L^∞-ERROR ANALYSIS FOR A SYSTEM OF QUASIVARIATIONAL INEQUALITIES WITH NONCOERCIVE OPERATORS

MESSAOUD BOULBRACHENE AND SAMIRA SAADI

Received 11 July 2005; Revised 14 November 2005; Accepted 18 December 2005

This paper deals with a system of elliptic quasivariational inequalities with noncoercive operators. Two different approaches are developed to prove L^∞-error estimates of a continuous piecewise linear approximation.

1. Introduction

We are interested in the finite element approximation in the L^∞ norm of the following system of quasivariational inequalities (QVIs): find $U = (u^1,\dots,u^J) \in (H_0^1(\Omega))^J$ satisfying

$$\begin{gathered} a^i(u^i, v - u^i) \geqq (f^i, v - u^i) \quad \forall v \in H_0^1(\Omega), \\ u^i \leq (MU)^i, \quad u^i \geq 0,\ v \leq (MU)^i. \end{gathered} \tag{1.1}$$

Here, Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$, with boundary $\partial\Omega$, $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$, for $i = 1,\dots,J$, $a^i(u,v)$ is a continuous bilinear form on $H^1(\Omega) \times H^1(\Omega)$, and f^i is a regular function.

Problem (1.1) arises in the management of energy production problems where J power generation machines are involved (see [2] and the references therein). In the case studied here, $(MU)^i$ represents a "cost function" and the prototype encountered is

$$(MU)^i = k + \inf_{\mu \neq i} u^\mu, \quad i = 1,\dots,J. \tag{1.2}$$

In (1.2), k represents the switching cost. It is positive when the unit is "turn on" and equal to zero when the unit is "turn off." Note also that operator M provides the coupling between the unknowns $u^1,\dots,u^J$.

In the present paper we are interested in the noncoercive problem. To handle such a situation, one can transform problem (1.1) into the following auxiliary system of QVIs:

Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2006, Article ID 15704, Pages 1–13
DOI 10.1155/JIA/2006/15704

find $U = (u^1,\dots,u^J) \in (H_0^1(\Omega))^J$ such that

$$b^i(u^i, v-u^i) \geqq (f^i + \lambda u^i, v-u^i) \quad \forall v \in H_0^1(\Omega),$$
$$u^i \le (MU)^i, \quad u^i \ge 0,\ v \le (MU)^i, \tag{1.3}$$

where, for $\lambda > 0$ large enough,

$$b^i(u,v) = a^i(u,v) + \lambda(v,v) \tag{1.4}$$

is a strongly coercive bilinear form, that is,

$$b^i(v,v) \ge \gamma \|v\|^2_{H^1(\Omega)}, \quad \gamma > 0,\ \forall v \in H^1(\Omega). \tag{1.5}$$

Naturally, the structure of problem (1.1) is analogous to that of the classical obstacle problem where the obstacle is replaced by an implicit one depending on the solution sought. The term quasivariational inequality being chosen is a result of this remark.

In [5], a quasi-optimal L^∞-error estimate was established for the coercive problem. This result was then extended to the noncoercive case (cf. [3, 4]).

In this paper two new approaches are proposed to prove the L^∞ convergence order for the noncoercive problem. The first approach consists of characterizing both the continuous and the finite element solutions as fixed points of *contractions* in L^∞.

The second one which is of *algorithmic type* stands on an algorithm generated by solving a sequence of coercive systems of QVIs. This algorithm is shown to converge geometrically to the solution of system (1.1).

It is worth mentioning that the *second approach* may be very useful for computational purposes.

It should also be mentioned that none of [3, 4] provides a computational scheme, even though they both contain the same approximation order as the one derived by the first approach presented in this paper.

The paper is organized as follows. In Section 2, we lay down some necessary preliminaries. In Section 3, we state the continuous problem, recall existence, uniqueness, and regularity of a solution, and characterize the solution as the unique fixed point of a contraction. In Section 4, we give analogous qualitative properties for the discrete problem, and characterize its solution as the unique fixed point of a contraction. In Section 5, we develop, separately, the two approaches and show that they both converge quasi-optimally in the L^∞ norm.

2. Preliminaries

2.1. Assumptions and notations. We are given functions $a^i_{jk}(x)$, $a^i_k(x)$, $a^i_0(x)$, $1 \le i \le J$, sufficiently smooth functions such that

$$\sum_{1\le j,k\le N} a^i_{jk}(x)\xi_j\xi_k \geqq \alpha|\zeta|^2, \quad \zeta \in \mathbb{R}^N,\ \alpha > 0,$$
$$a^i_0(x) \geqq \beta > 0, \quad (x \in \Omega). \tag{2.1}$$

We define the bilinear forms: for all $u,v \in H_0^1(\Omega)$,

$$a^i(u,v) = \int_\Omega \left(\sum_{1\le j,k\le N} a^i_{jk}(x) \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_k} + \sum_{k=1}^{N} a^i_k(x)\frac{\partial u}{\partial x_k} v + a^i_0(x)uv \right) dx. \tag{2.2}$$

We are also given right-hand sides f^i such that $f^i \in L^\infty(\Omega)$ and $f^i \ge f_0 > 0$ for $i = 1,\dots,J$.

