THE SYSTEM OF GENERALIZED SET-VALUED EQUILIBRIUM PROBLEMS

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We introduce new and interesting model of system of generalized set-valued equilibrium problems which generalizes and unifies the system of set-valued equilibrium problems, the system of generalized implicit vector variational inequalities, the system of generalized vector and vector-like variational inequalities introduced by Ansari et al. (2002), the system of generalized vector variational inequalities presented by Allevi et al. (2001), the system of vector equilibrium problems and the system of vector variational inequalities given by Ansari et al. (2000), the system of scalar variational inequalities presented by Ansari Yao (1999, 2000), Bianchi (1993), Cohen and Caplis (1988), Konnov (2001), and Pang (1985), the system of Ky-Fan variational inequalities proposed bt Deguire et al. (1999) as well as a variety of equilibrium problems in the literature. Several existence results of a solution for the system of generalized set-valued equilibrium problems will be shown.

1. Introduction

Throughout this paper, let I be an index set. For each $i \in I$, let Z_i be a topological vector space. $C_i : X \to 2^{Z_i}$ be a set-valued mapping such that $C_i(x)$ is a closed pointed and convex cone with $\operatorname{int} C_i(x) \neq \emptyset$ for each $x \in X$, where $\operatorname{int} A$ denotes the interior of the set A. For each $i \in I$, let E_i and F_i be two locally convex Hausdorff topological vector spaces. Consider two family of nonempty compact convex subsets $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$ with $X_i \subset E_i$ and $Y_i \subset F_i$. Let $E = \prod_{i\in I} E_i$, $X = \prod_{i\in I} X_i$, $F = \prod_{i\in I} F_i$ and $Y = \prod_{i\in I} Y_i$. An element of the set $X^i = \prod_{j\in I\setminus i} X_i$ will be denoted by x^i, therefore, $x \in X$ will be written as $x = (x^i, x_i) \in X^i \times X_i$. An element of the set Y will be denoted by $y = \prod_{i\in I} y_i$, where y_i is an element of the set Y_i. Let $T_i : X \to 2^{Y_i}$ and $\Psi_i : X \times Y_i \times X_i \to 2^{Z_i}$ be set-valued mappings.

The system of generalized set-valued equilibrium problems (in short, SGSEP) which is a family of generalized set-valued equilibrium problems defined on a product set will be introduced as follows: The (SGSEP) is to find $(\overline{x}, \overline{y_i})$ in $X \times Y_i$ such that for each $i \in I$,

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$\overline{x_i} \in X_i$, $\overline{y_i} \in T_i(\overline{x})$ and

$$\Psi_i(\overline{x},\overline{y_i},z_i) \not\subset -\operatorname{int} C_i(\overline{x}), \quad \forall z_i \in X_i. \tag{1.1}$$

The (SGSEP) is a new, interesting, meaningful and general mathematical model, which contains many mathematical models as special cases, for some examples.

(i) For each $i \in I$, if $Y_i = \{\overline{y_i}\}$ and $T_i(x) = \{\overline{y_i}\}$ for all $x \in X$, define a function $\Phi_i : X \times X_i \to Z_i$ as $\Phi_i(x,z_i) = F_i(x,\overline{y_i},z_i)$, then the (SGSEP) reduces to the system of set-valued equilibrium problems (in short, SSEP), which is to find $\overline{x}$ in X such that for each $i \in I$,

$$\Phi_i(\overline{x},z_i) \not\subset -\operatorname{int} C_i(\overline{x}), \quad \forall z_i \in X_i. \tag{1.2}$$

(ii) For each $i \in I$, let $L(E_i,F_i)$ denote the continuous linear operators from E_i to F_i, and $V_i : X \to 2^{L(E_i,F_i)}$ be a set-valued mapping, let $\psi_i : L(E_i,F_i) \times X_i \times X_i \to Z_i$ be a vector-valued mapping. Then a special case of the (SSEP) is the system of generalized implicit vector variational inequalities (in short, SGIVVI), which is to find $\overline{x} \in X$ such that for each $i \in I$,

$$\forall y_i \in X_i, \quad \exists \overline{u_i} \in V_i(\overline{x}) : \psi_i(\overline{u_i},\overline{x_i},y_i) \notin -\operatorname{int} C_i(\overline{x}). \tag{1.3}$$

(iii) For each $i \in I$, let $\eta_i : X_i \times X_i \to E_i$ be a bifunction, then a special case of the (SGIVVI) is the system of generalized vector variational-like inequalities (in short, SGVVLI), which is to find $\overline{x} \in X$ such that for each $i \in I, \forall y_i \in X_i$, $\exists \overline{u_i} \in V_i(\overline{x}) : \langle \overline{u_i}, \eta_i(y_i,\overline{x_i})\rangle \notin -\operatorname{int} C_i(\overline{x})$.

