GENERALIZED VECTOR QUASI-EQUILIBRIUM PROBLEMS WITH SET-VALUED MAPPINGS

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Received 26 October 2005; Revised 5 March 2006; Accepted 12 April 2006

A new mathematical model of generalized vector quasiequilibrium problem with set-valued mappings is introduced, and several existence results of a solution for the generalized vector quasiequilibrium problem with and without Φ-condensing mapping are shown. The results in this paper extend and unify those results in the literature.

1. Introduction

Throughout this paper, let Z, E, and F be topological vector spaces, let $X \subseteq E$ and $Y \subseteq F$ be nonempty, closed, and convex subsets. Let $D : X \to 2^X$, $T : X \to 2^Y$ and $\Psi : X \times Y \times X \to 2^Z$ be set-valued mappings, and let $C : X \to 2^Z$ be a set-valued mapping such that $C(x)$ is a closed pointed and convex cone with $\operatorname{int} C(x) \neq \emptyset$ for each $x \in X$, where $\operatorname{int} C(x)$ denotes the interior of the set $C(x)$. Then the generalized vector quasi-equilibrium problem with set-valued mappings (GVQEP) is to find $(\overline{x}, \overline{y})$ in $X \times Y$ such that

$$\overline{x} \in D(\overline{x}), \quad \overline{y} \in T(\overline{x}), \quad \Psi(\overline{x}, \overline{y}, z) \nsubseteq -\operatorname{int} C(\overline{x}), \quad \forall z \in D(\overline{x}). \tag{1.1}$$

The GVQEP is a new, interesting, meaningful, and general mathematical model, which contains many mathematical models as special cases, for some examples, we have the following.

(i) If Ψ is replaced by a single-valued function $f : X \times Y \times X \to Z$ and $C(x) = C$ for all $x \in X$, then the GVQEP reduces to finding $(\overline{x}, \overline{y})$ in $X \times Y$ such that

$$\overline{x} \in D(\overline{x}), \quad \overline{y} \in T(\overline{x}), \quad f(\overline{x}, \overline{y}, z) \notin -\operatorname{int} C, \quad \forall z \in D(\overline{x}). \tag{1.2}$$

It was investigated by Chiang et al. in [7].

Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2006, Article ID 69252, Pages 1–12
DOI 10.1155/JIA/2006/69252

If Ψ is replaced by a scalar function $f: X \times Y \times X \to R$ and $C(x) = \{r \in R : r \geq 0\}$ for all $x \in X$, then the GVQEP reduces to finding $(\overline{x}, \overline{y})$ in $X \times Y$ such that

$$\overline{x} \in D(\overline{x}), \quad \overline{y} \in T(\overline{x}), \quad f(\overline{x}, \overline{y}, z) \geq 0, \quad \forall z \in D(\overline{x}). \tag{1.3}$$

This was investigated in [5, 6, 12, 13, 19] and contains the generalized quasi-variational inequality in [4, 8, 20, 21] as a special case.

(ii) If $D(x) = X$ for all $x \in X$ and $f = -\Psi$, then the GVQEP reduces to finding $(\overline{x}, \overline{y})$ in $X \times Y$ such that

$$\overline{x} \in X, \quad \overline{y} \in T(\overline{x}), \quad f(\overline{x}, \overline{y}, z) \notin \operatorname{int} C(\overline{x}), \quad \forall z \in X. \tag{1.4}$$

It is the model of GVEP3 by Fu and Wan in [10]. Fu and Wan also introduce another kind of general vector equilibrium problem (i.e., GVEP1 in [10]) which is to find $\overline{x}$ in X such that for all $z \in X$, $\exists \overline{y} \in T(\overline{x})$, $f(\overline{x}, \overline{y}, z) \notin \operatorname{int} C(\overline{x})$.

(iii) If $Y = \{\overline{y}\}$ and $T(x) = \{\overline{y}\}$ for all $x \in X$, define a function $\varphi: X \times X \to 2^Z$ as $\varphi(x,z) = F(x, \overline{y}, z)$, then the GVQEP reduces to finding $\overline{x}$ in X such that

$$\overline{x} \in D(\overline{x}), \quad \varphi(\overline{x}, z) \nsubseteq -\operatorname{int} C(\overline{x}), \quad \forall z \in D(\overline{x}). \tag{1.5}$$

This was studied in [1, 16].