2.2. Elliptic quasivariational inequalities. Let $f \in L^\infty(\Omega)$ such that $f > f_0 > 0$, M a nondecreasing operator from $L^\infty(\Omega)$ into itself, and $b(u,v)$ a bilinear form of the same form as those defined in (1.4). The following problem is called an elliptic quasivariational inequality (QVI): find $u \in \mathbb{K}(u)$ such that

$$b(u,v-u) \geqq (f,v-u) \quad \forall v \in \mathbb{K}(u), \tag{2.3}$$

where $\mathbb{K}(u) = \{v \in H_0^1(\Omega) \text{ such that } v \le Mu \text{ a.e.}\}$.

Thanks to [2], the QVI (2.3) has a unique solution. Moreover, this solution enjoys some important qualitative properties.

2.2.1. A Monotonicity property. Let f, $\tilde{f}$ in $L^\infty(\Omega)$ and $u = \sigma(f,MU)$, $\tilde{u} = \sigma(\tilde{f},M\tilde{u})$ be the corresponding solutions of (2.3). Then we have the following comparison principle.

Proposition 2.1. *If $f \ge \tilde{f}$ then $u \ge \tilde{u}$.*

Proof. Let u^0 and $\tilde{u}^0$ be the respective solutions to equations

$$\begin{aligned} b(u^0,v) &= (f,v) \quad \forall v \in H_0^1(\Omega), \\ b(\tilde{u}^0,v) &= \left(\tilde{f},v\right) \quad \forall v \in H_0^1(\Omega). \end{aligned} \tag{2.4}$$

Now let us associate with u and $\tilde{u}$ the respective decreasing sequences

$$u^{n+1} = \sigma(f,Mu^n), \qquad \tilde{u}^{n+1} = \sigma(\tilde{f},M\tilde{u}^n). \tag{2.5}$$

Then the following assertion holds:

$$\text{if } f \ge \tilde{f} \quad \text{then } u^n \ge \tilde{u}^n. \tag{2.6}$$

Indeed, since $f \ge \tilde{f}$ and M is nondecreasing, we have $u^0 \ge \tilde{u}^0$. So, $MU^0 \ge M\tilde{u}^0$, and thus applying standard comparison results in elliptic variational inequalities, we get

$$u^1 \ge \tilde{u}^1. \tag{2.7}$$

Now assume that $u^{n-1} \ge \tilde{u}^{n-1}$. Then, as $f \ge \tilde{f}$, applying the same comparison argument as before, we get

$$u^n \ge \tilde{u}^n. \tag{2.8}$$

Finally, passing to the limit $(n \to \infty)$ as in [2, pages 342–358], we get $u \ge \tilde{u}$. $\square$

The solution of QVI (2.3) is Lipschitz continuous with respect to the right-hand side.

2.2.2. A Lipschitz dependence property

Proposition 2.2. *Let Proposition 2.1 hold. Then,*

$$\|u-\widetilde{u}\|_{L^\infty(\Omega)} \le \frac{1}{\lambda+\beta}\|f-\widetilde{f}\|_{L^\infty(\Omega)}. \tag{2.9}$$

Proof. Let us set

$$\Phi = \frac{1}{\lambda+\beta}\|f-\widetilde{f}\|_{L^\infty(\Omega)}. \tag{2.10}$$

Then, since $a_0^i(x) \geqq \beta > 0$, we get

$$\begin{aligned} f &\le \widetilde{f} + \|f-\widetilde{f}\|_{L^\infty(\Omega)} \\ &\le \widetilde{f} + \frac{a_0(x)+\lambda}{\lambda+\beta}\|f-\widetilde{f}\|_{L^\infty(\Omega)} \\ &\le \widetilde{f} + \left(a_0(x)+\lambda\right)\Phi. \end{aligned} \tag{2.11}$$

So, due to Proposition 2.1, we obtain

$$u \le \widetilde{u} + \Phi. \tag{2.12}$$

Likewise, interchanging the roles of f and $\widetilde{f}$, we similarly get

$$\widetilde{u} \le u + \Phi \tag{2.13}$$

which completes the proof. □

Remark 2.3. The above monotonicity and Lipschitz continuity results stay true in the discrete case provided a discrete maximum principle is satisfied (see Section 3).