(iv) A special case of the (SGVVLI) is the system of generalized vector variational inequalities (in short, SGVVI), which is to find $\overline{x} \in X$ such that for each $i \in I$,

$$\forall y_i \in X_i, \quad \exists \overline{u_i} \in V_i(\overline{x}) : \langle \overline{u_i}, y_i - \overline{x_i}\rangle \notin -\operatorname{int} C_i(\overline{x}), \tag{1.4}$$

and if $C_i(x) = C$ for all $x \in X$, then the (SGVVI) becomes the problem considered by Allevi et al. in [1] with relative pseudomonotonicity. If V_i is a single-valued function and $C_i(x) = C$ for all $x \in X$, then the (SGVVI) reduces to the system of vector variational inequalities (in short, SVVI), which is to find $\overline{x} \in X$ such that for each $i \in I$,

$$\langle V_i(\overline{x}), y_i - \overline{x_i}\rangle \notin -\operatorname{int} C, \quad \forall y_i \in X_i. \tag{1.5}$$

Moreover, if $Z_i = R$ and $C = R^+ = \{r \in R : r \geq 0\}$, then the (SVVI) reduces to the system of (scalar) variational inequalities (in short, SVI), which is to find $\overline{x} \in X$ such that for each $i \in I$,

$$\langle V_i(\overline{x}), y_i - \overline{x_i}\rangle \geq 0, \quad \forall y_i \in X_i. \tag{1.6}$$

The (SVI) was studied by Pang [27], Cohen and Chaplais [15], Ansari and Yao [5, 7], Bianchi [9] and Konnov [21].

(v) For each $i \in I$, if Φ_i is replaced by a single-valued mapping $\varphi_i : X \times X_i \to Z_i$, then the (SSEP) reduces to a system of vector equilibrium problems (in short, SVEP), which is to find $\overline{x}$ in X such that for each $i \in I$,

$$\varphi_i(\overline{x}, z_i) \notin -\operatorname{int} C_i(\overline{x}), \quad \forall z_i \in X_i. \tag{1.7}$$

For each $i \in I$, let $f_i : X \to Z_i$ be a vector-valued function. If for each $i \in I$,

$$\varphi_i(x, z_i) = f_i(x^i, z_i) - f_i(x), \tag{1.8}$$

then the (SVEP) is equivalent to the generalized Nash equilibrium problem (in short, GNEP), which is to find $\overline{x} \in X$ such that for each $i \in I$, $f_i(\overline{x^i}, z_i) - f_i(\overline{x}) \notin -\operatorname{int} C(\overline{x})$, $\forall z_i \in X_i$.

The (SSEP), the (SGIVVI), the (SGVVLI), the (SGVVI), the (SVEP) and the (GNEP) were introduced and studied by Ansari et al. in [4].

If $Z_i = Z$, $C_i(x) = C$ for each $i \in I$ and for all $x \in X$, then the (SVEP) becomes the problem studied by Ansari et al. in [3] and contains the system of vector optimization problems, the Nash equilibrium problem for vector-valued functions and the (SVVI) as special cases.

If $Z_i = R$, $C_i(x) = \{r \in R : r \leq \lambda\}$ for each $i \in I$ and for all $x \in X$, then the (SVEP) reduces the system of (scalar) Ky-Fan variational inequalities which is to find $\overline{x} \in X$ such that for each $i \in I$,

$$\varphi_i(\overline{x}, z_i) \leq \lambda, \quad \forall z_i \in X_i. \tag{1.9}$$

This problem was studied by Deguire et al. [16].

(vi) If $I = 1$, then the (SGSEP) reduces to the generalized set-valued equilibrium problem (in short, GSEP), which is to find $\overline{x} \in X$ and $\overline{y} \in T(\overline{x})$ such that

$$\Psi(\overline{x}, \overline{y}, z) \not\subset -\operatorname{int} C(\overline{x}), \quad \forall z \in X. \tag{1.10}$$

This problem was introduced and studied by Fu and Wan [18].