In this paper, by some maximal element theorems, we prove the existence results of a solution for the GVQEP with and without Φ-condensing mappings, and we also present some existence results of a solution for the special cases of the GVQEP. The results in this paper extend and unify those results in [1, 5–7, 10, 12, 13, 15, 19] and the references therein.

2. Preliminaries

In this section, we recall some definitions and some well-known results we need.

Definition 2.1 (see [2]). Let $C: X \to 2^Z$ be a set-valued mapping with $\operatorname{int} C(x) \neq \emptyset$ for all $x \in X$. Let $\varphi: X \times X \to 2^Z$ be a set-valued mapping. Then $\varphi(x,z)$ is said to be C_x-quasiconvex-like if for all $x \in X$, $y_1, y_2 \in X$, and $\alpha \in [0,1]$, either

$$\varphi(x, \alpha y_1 + (1-\alpha) y_2) \subseteq \varphi(x, y_1) - C(x) \tag{2.1}$$

or

$$\varphi(x, \alpha y_1 + (1-\alpha) y_2) \subseteq \varphi(x, y_2) - C(x). \tag{2.2}$$

Definition 2.2 (see [15]). Let $C: X \to 2^Z$ be a set-valued mapping such that $C(x)$ is a closed pointed and convex cone with $\operatorname{int} C(x) \neq \emptyset$ for each $x \in X$. Then the set-valued mapping $\varphi: X \times X \to 2^Z$ is called to be C_x-0-diagonally quasiconvex if for any finite set

$\{z_1,z_2,\dots,z_n\}$ in X, and for all $x\in X$ with $x\in \mathrm{Co}\{z_1,z_2,\dots,z_n\}$, there exists some $j\in \{1,2,\dots,n\}$ such that $\varphi(x,z_j)\nsubseteq -\mathrm{int}\,C(x)$.

Remark 2.3. (i) It is clear that if $Z=R$ and $C(x)=\{r\in R: r\geq 0\}$ for all $x\in X$, and φ is a single-valued function, then the $C(x)$-0-diagonal quasiconvexity of φ reduces to the 0-diagonal quasiconvexity in [22, 23], here $\gamma=0$.

(ii) The following example shows that the C_x-0-diagonal quasiconvexity of a set-valued map F is a true generalization of C_x-convex-likeness of the same F.

Let E be a real normed space with dual space E^*, $X\subset E$, $Z=R$. Let $\|\bullet\|$ denote the norm on E. Let $C: X\to 2^Z$ be defined as $C(x)=[\|x\|,+\infty)$, for all $x\in X$, and let $[e_1,e_2]$ denote the line segment joining e_1 and e_2. Choose $p\in E^*$, we define a set-valued map $F: X\times X\to 2^Z$ as

$$F(x,z)=\{\langle u,z-x\rangle : u\in[-2\|x\|\|z\|p,-\|x\|\|z\|p]\},\quad \forall x\in X. \tag{2.3}$$

Then, F is C_x-0-diagonal quasiconvex in the second argument. Otherwise, there exists finite set $\{z_1,z_2,\dots,z_n\}\subseteq X$, and there is $x\in X$ with $x=\sum_{j=1}^n\alpha_j z_j$ ($\alpha_j\geq 0$, $\sum_{j=1}^n\alpha_j=1$) such that for all $j=1,2,\dots,n$, $F(x,z_j)\subseteq -\mathrm{int}\,C(x)$. Then for each j, for all $\lambda_j\in[0,1]$, we have

$$\langle \lambda_j(-2\|x\|\|z_j\|p)+(1-\lambda_j)(-\|x\|\|z_j\|p),z_j-x\rangle < -\|x\|\leq 0. \tag{2.4}$$

It follows that

$$\langle p,z_j-x\rangle > 0,\quad j=1,2,\dots,n. \tag{2.5}$$

Then we have

$$0<\sum_{j=1}^n\alpha_j\langle p,z_j-x\rangle=\langle p,x-x\rangle=0, \tag{2.6}$$

a contradiction.

Definition 2.4 (see [3, 14]). Let E and F be two topological spaces and let $T: E\to 2^F$ be a set-valued mapping.

(1) A subset $X\subseteq E$ is said to be compactly open (resp., compactly closed) in E if for any nonempty compact subset K of E, $X\cap K$ is open (resp., closed) in K.

