x) = E(x) \equiv 0$, and $C(x) = \lambda$, we obtain

$$(2\lambda - 2)||P'||^2 \leq ||P''||^2 + \lambda^2||P||^2 \tag{3.17}$$

for any λ and any polynomial $P(x)$, where equality holds if and only if $P(x) = CH_n(x)$, $n := \deg(P)$.

When $\lambda = 2n$, (3.17) becomes (3.16). We also have from (3.15)

$$||P'||^2 \leq ||P||^2 + \sqrt{||P||^2 + ||P''||^2}. \tag{3.18}$$

Replacing $P(x)$ by $P'(x)$ in (3.17) and then applying (3.17), we obtain

$$(2\mu - 2)(2\lambda - 2)||P''||^2 \leq (2\mu - 2)||P^{(3)}||^2 + \lambda^2(||P''||^2 + \mu^2||P||^2),$$

that is,

$$(4(\mu - 1)(\lambda - 1) - \lambda^2)||P''||^2 \leq 2(\mu - 1)||P^{(3)}||^2 + \lambda^2\mu^2||P||^2 \tag{3.19}$$

for any constants λ, μ and any polynomial $P(x)$, where equality holds if and only if $P(x) = CH_n(x)$, $\mu = 2n$, and $\lambda = 2(n-1)$, $n := \deg(P)$.

When $\mu = 2n$ and $\lambda = 2(n-1)$ $(n \geq 1)$, (3.19) becomes

$$||P''||^2 \leq \frac{(2n-1)||P^{(3)}||^2}{2(3n^2 - 6n + 2)} + \frac{4n^2(n-1)^2||P||^2}{3n^2 - 6n + 2}, \tag{3.20}$$

which was first obtained by Varma [13, Inequality (1.15)] for polynomials of degree $\leq n$. Equality in (3.20) holds if and only if $P(x) = CH_n(x)$.

EXAMPLE 3.2 Let $\sigma = w(x)dx$, $w(x) = |x|^{2\mu}e^{-x^2}$ $(\mu > -\frac{1}{2})$. Then σ is a positive-definite semiclassical moment functional satisfying

$$(x\sigma)' = (2\mu + 1 - 2x^2)\sigma$$

and

$$(x^2\sigma)' = 2[(\mu + 1)x - x^3]\sigma.$$

The corresponding OPS is the generalized Hermite polynomials $\{H_n^{(\mu)}(x)\}_{n=0}^{\infty}$ satisfying

$$x^2y''(x) + 2(\mu x - x^3)y'(x) + (2nx^2 - \theta_n)y(x) = 0,$$

where $\theta_{2m} = 0$ and $\theta_{2m+1} = 2\mu$, $m \geq 0$ (see [3]).

If we take $A(x) = x^2$, $B(x) = 2[(\mu + 1)x - x^3]$, $C(x) = 2nx^2 - \theta_n$, and $D(x) \equiv E(x) \equiv 0$ in (3.6), then we have

$$\begin{aligned}
2\int_{-\infty}^{\infty} [(2n - 3)x^2 + \mu + 1 - \theta_n](xP'(x))^2 w(x)dx \\
\leq \int_{-\infty}^{\infty} (x^2P''(x))^2 w(x)dx \\
+ \int_{-\infty}^{\infty} [4nx^2((n-2)x^2 + 2\mu + 3 - \theta_n) + \theta_n^2]P^2(x)w(x)dx,
\end{aligned}$$

where equality holds if and only if $P(x) = CH_n^{(\mu+1)}(x)$.

If we take $A(x) = x^2$, $B(x) = 2(\mu x - x^3)$, $C(x) = 2nx^2 - \theta_n$, $D(x) = 2x$, and $E(x) = 4\mu + 2 - 4x^2$ in (3.6), then we have

$$\begin{aligned}
\int_{-\infty}^{\infty} A(x,n)(xP'(x))^2 w(x)dx \leq \int_{-\infty}^{\infty} B(x,n)P^2(x)w(x)dx \\
+ \int_{-\infty}^{\infty} (x^2P''(x))^2 w(x)dx,
\end{aligned}$$

where

$$A(x,n) := (4n-10)x^2 + 6\mu - 2\theta_n;$$
$$B(x,n) := (4n^2-16n)x^4 + [24n + (4-4n)\theta_n + 16\mu n]x^2 + \theta_n(\theta_n - 4\mu - 2)$$

and equality holds if and only if $P(x) = CH_n^{(\mu)}(x)$.

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