If $I = 1$, then the (SSEP) reduces to the set-valued equilibrium problem (in short, SEP), which is to find $\overline{x} \in X$ such that

$$\Phi(\overline{x}, z) \not\subset -\operatorname{int} C(\overline{x}), \quad \forall z \in X. \tag{1.11}$$

The (SEP) was studied by Ansari et al. [2], Ansari and Yao [6], Konnov and Yao [22], Lin et al. [23], Oettli and Schlager [26], and the (SEP) contains the vector equilibrium problem in [10, 13, 19, 23, 25, 28] and the equilibrium problem in [11, 12] as special cases.

In this paper, some existence results of a solution for the (SGSEP) will be shown. These results improve and generalize the main results in [3, 4].

2. Basic definitions

In order to prove the main results, it is need to introduce the following new definitions.

Definition 2.1. Let $C_i : X \to 2^{Z_i}$ be a set-valued mapping such that $C_i(x)$ is a closed pointed and convex cone with $\operatorname{int} C_i(x) \neq \emptyset$ for each $x \in X$. Then the set-valued mapping $\Phi : X \times X_i \to 2^{Z_i}$ is called to be $C_i(x)$-0-partially diagonally quasiconvex if, for any finite set $\{z_{i_1}, z_{i_2}, \dots, z_{i_n}\} \in X_i$, and for all $x = (x^i, x_i) \in X$ with $x_i \in \operatorname{Co}\{z_{i_1}, z_{i_2}, \dots, z_{i_n}\}$, there exists some j in $\{1, 2, \dots, n\}$ such that $\Phi(x, z_{i_j}) \not\subset -\operatorname{int} C_i(x)$.

It is clear that if $Z_i = R$ and $C_i(x) = \{r \in R : r \geq 0\}$ for all $x \in X$, and Φ_i is a single-valued function, then $C_i(x)$-0-partially diagonally quasiconvexity of Φ_i reduces to the 0-partially diagonally quasiconvex (i.e., [14, Definition 3]), furthermore, let $I = \{1\}$, [14, Definition 3] reduces to the γ-diagonal quasiconvexity in [31, 32], here $\gamma = 0$.

It is need to recall the following definitions for set-valued mappings in [6, 8].

Definition 2.2. Let E and Z be topological spaces, $X \subset E$ a nonempty convex set. Let $C : X \to 2^Z$ be a set-valued mapping with $\operatorname{Int} C(x) \neq \emptyset$ for all $x \in X$ and $\Phi : X \times X \to 2^Z$ be a set-valued mapping. Then $\Phi(x, z)$ is said to be $C(x)$-quasiconvex-like if, for all $x \in X$, $y_1, y_2 \in X$, and $\alpha \in [0, 1]$, we have either

$$\Phi(x, \alpha y_1 + (1 - \alpha) y_2) \subset \Phi(x, y_1) - C(x) \tag{2.1}$$

or

$$\Phi(x, \alpha y_1 + (1 - \alpha) y_2) \subset \Phi(x, y_2) - C(x). \tag{2.2}$$

Definition 2.3. Let X and Y be two topological spaces and $T : X \to 2^Y$ be a set-valued mapping.

(1) T is said to be upper semicontinuous if the set $\{x \in X : T(x) \subset V\}$ is open in X for every open subset V of Y.

(2) T is said to have open lower sections if the set $T^{-1}(y) = \{x \in X : y \in T(x)\}$ is open in X for each $y \in Y$.

3. Existence results

Some existence results of a solution for the (SGSEP) are first be shown as follows.

Theorem 3.1. *Let I be an index set and I be countable. For each $i \in I$, let Z_i be a real topological vector space, E_i and F_i be two locally convex Hausdorff topological vector spaces, $X_i \subset E_i$ be a nonempty, convex and metrizable set and $Y_i \subset F_i$ be a nonempty, compact, convex and metrizable set, let $C_i : X \to 2^{Z_i}$ be a set-valued mapping such that $C_i(x)$ is a closed pointed and convex cone with $\operatorname{int} C_i(x) \neq \emptyset$ for each $x \in X$, $T_i : X \to 2^{Y_i}$ and $\Psi_i : X \times Y_i \times X_i \to 2^{Z_i}$ be set-valued mappings. For each $i \in I$, assume that the following.*