(2) T is said to be upper semicontinuous if the set $\{x\in E: T(x)\subseteq V\}$ is open in E for every open subset V of F.

(3) T is said to have open (resp., compactly open) lower sections if the set $T^{-1}(y)=\{x\in E: y\in T(x)\}$ is open (resp., compactly open) in E for each $y\in F$.

Remark 2.5. Clearly each open (resp., closed) subset of E is compactly open (resp., compactly closed), and the converse is not true in general.

Definition 2.6 (see [9]). Let E be a Hausdorff topological space and L a lattice with least element, denoted by 0. A map $\Phi: 2^E\to L$ is a measure of noncompactness provided that the following conditions hold for all $M,N\in 2^E$:

(i) $\Phi(M) = 0$ if and only if M is precompact (i.e., it is relatively compact);
(ii) $\Phi(\overline{\text{Co}}M) = \Phi(M)$, where $\overline{co}M$ denotes the convex closure of M;
(iii) $\Phi(M \cup N) = \max\{\Phi(M), \Phi(N)\}$.

Definition 2.7 (see [9]). Let $\Phi : 2^E \to L$ be a measure of noncompactness on E and $D \subseteq E$. A set-valued mapping $T : D \to 2^E$ is called Φ-condensing provided that if $M \subseteq D$ with $\Phi(T(M)) \geq \Phi(M)$, then M is relatively compact.

Remark 2.8. Note that every set-valued mapping defined on a compact set is Φ-condensing for any measure of noncompactness Φ. If E is locally convex and $T : D \to 2^E$ is a compact set-valued mapping (i.e., $T(X)$ is precompact), then T is Φ-condensing for any measure of noncompactness Φ. It is clear that if $T : D \to 2^E$ is Φ-condensing and $T^* : D \to 2^E$ satisfies $T^*(x) \subseteq T(x)$ for all $x \in D$, then T^* is also Φ-condensing.

Let CoA denote the convex hull of the set A.

Lemma 2.9 (see [11]). *Let X be a nonempty convex subset of a Hausdorff topological vector space E and let $S : X \to 2^X$ be a set-valued mapping such that for each $x \in X$, $x \notin \text{Co}(S(x))$ and for each $y \in X$, $S^{-1}(y)$ is open in X. Suppose further that there exist a nonempty compact subset N of X and a nonempty compact convex subset B of X such that $\text{Co}(S(x)) \cap B \neq \emptyset$ for all $x \in X \setminus N$. Then there exists a point $\overline{x} \in X$ such that $S(\overline{x}) = \emptyset$.*

Lemma 2.10 (see [14]). *Let X be a nonempty closed and convex subset of a locally convex topological vector space E and let $\Phi : 2^E \to L$ be a measure of noncompactness on E. Suppose that $S : X \to 2^X$ is a set-valued mapping such that the following conditions are satisfied:*
(i) *for each $x \in X$, $x \notin S(x)$;*
(ii) *for each $y \in X$, $S^{-1}(y)$ is compactly open in X;*
(iii) *the set-valued mapping $S : X \to 2^X$ is Φ-condensing. Then there exists $\overline{x} \in X$ such that $S(\overline{x}) = \emptyset$.*

3. Existence results

Some existence results of a solution for the GVQEP without Φ-condensing mappings are first shown.

Theorem 3.1. *Let Z be a topological vector space, let E and F be two Hausdorff topological vector spaces, let $X \subseteq E$ and $Y \subseteq F$ be nonempty and convex subsets, let $C : X \to 2^Z$ be a set-valued mapping such that $C(x)$ is a closed pointed and convex cone with $\text{int}\, C(x) \neq \emptyset$ for each $x \in X$, let $D : X \to 2^X$ and $T : X \to 2^Y$ be set-valued mappings with nonempty convex values and open lower sections, and the set $W = \{(x,y) \in X \times Y : x \in D(x)$ and $y \in T(x)\}$ is closed in $X \times Y$. Let $\Psi : X \times Y \times X \to 2^Z$ be a set-valued mapping. Assume that*
(i) *$M = Z \setminus (-\text{int}\, C) : X \to 2^Z$ is upper semicontinuous;*
(ii) *for each $y \in Y$, $\Psi(x,y,z)$ is C_x-0-diagonally quasiconvex;*
(iii) *for all $z \in X$, $(x,y) \mapsto \Psi(x,y,z)$ is upper semicontinuous on $X \times Y$ with compact values;*
(iv) *there exist nonempty and compact subsets $N \subseteq X$ and $K \subseteq Y$ and nonempty, compact, and convex subsets $B \subseteq X$, $A \subseteq Y$ such that for all $(x,y) \in X \times Y \setminus N \times K$ $\exists \overline{u} \in B$, $\overline{v} \in A$ satisfying $\overline{u} \in D(x)$, $\overline{v} \in T(x)$ and $\Psi(x,y,\overline{u}) \subseteq -\text{int}\, C(x)$.*