3. The continuous problem

3.1. The continuous system of QVIs. The existence of a unique solution to system (1.1) can be proved as in [2, pages 342–358]. Indeed, let $L_+^\infty(\Omega)$ denote the positive cone of $L^\infty(\Omega)$ and consider $\mathbb{H}^+ = (L_+^\infty(\Omega))^J$ equipped with the norm

$$\|V\|_\infty = \max_{1\le i\le J}\left\|v^i\right\|_{L^\infty(\Omega)}. \tag{3.1}$$

Consider the mapping

$$\begin{aligned} T : \mathbb{H}^+ &\longrightarrow \mathbb{H}^+, \\ W \longrightarrow TW &= \zeta = \left(\zeta^1,\ldots,\zeta^J\right), \end{aligned} \tag{3.2}$$

where $\zeta^i = \sigma(f^i + \lambda w^i, (MW)^i) \in H_0^1(\Omega)$ solves the following variational inequality (VI):

$$\begin{gathered} b^i(\zeta^i, v - \zeta^i) \geqq (f^i + \lambda w^i, v - \zeta^i) \quad \forall v \in H_0^1(\Omega), \\ \zeta^i \leq (MW)^i, \quad \zeta^i \geq 0, \quad v \leq (MW)^i. \end{gathered} \tag{3.3}$$

Problem (3.3), being a coercive VI, thanks to [1], has one and only one solution.

Consider now $\bar{U}^0 = (\bar{u}^{1,0}, \dots, \bar{u}^{J,0})$, where $\bar{u}^{i,0}$ is solution to the following variational equation:

$$a^i(\bar{u}^{i,0}, v) = (f^i, v) \quad \forall v \in H_0^1(\Omega). \tag{3.4}$$

Thanks to [2], problem (3.4) has a unique solution. Moreover, $\overline{u}^{i,0} \in W^{2,p}(\Omega)$; $2 \leq p < \infty$.

The mapping T possesses the following properties.

Proposition 3.1 (cf.[2]). *T is increasing, and concave and satisfies $TW \leq \bar{U}^0$ such that $W \leq \bar{U}^0$.*

Algorithm 3.2. Starting from $\bar{U}^0$ defined in (3.4) (resp., $\underline{U}^0 = (0, \dots, 0)$), we define a *decreasing sequence*

$$\bar{U}^{n+1} = T\bar{U}^n, \quad n = 0, 1, \dots, \tag{3.5}$$

(resp., an *increasing sequence*)

$$\underline{U}^{n+1} = T\underline{U}^n, \quad n = 0, 1, \dots. \tag{3.6}$$

It is clear that in view of (3.2), (3.3), the components of the vectors $\bar{U}^n$ and $\underline{U}^n$ are solutions of VIs.

Theorem 3.3. *Let Proposition 3.1 hold; then, the sequences $(\bar{U}^n)$ and $(\underline{U}^n)$ remain in the sector $\langle 0, \bar{U}^0 \rangle$. Moreover, they converge monotonically to the unique solution of system (1.1).*

Proof. See [2, pages 342–358]. □

3.1.1. Regularity of the solution of system (1.1).

Theorem 3.4 [2, page 453]. *Assume $a^i_{jk}(x)$ in $C^{1,\alpha}(\bar{\Omega})$, $a^i(x)$, $a^i_0(x)$, and f^i in $C^{0,\alpha}(\bar{\Omega})$, $\alpha > 0$. Then $(u^1, \dots, u^J) \in (W^{2,p}(\Omega))^J$; $2 \leq p < \infty$.*

3.2. Characterization of the solution of system (1.1) as a fixed point of a contraction. Consider the following mapping:

$$\begin{gathered} \mathbb{T} : \mathbb{H}^+ \longrightarrow \mathbb{H}^+, \\ W \longrightarrow \mathbb{T}W = Z, \end{gathered} \tag{3.7}$$

where $Z = (z^1,\dots,z^J)$ is solution to the coercive system of QVIs below:

$$\begin{gathered} b^i(z^i, v - z^i) \geqq (f^i + \lambda w^i, v - z^i) \quad \forall v \in H_0^1(\Omega), \\ z^i \le (MZ)^i, \quad z^i \ge 0, \quad v \le (MZ)^i. \end{gathered} \tag{3.8}$$

Thanks to [2], problem (3.8) has one and only one solution.

Theorem 3.5. *Under conditions of Proposition 2.2, the mapping $\mathbb{T}$ is a contraction on $\mathbb{H}^+$, that is,*

$$\|\mathbb{T}W - \mathbb{T}\widetilde{W}\|_\infty \le \frac{\lambda}{\lambda+\beta}\|W - \widetilde{W}\|_\infty. \tag{3.9}$$

Therefore, $\mathbb{T}$ admits a unique fixed point which coincides with the solution U of the system of QVIs (1.1).

Proof. Let $W, \widetilde{W} \in \mathbb{H}^+$, and let $Z = \mathbb{T}W$, $\widetilde{Z} = \mathbb{T}\widetilde{W}$ be the corresponding solutions to system of QVIs (3.8) with right-hand sides $F^i = f^i + \lambda w^i$ and $\widetilde{F}^i = f^i + \lambda \widetilde{w}^i$, respectively.