(i) *$M_i = Z_i \backslash (-\operatorname{int} C_i) : X \to 2^{Z_i}$ is upper semicontinuous.*

(ii) *For each $y_i \in X_i$, $\Psi_i(x, y_i, z_i)$ is $C_i(x)$-0-partially diagonally quasiconvex.*

(iii) *For all $z_i \in X_i$, $(x, y_i) \mapsto \Psi_i(x, y_i, z_i)$ is upper semicontinuous on $X \times Y_i$ with compact values.*

(iv) $T_i : X \to 2^{Y_i}$ *is an upper semicontinous set-valued mapping with nonempty compact values.*

(v) *For each* $i \in I$*, there exists a nonempty compact subset* $K_i \subset X_i$ *and a compact convex set* $B_i \subset X_i$*; let* $K = \prod_{i\in I} K_i \subset X$ *and* $B = \prod_{i\in I} B_i \subset X$ *such that, for each* $x \in X \setminus K$*, there exists* $z_i{}^* \in B_i$ *such that* $\Psi_i(x, y_i, z_i^*) \subset -\operatorname{int} C_i(x), \forall y_i \in T_i(x)$*.*

Then, there exists $(\overline{x}, \overline{y_i}) = (\overline{x^i}, \overline{x_i}, \overline{y_i})$ *in* $K \times Y_i$ *such that for each* $i \in I$*,*

$$\overline{x_i} \in K_i, \qquad \overline{y_i} \in T_i(\overline{x}) : \Psi_i(\overline{x}, \overline{y_i}, z_i) \not\subset -\operatorname{int} C_i(\overline{x}), \quad \forall z_i \in X_i. \tag{3.1}$$

Proof

Case 1. For each $i \in I$, the set X_i is a compact set.

For each $i \in I$, define a set-valued mapping $P_i : X \times Y_i \to 2^{X_i}$ by

$$P_i(x, y_i) = \{z_i \in X_i : \Psi_i(x, y_i, z_i) \subset -\operatorname{int} C_i(x)\}, \quad \forall (x, y_i) \in X \times Y_i. \tag{3.2}$$

It is need to prove that $x_i \notin \operatorname{Co}(P_i(x, y_i))$ for all $(x, y_i) = (x^i, x_i, y_i) \in X \times Y_i$, where $\operatorname{Co} A$ denotes the convex hull of the set A. To see this, suppose, by way of contradiction, that there exist some $i \in I$ and some point $(\overline{x}, \overline{y_i}) \in X \times Y_i$ such that $\overline{x_i} \in \operatorname{Co}(P_i(\overline{x}, \overline{y_i}))$. Then there exist finite points $z_{i_1}, z_{i_2}, \ldots, z_{i_n}$ in X_i, and $\alpha_j \geq 0$ with $\sum_{j=1}^n \alpha_j = 1$ such that $\overline{x} = \sum_{j=1}^n \alpha_j z_{i_j}$ and $z_{i_j} \in P_i(\overline{x}, \overline{y_i})$ for all $j = 1, 2, \ldots, n$. That is, $\Psi_i(\overline{x}, \overline{x_i}, z_{i_j}) \in -\operatorname{int} C_i(\overline{x})$, $j = 1, 2, \ldots, n$, which contradicts the fact that $\Psi_i(x, \overline{y_i}, z_i)$ is $C_i(x)$-0-partially diagonally quasi-convex.

Now, it is need to prove that the set $P_i^{-1}(z_i) = \{(x, y_i) \in X \times Y_i : \Psi_i(x, y_i, z_i) \subset -\operatorname{int} C_i(x)\}$ is open for each $i \in I$ and for each $z_i \in X_i$. That is, P_i has open lower sections on $X \times Y_i$. It is only need to prove that $Q_i(z_i) = \{(x, y_i) \in X \times Y_i : \Psi_i(x, y_i, z_i) \not\subset -\operatorname{int} C_i(x)\}$ is closed for all $z_i \in X_i$. In fact, consider a net $(x_t, y_{i_t}) \in Q_i(z_i)$ such that $(x_t, y_{i_t}) \to (x, y_i) \in X \times Y_i$. Since $(x_t, y_{i_t}) \in Q_i(z_i)$, there exists $u_t \in \Psi_i(x_t, y_{i_t}, z_i)$ such that $u_t \notin -\operatorname{int} C_i(x_t)$. From the upper semicontinuity and compact values of Ψ_i on $X \times Y_i$ and [29, Proposition 1], it suffices to find a subset $\{u_{t_j}\}$ which converges to some $u \in \Psi_i(x, y_i, z_i)$, where $u_{t_j} \in \Psi_i(x_{t_j}, y_{i_{t_j}}, z_i)$. Since $(x_{t_j}, y_{i_{t_j}}) \to (x, y_i)$, by [8, Proposition 7, page 110] and the upper semicontinuity of M_i, it follows that $u \notin -\operatorname{int} C_i(x)$, and hence $(x, y_i) \in Q_i(z_i)$, $Q_i(z_i)$ is closed.