Then, there exists $(\overline{x},\overline{y})$ *in* $X\times Y$ *such that*

$$\overline{x}\in D(\overline{x}),\quad \overline{y}\in T(\overline{x}),\quad \Psi(\overline{x},\overline{y},z)\nsubseteq -\operatorname{int}C(\overline{x}),\quad \forall z\in D(\overline{x}). \tag{3.1}$$

That is, the solution set of the GVQEP is nonempty.

Proof. Define a set-valued mapping $P: X\times Y\to 2^X$ by

$$P(x,y)=\{z\in X:\Psi(x,y,z)\subseteq -\operatorname{int}C(x)\},\quad \forall (x,y)\in X\times Y. \tag{3.2}$$

It is needed to prove that $x\notin \operatorname{Co}(P(x,y))$ for all $(x,y)\in X\times Y$. To see this, suppose, by way of contradiction, that there exist some points $(\overline{x},\overline{y})\in X\times Y$ such that $\overline{x}\in\operatorname{Co}(P(\overline{x},\overline{y}))$. Then there exist finite points $z_1,z_2,\dots,z_n$ in X, and $\alpha_j\geq 0$ with $\sum_{j=1}^n\alpha_j=1$ such that $\overline{x}=\sum_{j=1}^n\alpha_j z_j$ and $z_j\in P(\overline{x},\overline{y})$ for all $j=1,2,\dots,n$. That is, $\Psi(\overline{x},\overline{y},z_j)\subseteq -\operatorname{int}C(\overline{x})$, $j=1,2,\dots,n$, which contradicts the fact that $\Psi(x,\overline{y},z)$ is C_x-0-diagonal quasiconvex. Therefore, $x\notin\operatorname{Co}(P(x,y))$, for all $(x,y)\in X\times Y$. Now, it is needed to prove that the set $P^{-1}(z)=\{(x,y)\in X\times Y:\Psi(x,y,z)\subseteq -\operatorname{int}C(x)\}$ is open for each $z\in X$. That is, P has open lower sections on $X\times Y$. It is only needed to prove that $Q(z)=\{(x,y)\in X\times Y:\Psi(x,y,z)\nsubseteq -\operatorname{int}C(x)\}$ is closed for all $z\in X$. In fact, consider a net $(x_t,y_t)\in Q(z)$ such that $(x_t,y_t)\to(x,y)\in X\times Y$. Since $(x_t,y_t)\in Q(z)$, there exists $u_t\in\Psi(x_t,y_t,z)$ such that $u_t\notin -\operatorname{int}C(x_t)$. From the upper semicontinuity and compact values of Ψ on $X\times Y$ and [17, Proposition 1], it suffices to find a subset $\{u_{t_j}\}$ which converges to some $u\in\Psi(x,y,z)$, where $u_{t_j}\in\Psi(x_{t_j},y_{t_j},z)$. Since $(x_{t_j},y_{t_j})\to(x,y)$, by [3, Proposition 7, page 110] and the upper semicontinuity of M, it follows that $u\notin -\operatorname{int}C(x)$, and hence $(x,y)\in Q(z)$, $Q(z)$ is closed.

Hence, P has open lower sections, and by [18, Lemma 2], we know that $\operatorname{Co}P: X\times Y\to 2^X$ also has open lower sections. Also define another set-valued mapping $S: X\times Y\to 2^{X\times Y}$ by

$$S(x,y)=\begin{cases}[D(x)\cap\operatorname{Co}P(x,y)]\times T(x) & \text{if } (x,y)\in W,\\ D(x)\times T(x) & \text{if } (x,y)\notin W.\end{cases} \tag{3.3}$$

Then, it is clear that for all $(x,y)\in X\times Y$, $S(x,y)$ is convex, and $(x,y)\notin\operatorname{Co}(S(x,y))=S(x,y)$.