Let us also denote

$$z^i = \sigma(F^i, (MZ)^i), \qquad \widetilde{z}^i = \sigma\left(\widetilde{F}^i, (M\widetilde{Z})^i\right). \tag{3.10}$$

Then, making use of Proposition 2.2, we immediately get

$$||z^i - \widetilde{z}^i||_{L^\infty(\Omega)} \le \frac{\lambda}{\lambda+\beta}||w^i - \widetilde{w}^i||_{L^\infty(\Omega)} \tag{3.11}$$

and, consequently,

$$\begin{aligned} \|\mathbb{T}W - \mathbb{T}\widetilde{W}\|_\infty &= \|Z - \widetilde{Z}\|_\infty \\ &= \max_{1\le i\le J}||z^i - \widetilde{z}^i||_{L^\infty(\Omega)} \\ &\le \max_{1\le i\le J}\left(\frac{\lambda}{\lambda+\beta}\right)||z^i - \widetilde{z}^i||_{L^\infty(\Omega)} \\ &\le \left(\frac{\lambda}{\lambda+\beta}\right)\max_{1\le i\le J}||z^i - \widetilde{z}^i||_{L^\infty(\Omega)} \\ &\le \frac{\lambda}{\lambda+\beta}\|W - \widetilde{W}\|_\infty, \end{aligned} \tag{3.12}$$

which completes the proof. □

3.3. Another iterative scheme for system (1.1). In view of the above result, it is natural to associate with the solution of system of QVIs (1.1) the following algorithm.

Let $\widehat{U}^0 = (\widehat{u}_1^0,\dots,\widehat{u}_J^0)$ such that $\widehat{u}_i^0$ solves the equation

$$b(\widehat{u}_i^0, v) = (f, v) \quad \forall v \in H_0^1(\Omega). \tag{3.13}$$

Algorithm 3.6. Starting from $\hat{U}^0$ (resp., $\check{U}_0 = 0$), we define a *decreasing sequence*

$$\hat{U}^n = \mathbb{T}\hat{U}^{n-1}, \quad n = 1,2,\ldots, \tag{3.14}$$

(resp., an *increasing sequence*)

$$\check{U}^n = \mathbb{T}\check{U}^{n-1}, \quad n = 1,2,\ldots. \tag{3.15}$$

Note that unlike sequences (3.5), (3.6), the components of $\hat{U}^n = (\hat{u}_1^n,\ldots,\hat{u}_J^n)$ and $\check{U}^n = (\check{u}_1^n,\ldots,\check{u}_J^n)$ solve coercive *QVIs*

$$\begin{gathered}
b^i(\hat{u}_i^n, v - \hat{u}_i^n) \geqq (f^i + \lambda\hat{u}_i^{n-1}, v - \hat{u}_i^n) \quad \forall v \in H_0^1(\Omega), \\
\hat{u}_i^n \leq (M\hat{U}^n)^i, \quad \hat{u}_i^n \geq 0,\ v \leq (M\hat{U}^n)^i; \\
b^i(\check{u}_i^n, v - \check{u}_i^n) \geqq (f^i + \lambda\check{u}_i^n, v - \check{u}_i^n) \quad \forall v \in H_0^1(\Omega), \\
\check{u}_i^n \leq (M\check{U}^n)^i, \quad \check{u}_i^n \geq 0,\ v \leq (M\check{U}^n)^i.
\end{gathered} \tag{3.16}$$

Theorem 3.7. *Let $\rho = \lambda/(\lambda+\beta)$. Then, under conditions of Theorem 3.5, the sequences $(\hat{U}^n)$ and $(\check{U}^n)$ remain in the sector $\langle 0, \hat{U}^0\rangle$ and converge geometrically to the unique solution U of (1.1), that is,*

$$||\hat{U}^n - U||_\infty \leq \rho^n ||\hat{U}^0 - U||_\infty, \tag{3.17}$$

$$||\check{U}^n - U||_\infty \leq \rho^n ||\hat{U}^0 - U||_\infty. \tag{3.18}$$

Proof. Let us prove (3.17). The proof of (3.18) is similar.

For $n = 1$, we have

$$||\hat{U}^1 - U||_\infty = ||\mathbb{T}\hat{U}^0 - U||_\infty = ||\mathbb{T}\hat{U}^0 - \mathbb{T}U||_\infty \leq \rho^n ||\hat{U}^0 - U||_\infty. \tag{3.19}$$

Assume

$$||\hat{U}^{n-1} - U||_\infty \leq \rho^{n-1} ||\hat{U}^0 - U||_\infty. \tag{3.20}$$

Then,

$$||\hat{U}^n - U||_\infty = ||\mathbb{T}\hat{U}^{n-1} - \mathbb{T}U||_\infty \leq \rho ||\hat{U}^{n-1} - U||_\infty. \tag{3.21}$$

Thus

$$||\hat{U}^n - U||_\infty \leq \rho\rho^{n-1} ||\hat{U}^0 - U||_\infty \leq \rho^n ||\hat{U}^0 - U||_\infty. \tag{3.22}$$

□

4. The discrete problem

Let Ω be decomposed into triangles and let τ_h denote the set of all those elements; $h > 0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform.

Let $\mathbb{V}_h$ denote the standard piecewise linear finite element space, and let $\mathbb{B}^i$, $1 \leq i \leq J$, be the matrices with generic coefficients $b^i(\varphi_l, \varphi_s)$, where φ_s, $s = 1, 2, \ldots$, and $m(h)$ are the nodal basis functions. Let also r_h be the usual interpolation operator.