For each $i \in I$, also define another set-valued mapping $G_i : X \times Y_i \to 2^{X_i}$ by

$$G_i(x, y_i) = \operatorname{Co}(P_i(x, y_i)), \quad \forall (x, y_i) \in X \times Y_i. \tag{3.3}$$

Let $W_i = \{(x, y_i) \in X \times Y_i : G_i(x, y_i) \neq \emptyset\}$. Since P_i has open lower sections in X, and by [30, Lemma 5], we know that $\operatorname{Co}(P_i)$ has open lower sections in $X \times Y_i$. Then, for each $z_i \in X_i$, $G_i^{-1}(z_i) = (\operatorname{Co} P_i)^{-1}(z_i)$ is open, that is, G_i also has open lower sections in $X \times Y_i$. Hence, $W_i = \cup_{z_i \in X_i} G_i{}^{-1}(z_i)$ is an open set in $X \times Y_i$. Then, the set-valued mapping $G_i|_{W_i}$: $W_i \to 2^{X_i}$ has open lower sections in W_i, and for all $(x, y_i) \in W_i$, $G_i(x, y_i)$ is nonempty and convex. Also, since $X \times Y_i$ is metrizable space [20, page 50], W_i is paracompact [24, page 831]. Hence, by [30, Lemma 6], there is a continuous function $s_i : W_i \to X_i$ such that

$s_i(x,y_i) \in G_i(x,y_i)$ for all $(x,y_i) \in W_i$. Define $H_i : X \times Y_i \to 2^{X_i}$ by

$$H_i(x,y_i) = \begin{cases} \{s_i(x,y_i)\} & \text{if } (x,y_i) \in W_i, \\ X_i & \text{if } (x,y_i) \notin W_i. \end{cases} \tag{3.4}$$

It is easy to prove that H_i is upper semicontinuous.

Now define a set-valued mapping $\Gamma : X \times Y \to 2^{X\times Y}$ by $\Gamma(x,y) = (\prod_{i\in I} H_i(x,y_i), \prod_{i\in I} T_i(x))$, for each $(x,y) \in X \times Y$. By [17, Lemma 3], Γ is upper semicontinuous. Since for each $(x,y) \in X \times Y$, $H(x,y)$ is convex, closed, and nonempty, by [17, Theorem 1], there is $(\overline{x},\overline{y}) \in X \times Y$ such that $(\overline{x},\overline{y}) \in H(\overline{x},\overline{y})$. Note that for each $i \in I$, $(\overline{x},\overline{y_i}) \notin W_i$. Otherwise, there is some $i \in I$ such that $(\overline{x},\overline{y_i}) \in W_i$, then $\overline{x_i} = s_i(\overline{x},\overline{y_i}) \in \mathrm{Co}(P_i(\overline{x},\overline{y_i}))$, which contradicts $x_i \notin \mathrm{Co}(P_i(x,y_i))$ for all $(x,y_i) = (x^i,x_i,y_i) \in X \times Y_i$. Thus $\overline{x_i} \in X_i$, $\overline{y_i} \in T_i(\overline{x})$ and $G_i(\overline{x},\overline{y_i}) = \emptyset$ for each $i \in I$. That is, $\overline{x} \in X$, $\overline{y_i} \in T_i(\overline{x})$ and $\mathrm{Co}(P_i(\overline{x},\overline{y_i})) = \emptyset$ for each $i \in I$, which implies $\overline{x} \in X$, $\overline{y_i} \in T_i(\overline{x})$ and $P_i(\overline{x},\overline{y_i}) = \emptyset$ for each $i \in I$. Consequently, there exists $(\overline{x},\overline{y_i})$ in $X \times Y_i$ such that for each $i \in I$, $\overline{x} \in X$ and $\overline{y_i} \in T_i(\overline{x})$: $\Psi_i(\overline{x},\overline{y_i},z_i) \not\subset -\operatorname{int} C_i(\overline{x}), \forall z_i \in X_i$.

Case 2. X_i is not a compact set.

For each $i \in I$, let $\{z_{i_1},\dots,z_{i_k}\}$ be a finite subset of X_i. Let $\Lambda_i = \mathrm{Co}(B_i \cup \{z_{i_1},\dots,z_{i_k}\})$. Then, for each $i \in I$, Λ_i is compact and convex. By Case 1, there exists $\overline{x} \in \Lambda = \prod_{i\in I}\Lambda_i$ and $\overline{y_i} \in T_i(\overline{x})$ for each $i \in I$ such that, for each $i \in I$, $\Psi_i(\overline{x},\overline{y_i},z_i) \not\subset -\operatorname{int} C_i(\overline{x})$, for all $z_i \in \Lambda_i$. From $B \subset \Lambda$ and assumption (v) it follows that $\overline{x} \in K$. In particular, we have, $(\overline{x},\overline{y}) \in K \times Y$ such that, for each $i \in I$, $\Psi_i(\overline{x},\overline{y_i},z_{i_j}) \not\subset -\operatorname{int} C_i(\overline{x})$, for all $j = 1,2,\dots,k$. By (vi) and [8, Proposition 7, page 110], the set $\{(x,y_i) \in K \times Y_i : y_i \in T_i(x)\}$ is closed in $K \times Y_i$. Hence, for each $i \in I$ and for all $z_i \in X_i$, $\Delta(z_i) = \{(x,y_i) \in K \times Y_i : F_i(x,y_i,z_i) \not\subset -\operatorname{int} C_i(x),\ y_i \in T_i(x)\} = Q_i(z_i) \cap \{(x,y_i) \in K \times Y_i : y_i \in T_i(x)\}$ is closed in $K \times Y_i$. Since every finite subfamily of closed sets $\Delta(z_i)$ in compact set $K \times Y_i$ has a nonempty intersection, for each $i \in I$, $\cap_{z_i\in X_i}\Delta(z_i) \neq \emptyset$. Thus, there exists $(\overline{x},\overline{y}) \in K \times Y$ such that, for each $i \in I$, $\overline{x_i} \in K_i$, $\overline{y_i} \in T_i(\overline{x})$, such that $\Psi_i(\overline{x},\overline{y_i},z_i) \not\subset -\operatorname{int} C_i(\overline{x})$, for all $z_i \in X_i$. This completes the proof. □

Theorem 3.2. *If we replace, in Theorem 3.1, condition* (ii) *by the following conditions.*

(ii(a)) *For all $x = (x^i,x_i) \in X$, for all $y_i \in Y_i$, $\Psi_i(x,y_i,x_i) \not\subset -\operatorname{int} C_i(x)$.*

(ii(b)) *For each $(x,y_i) \in X \times Y_i$, the set $P_i(x,y_i) = \{z_i \in X_i : \Psi_i(x,y_i,z_i) \subset -\operatorname{int} C_i(x)\}$ is a convex set.*

Then, the conclusion of Theorem 3.1 still holds.

Proof. By Theorem 3.1, it is only need to prove that $\Psi_i(x,y_i,z_i)$ is $C_i(x)$-0-partially diagonally quasiconvex for each $i \in I$ and for all $y_i \in X_i$. If not, then there exist some $i \in I$ and $y_i \in X_i$, some finite set $\{z_{i_1},z_{i_2},\dots,z_{i_n}\}$ in X_i, and some point $x = (x^i,x_i) \in X$ with $x_i \in \mathrm{Co}\{z_{i_1},z_{i_2},\dots,z_{i_n}\}$. Then, for each $j = 1,2,\dots,n$, $\Psi_i(x,y_i,z_{i_j}) \subset -\operatorname{int} C_i(x)$. Since $P_i(x,y_i) = \{z_i \in X_i : \Psi_i(x,y_i,z_i) \subset -\operatorname{int} C_i(x)\}$ is a convex set, $x_i \in P_i(x,y_i)$, that is, $\Psi_i(x,y_i,x_i) \subset -\operatorname{int} C_i(x)$, which contradicts to the condition (ii). This completes the proof. □

THEOREM 3.3. *If we replace, in Theorem 3.2, condition* (ii(b)) *by the following condition.*
(ii(c)) *For each* $y_i \in Y_i$, $\Psi_i(x, y_i, z_i)$ *is* $C_i(x)$*-convex-like.*
Then, the conclusion of Theorem 3.2 still holds.