Definition 4.1. A real $n \times n$ matrix $B = [b_{ij}]$ with $b_{ij} \leq 0$ for all $i \neq j$ is an *M-matrix* if B is nonsingular and $B^{-1} \geq 0$.

The discrete maximum principle assumption (d.m.p.). We assume that the matrices $\mathbb{B}^i$ are M-matrices (cf. [6]).

4.1. Discrete elliptic quasivariational inequalities. The discrete counterpart of QVI (2.3) reads as follows: find $u_h \in \mathbb{K}_h(u_h)$ such that

$$b(u_h, v - u_h) \geqq (f, v - u_h) \quad \forall v \in \mathbb{K}_h(u_h), \tag{4.1}$$

where $\mathbb{K}_h(u_h) = \{v \in \mathbb{V}_h \text{ such that } v \leq r_h M U_h\}$.

Next we will state properties for the solution of (4.1) which are the direct discrete counterparts of those given in Propositions 2.1 and 2.2. We will omit their respective proofs as these are very similar to those of the continuous case.

4.1.1. A discrete monotonicity property. Let f, $\tilde{f}$ be in $L^\infty(\Omega)$ and $u_h = \sigma_h(f, MU_h)$, $\tilde{u}_h = \sigma_h(\tilde{f}, M\tilde{u}_h)$ the corresponding solutions to (4.1). Then, under the d.m.p., we have the following discrete comparison result.

Proposition 4.2. *If $f \geq \tilde{f}$, then $\sigma_h(f, MU_h) \geq \sigma_h(\tilde{f}, M\tilde{u}_h)$.*

4.1.2. A discrete Lipschitz dependence property.

Proposition 4.3. *Let Proposition 4.2 hold. Then,*

$$||u_h - \tilde{u}_h||_{L^\infty(\Omega)} \leq \frac{1}{\lambda + \beta} \|f - \tilde{f}\|_{L^\infty(\Omega)}. \tag{4.2}$$

4.2. The discrete system of QVIs. We define the discrete system of QVIs as follows: find $U_h = (u_h^1, \ldots, u_h^J) \in (\mathbb{V}_h)^J$ such that

$$\begin{aligned} &a^i(u_h^i, v - u_h^i) \geqq (f^i, v - u_h^i) \quad \forall v \in \mathbb{V}_h, \\ &u_h^i \leq r_h(MU_h)^i, \quad u_h^i \geq 0,\ v \leq r_h(MU_h)^i. \end{aligned} \tag{4.3}$$

Similarly to the continuous problem, the above problem can be transformed into the following: find $U_h = (u_h^1, \ldots, u_h^J) \in (\mathbb{V}_h)^J$ solution to the equivalent system

$$\begin{aligned} &b^i(u_h^i, v - u_h^i) \geqq (f^i + \lambda u_h^i, v - u_h^i) \quad \forall v \in \mathbb{V}_h, \\ &u_h^i \leq r_h(MU_h)^i, \quad u_h^i \geq 0, \quad v \leq r_h(MU_h)^i. \end{aligned} \tag{4.4}$$

The existence of a unique solution to system (4.3) can be shown very similarly to that of the continuous case provided the discrete maximum principle (d.m.p.) is satisfied. The key idea consists of associating with the above system the following fixed point mapping:

$$T_h : \mathbb{H}^+ \longrightarrow (\mathbb{V}_h)^J,$$
$$W \longrightarrow T_h W = \zeta_h = (\zeta_h^1, \dots, \zeta_h^J), \tag{4.5}$$

where $\zeta_h^i = \sigma_h(f^i + \lambda w^i, (MW)^i)$ is the solution of the following discrete VI:

$$b^i(\zeta_h^i, v - \zeta_h^i) \geqq (f^i + \lambda w^i, v - \zeta_h^i) \quad \forall v \in \mathbb{V}_h,$$
$$\zeta_h^i \leq r_h(MW)^i, \quad \zeta_h^i \geq 0,\ v \leq r_h(MW)^i. \tag{4.6}$$

Let $\bar{U}_h^0 = (\bar{u}_h^{1,0}, \dots, \bar{u}_h^{J,0})$ be the discrete analogue of $\bar{U}^0$ defined in (3.4):

$$a^i(\bar{u}_h^{i,0}, v) = (f^i, v) \quad \forall v \in \mathbb{V}_h. \tag{4.7}$$

Then, T_h possesses analogous properties to those enjoyed by mapping T (see Proposition 3.1).

Proposition 4.4. *T_h is increasing, concave on $\mathbb{H}^+$ and satisfies $T_h W \leq \bar{U}^0$ for all $W \leq \bar{U}_h^0$.*

Algorithm 4.5. Starting from $\bar{U}_h^0$ solution of (4.7), (resp., $\underline{U}_h^0 = (0, \dots, 0)$), we define a *discrete decreasing sequence*

$$\bar{U}_h^{n+1} = T_h \bar{U}_h^n, \quad n = 0, 1, \dots, \tag{4.8}$$

(resp., a *discrete increasing sequence*)

$$\underline{U}_h^{n+1} = T_h \underline{U}_h^n, \quad n = 0, 1, \dots. \tag{4.9}$$

Theorem 4.6. *Let Proposition 4.4 hold, then, the sequences $(\bar{U}_h^n)$ and $(\underline{U}_h^n)$ remain in the sector $\langle 0, \bar{U}_h^0 \rangle$. Moreover, they converge monotonically to the unique solution U_h of system of QVIs (4.3).*

4.3. Characterization of the solution of system (4.3) as a fixed point of a contraction. Similarly to the continuous problem, the solution of system (4.3) can be characterized as the unique fixed point of a contraction.

Indeed, consider the following mapping:

$$\mathbb{T}_h : \mathbb{H}^+ \longrightarrow (\mathbb{V}_h)^J,$$
$$W \longrightarrow \mathbb{T}_h W = Z_h = (z_h^1, \dots, z_h^J), \tag{4.10}$$

where $Z_h = (z_h^1, \dots, z_h^J)$ is solution to the discrete coercive system of QVIs:

$$b^i(z_h^i, v - z_h^i) \geqq (f + \lambda w^i, v - z_h^i) \quad \forall v \in \mathbb{V}_h,$$
$$z_h^i \leq r_h(MZ)^i, \quad z_h^i \geq 0,\ v \leq r_h(MZ)^i. \tag{4.11}$$

Then, making use of Proposition 4.3, we get the following.

Theorem 4.7. *The mapping $\mathbb{T}_h$ is a contraction on $\mathbb{H}^+$. That is,*

$$||\mathbb{T}_h W - \mathbb{T}_h \widetilde{W}||_\infty \leq \frac{\lambda}{\lambda+\beta} \| W - \widetilde{W} \|_\infty. \tag{4.12}$$

Therefore, there exists a unique fixed point which coincides with the solution U_h of the system of QVI (4.3).

Proof. It is very similar to that of the continuous case. □

4.4. Another iterative scheme for system (4.3). In view of the above result, it is natural to associate with the solution of system of QVIs (1.1) the following algorithm.

First, let $\widehat{U}_h^0 = (\widehat{u}_h^{1,0}, \ldots, \widehat{u}_h^{J,0})$ such that $\widehat{u}_h^{i,0}$ solves the equation

$$b^i(\widehat{u}_h^{i,0}, v) = (f, v) \quad \forall v \in \mathbb{V}_h. \tag{4.13}$$

Algorithm 4.8. Starting from $\widehat{U}_h^0$ (resp., $\check{U}_{0h} = 0$), we define a *decreasing sequence*

$$\widehat{U}_h^n = \mathbb{T}_h \widehat{U}_h^{n-1}, \quad n = 1, 2, \ldots, \tag{4.14}$$

(resp., an *increasing sequence*)

$$\check{U}_h^n = \mathbb{T}_h \check{U}^{n-1}, \quad n = 1, 2, \ldots. \tag{4.15}$$

Note that unlike sequences (4.8), (4.9), the components of both $\widehat{U}_h^n = (\widehat{u}_h^{1,n}, \ldots, \widehat{u}_h^{J,n})$ and $\check{U}_h^n = (\check{u}_h^{1,n}, \ldots, \check{u}_h^{J,n})$ solve discrete coercive QVIs, which are

$$\begin{gathered} b^i(\widehat{u}_h^{i,n}, v - \widehat{u}_h^{i,n}) \geqq (f^i + \lambda \widehat{u}_h^{i,n-1}, v - \widehat{u}_h^{i,n}) \quad \forall v \in \mathbb{V}_h, \\ \widehat{u}_h^{i,n} \leq r_h(M\widehat{U}_h^n)^i, \quad \widehat{u}_h^{i,n} \geq 0,\ v \leq r_h(M\widehat{U}_h^n)^i; \\ b^i(\check{u}_h^{i,n}, v - \check{u}_h^{i,n}) \geqq (f^i + \lambda \check{u}_h^{i,n}, v - \check{u}_h^{i,n}) \quad \forall v \in \mathbb{V}_h, \\ \check{u}_h^{i,n} \leq r_h(M\check{U}_h^n)^i, \quad \check{u}_h^{i,n} \geq 0,\ v \leq r_h(M\check{U}_h^n)^i. \end{gathered} \tag{4.16}$$

Theorem 4.9. *Let $\rho = \lambda/(\lambda+\beta)$. Then, under conditions of Theorem 4.7, the sequences $(\widehat{U}_h^n)$ and $(\check{U}_h^n)$ remain in the sector $\langle 0, \widehat{U}_h^0 \rangle$ and converge geometrically to the unique solution U_h of (4.3), that is,*

$$\begin{gathered} ||\widehat{U}_h^n - U_h||_\infty \leq \rho^n ||\widehat{U}_h^0 - U_h||_\infty, \\ ||\check{U}_h^n - U_h||_\infty \leq \rho^n ||\widehat{U}_h^0 - U_h||_\infty. \end{gathered} \tag{4.17}$$

Proof. The proof is similar to that of the continuous case. □

5. L^∞-error analysis

We now turn to the L^∞-error analysis. For that purpose, we will give two different approaches.