Proof. For each $i \in I$, define a set-valued mapping $P_i : X \times Y_i \to 2^{X_i}$ by $P_i(x, y_i) = \{z_i \in X_i : \Psi_i(x, y_i, z_i) \subset -\operatorname{int} C_i(x)\}, \forall (x, y_i) \in X \times Y_i$. It is easy to prove that for each $i \in I$ and for each $(x, y_i) \in X \times Y_i$, the set $P_i(x, y_i)$ is a convex set. By Theorem 3.2, the conclusion of Theorem 3.3 holds. This completes the proof. □

Then, some existence results for the special cases of the (SGSEP) will be considered.

COROLLARY 3.4. *Let I be an index set and I be countable. For each* $i \in I$, *let* Z_i *be a real topological vector space,* E_i *be a locally convex Hausdorff topological vector space,* $X_i \subset E_i$ *be a nonempty, convex and metrizable set, let* $C_i : X \to 2^{Z_i}$ *be a set-valued mapping such that* $C_i(x)$ *is a closed pointed and convex cone with* $\operatorname{int} C_i(x) \neq \emptyset$ *for each* $x \in X$, *let* $\Phi_i : X \times X_i \to 2^{Z_i}$ *be a set-valued mapping. For each* $i \in I$, *assume that the following.*

(i) *The set-valued mapping* $M_i = Z_i \backslash (-\operatorname{int} C_i) : X \to 2^{Z_i}$ *is upper semicontinuous.*

(ii) $\Phi_i(x, z_i)$ *is* $C_i(x)$*-0-partially diagonally quasiconvex.*

(iii) *For all* $z_i \in X_i$, *the map* $x \mapsto \Phi_i(x, z_i)$ *is upper semicontinuous on X with compact values.*

(vi) *There exists a nonempty compact subset* $K_i \subset X_i$ *and a compact convex set* $B_i \subset X_i$; *let* $K = \prod_{i\in I} K_i \subset X$ *and* $B = \prod_{i\in I} B_i \subset X$ *such that, for each* $x \in X \setminus K$, *there exists* $z_i{}^* \in B_i$ *such that* $\Phi_i(x, z_i{}^*) \subset -\operatorname{int} C_i(x)$.

Then, there exists $\overline{x}$ *in K such that for each* $i \in I$, $\Phi_i(\overline{x}, z_i) \not\subset -\operatorname{int} C_i(\overline{x}), \forall z_i \in X_i$.

Proof. For each $i \in I$, Let $Y_i = \{\overline{y_i}\}$ and define a set-valued mapping $T_i : X \to 2^{Y_i}$ as $T_i(x) = \{\overline{y_i}\}$ for all $x \in X$ and define another set-valued mapping $\Psi_i : X \times Y_i \times X_i$ as $\Psi_i(x, \overline{y_i}, z_i) = \Phi_i(x, z_i), \forall (x, \overline{y_i}, z_i) \in X \times Y_i \times X_i$. It is easy to see that all conditions of Theorem 3.1 are satisfied. Then the conclusion of Corollary 3.4 follows from Theorem 3.1. This completes the proof. □

COROLLARY 3.5. *If we replace, in Corollary 3.4, condition* (ii) *by the following conditions.*

(a) *For each* $x \in X$, $\{z_i \in X_i : \Phi_i(x, z_i) \subset -\operatorname{int} C_i(x)\}$ *is a convex set (or* $\Phi(x, z_i)$ *is* $C_i(x)$*-convex-like).*

(b) *For all* $x = (x^i, x_i) \in X$, $\Phi_i(x, x_i) \not\subset C_i(x)$.

Then, the conclusion of Corollary 3.4 still holds.

Remark 3.6. Theorems 3.1–3.3, Corollaries 3.4 and 3.5 improve and generalize [4, Theorems 2 and 3], [3, Theorems 2.1 and 2.2] with additional conditions of the metrizability of X_i.

Remark 3.7. By the results in this paper, it is easy to obtain the existence results for the other special cases of the (SGSEP), and they are omitted here.

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