5.1. The contraction approach. It stands on the characterization of the solutions of both the continuous and discrete systems (1.1) and (4.3) as fixed points of contractions.

First, let us introduce the following intermediate discrete coercive system of QVIs: find $\overline{Z}_h = (\bar{z}_h^1, \dots, \bar{z}_h^J)$ solution to

$$\begin{gathered} b(\bar{z}_h^i, v - \bar{z}_h^i) \geqq (f + \lambda u^i, v - \bar{z}_h^i) \quad \forall v \in \mathbb{V}_h, \\ \bar{z}_h^i \leq r_h(M\bar{Z}_h)^i, \quad \bar{z}_h^i \geq 0, \; v \leq r_h(M\bar{Z}_h)^i. \end{gathered} \tag{5.1}$$

Clearly, (5.1) is a coercive system whose right-hand side depends on $U = (u^1, \dots, u^J)$, the solution of system (1.1). So, in view of (4.10), (4.11), we readily have

$$\bar{Z}_h = \mathbb{T}_h U. \tag{5.2}$$

Therefore, using the result of [5], we get the following error estimate:

$$||\bar{Z}_h - U||_\infty \leq Ch^2 |\operatorname{Log} h|^3. \tag{5.3}$$

Theorem 5.1. *Let U and U_h be the solutions of systems (1.1) and (4.3), respectively. Then,*

$$||U - U_h||_\infty \leq Ch^2 |\operatorname{Log} h|^3. \tag{5.4}$$

Proof. In view of (5.2) and Theorems 3.5 and 4.7, we clearly have

$$U = \mathbb{T}U; \qquad U_h = \mathbb{T}_h U_h; \qquad \bar{Z}_h = \mathbb{T}_h U. \tag{5.5}$$

Then, using estimation (5.3), we get

$$||\mathbb{T}_h U - \mathbb{T}U||_\infty = ||\bar{Z}_h - U||_\infty \leq Ch^2 |\operatorname{Log} h|^3. \tag{5.6}$$

Therefore

$$\begin{aligned} ||U_h - U||_\infty &\leq ||U_h - \mathbb{T}_h U||_\infty + ||\mathbb{T}_h U - \mathbb{T}U||_\infty \\ &\leq ||\mathbb{T}_h U_h - \mathbb{T}_h U||_\infty + ||\mathbb{T}_h U - \mathbb{T}U||_\infty \\ &\leq \rho ||U - U_h||_\infty + Ch^2 |\operatorname{Log} h|^3. \end{aligned} \tag{5.7}$$

Thus

$$||U - U_h||_\infty \leq \frac{Ch^2 |\operatorname{Log} h|^3}{(1 - \rho)}. \tag{5.8}$$

□

5.2. The algorithmic approach. It combines the error estimate between the *n*th iterate of (3.14) and its discrete counterpart (4.15), and the geometrical convergence of those algorithms.

Let us first introduce the following sequence of discrete coercive systems of QVIs: find $\tilde{U}_h^n = (\tilde{u}_h^{1,n}, \dots, \tilde{u}_h^{J,n})$ such that

$$\begin{gathered} b^i(\tilde{u}_h^{i,n}, v - \tilde{u}_h^{i,n}) \geqq (f^i + \lambda \hat{u}^{i,n-1}, v - \tilde{u}_h^{i,n}) \quad \forall v \in \mathbb{V}_h, \\ \tilde{u}_h^{i,n} \le r_h(M\tilde{U}_h^n)^i, \quad \tilde{u}_h^{i,n} \ge 0,\ v \le r_h(M\tilde{U}_h^n)^i, \end{gathered} \tag{5.9}$$

where $\hat{U}_h^n = (\hat{u}_h^{1,n}, \dots, \hat{u}_h^{J,n})$ is the continuous sequence defined in (3.14), and $\tilde{U}_h^0 = \hat{U}_h^0$.

The following lemma plays a crucial role in the present approach.

Lemma 5.2.

$$||\hat{U}^n - \hat{U}_h^n||_\infty \le \left(\frac{1-\rho^{n+1}}{1-\rho}\right) \sum_{p=0}^{n} ||\hat{U}^p - \tilde{U}_h^p||_\infty. \tag{5.10}$$

Proof. $\mathbb{T}_h$ being a contraction, we have

$$\begin{aligned} ||\hat{U}^1 - \hat{U}_h^1||_\infty &\le ||\hat{U}^1 - \tilde{U}_h^1||_\infty + ||\tilde{U}_h^1 - \hat{U}_h^1||_\infty \\ &\le ||\hat{U}^1 - \tilde{U}_h^1||_\infty + ||\mathbb{T}_h\tilde{U}_h^0 - \mathbb{T}_h\hat{U}_h^0||_\infty \\ &\le ||\hat{U}^1 - \tilde{U}_h^1||_\infty + \rho||\tilde{U}_h^0 - \hat{U}_h^0||_\infty \\ &\le (1+\rho)\Big(||\hat{U}^1 - \tilde{U}_h^1||_\infty + ||\tilde{U}_h^0 - \hat{U}_h^0||_\infty\Big). \end{aligned} \tag{5.11}$$

Now assume that

$$||\hat{U}^{n-1} - \hat{U}_h^{n-1}||_\infty \le \left(\frac{1-\rho^n}{1-\rho}\right) \sum_{p=0}^{n-1} ||\hat{U}^p - \tilde{U}_h^p||. \tag{5.12}$$

Then, using, again, the fact that $\mathbb{T}_h$ is a contraction, we get

$$\begin{aligned} ||\hat{U}^n - \hat{U}_h^n||_\infty &\le ||\hat{U}^n - \tilde{U}_h^n||_\infty + ||\tilde{U}_h^n - \hat{U}_h^n||_\infty \\ &\le ||\hat{U}^n - \tilde{U}_h^n||_\infty + ||\mathbb{T}_h\hat{U}^{n-1} - \mathbb{T}_h\hat{U}_h^{n-1}||_\infty \\ &\le ||\hat{U}^n - \tilde{U}_h^n||_\infty + \rho||\hat{U}^{n-1} - \hat{U}_h^{n-1}||_\infty \\ &\le ||\hat{U}^n - \tilde{U}_h^n||_\infty + \rho(1+\rho+\cdots+\rho^{n-1}) \sum_{p=0}^{n} ||\hat{U}^p - \tilde{U}_h^p|| \\ &\le ||\hat{U}^n - \tilde{U}_h^n||_\infty + (1+\rho+\cdots+\rho^{n}) \sum_{p=0}^{n} ||\hat{U}^p - \tilde{U}_h^p|| \\ &\le \left(\frac{1-\rho^{n+1}}{1-\rho}\right) \sum_{p=0}^{n} ||\hat{U}^p - \tilde{U}_h^p|| \end{aligned} \tag{5.13}$$

which completes the proof. □

Theorem 5.3. *Let U and U_h be the solutions of systems (1.1) and (4.3), respectively. Then,*

$$||U - U_h||_\infty \le Ch^2 |\operatorname{Logh}|^4. \tag{5.14}$$

Proof. We have

$$\begin{aligned} \|U-U_h\|_\infty &\le \|U-\widehat{U}^n\|_\infty + \|\widehat{U}^n-\widehat{U}_h^n\|_\infty + \|\widehat{U}_h^n-U_h\|_\infty \\ &\le \rho^n\|\widehat{U}^0-U\|_\infty + \left(\frac{1-\rho^{n+1}}{1-\rho}\right)\sum_{p=0}^{n}\|\widehat{U}^p-\widetilde{U}_h^p\|_\infty + \rho^n\|\widehat{U}_h^0-U_h\|_\infty. \end{aligned} \tag{5.15}$$

Now, taking

$$\rho^n \le h^2, \tag{5.16}$$

we get

$$\|U-U_h\|_\infty \le Ch^2|\operatorname{Logh}|^4. \tag{5.17}$$

□

Remark 5.4. Clearly, the first approach provides a better approximation as the second one leads to a convergence order with an extra logarithmic factor.

References

[1] A. Bensoussan and J.-L. Lions, *Applications des inéquations variationnelles en contrôle stochastique*, Méthodes Mathématiques de l'Informatique, no. 6, Dunod, Paris, 1978.
[2] ———, *Impulse Control and Quasivariational Inequalities*, Gauthier-Villars, Montrouge, 1984.
[3] M. Boulbrachene, *Pointwise error estimate for a noncoercive system of quasi-variational inequalities related to the management of energy production*, Journal of Inequalities in Pure and Applied Mathematics **3** (2002), no. 5, 9, article 79.
[4] ———, *L^∞-error estimate for a noncoercive system of elliptic quasi-variational inequalities: a simple proof*, Applied Mathematics E-Notes **5** (2005), 97–102.
[5] M. Boulbrachene, M. Haiour, and S. Saadi, *L^∞-error estimate for a system of elliptic quasi-variational inequalities*, International Journal of Mathematics and Mathematical Sciences **2003** (2003), no. 24, 1547–1561.
[6] P. G. Ciarlet and P.-A. Raviart, *Maximum principle and uniform convergence for the finite element method*, Computer Methods in Applied Mechanics and Engineering **2** (1973), no. 1, 17–31.

Messaoud Boulbrachene: Department of Mathematics & Statistics, College of Science, Sultan Qaboos University, P.O. Box 36, Muscat 123, Sultanate of Oman
E-mail address: boulbrac@squ.edu.om

Samira Saadi: Departement de Mathématiques, Faculté des Sciences, Université d'Annaba, BP 12, Annaba 23000, Algeria
E-mail address: signor_2000@yahoo.fr