Since for all $(u,v)\in X\times Y$,

$$\begin{aligned}S^{-1}(u,v)=&\left[\operatorname{Co}P^{-1}(u)\cap(D^{-1}(u)\times Y)\cap(T^{-1}(v)\times Y)\right]\\ &\cup\left[(X\times Y\setminus W)\cap(D^{-1}(u)\times Y)\cap(T^{-1}(v)\times Y)\right],\end{aligned} \tag{3.4}$$

and $D^{-1}(u)\times Y$, $T^{-1}(v)\times Y$, $\operatorname{Co}P^{-1}(u)$, and $X\times Y\setminus W$ are open in $X\times Y$, we have $S^{-1}(u,v)$ open in $X\times Y$.

From condition (iv), there exist nonempty and compact subset $N\times K\subseteq X\times Y$ and nonempty, compact, and convex subset $B\times A\subseteq X\times Y$ such that for all $(x,y)\in X\times Y\setminus N\times K$, $\exists(\overline{u},\overline{v})\in S(x,y)\cap(B\times A)$. And so $\operatorname{Co}(S(x,y))\cap(B\times A)\neq\varnothing$. Hence, by Lemma 2.9, $\exists(\overline{x},\overline{y})\in X\times Y$ such that $S(\overline{x},\overline{y})=\varnothing$. Since for all $(x,y)\in X\times X$, $D(x)$ and $T(y)$ are nonempty, we have $(\overline{x},\overline{y}) \in W$ and $D(\overline{x}) \cap \operatorname{Co} P(\overline{x},\overline{y}) = \emptyset$. This implies that $(\overline{x},\overline{y}) \in W$ and $D(\overline{x}) \cap P(\overline{x},\overline{y}) = \emptyset$. Therefore,

$$\overline{x} \in D(\overline{x}), \quad \overline{y} \in T(\overline{x}), \quad \Psi(\overline{x},\overline{y},z) \nsubseteq -\operatorname{int} C(\overline{x}), \quad \forall z \in D(\overline{x}). \tag{3.5}$$

That is, the solution set of the GVQEP is nonempty. The proof is completed. □

Theorem 3.2. *Let Z be a topological vector space, let E and F be two Hausdorff topological vector spaces, let $X \subseteq E$ and $Y \subseteq F$ be nonempty and convex subsets, let $C : X \to 2^Z$ be a set-valued mapping such that $C(x)$ is a closed pointed and convex cone with $\operatorname{int} C(x) \neq \emptyset$ for each $x \in X$, let $D : X \to 2^X$ and $T : X \to 2^Y$ be set-valued mappings with nonempty convex values and open lower sections, and the set $W = \{(x,y) \in X \times Y : x \in D(x)$ and $y \in T(x)\}$ is closed in $X \times Y$. Let $\Psi : X \times Y \times X \to 2^Z$ be a set-valued mapping. Assume that*

(i) *$M = Z \backslash (-\operatorname{int} C) : X \to 2^Z$ is upper semicontinuous;*
(ii) *for all $x \in X$, for all $y \in Y$, $\Psi(x,y,x) \nsubseteq -\operatorname{int} C(x)$;*
(iii) *for each $(x,y) \in X \times Y$, the set $P(x,y) = \{z \in X : \Psi(x,y,z) \subseteq -\operatorname{int} C(x)\}$ is a convex set;*
(iv) *for all $z \in X$, $(x,y) \mapsto \Psi(x,y,z)$ is upper semicontinuous on $X \times Y$ with compact values;*
(v) *there exist nonempty and compact subsets $N \subseteq X$ and $K \subseteq Y$ and nonempty, compact, and convex subsets $B \subseteq X$, $A \subseteq Y$ such that for all $(x,y) \in X \times Y \backslash N \times K$ $\exists \overline{u} \in B$, $\overline{v} \in A$ satisfying $\overline{u} \in D(x)$, $\overline{v} \in T(x)$, and $\Psi(x,y,\overline{u}) \subseteq -\operatorname{int} C(x)$.*

Then, there exists $(\overline{x},\overline{y})$ in $X \times Y$ such that

$$\overline{x} \in D(\overline{x}), \quad \overline{y} \in T(\overline{x}), \quad \Psi(\overline{x},\overline{y},z) \nsubseteq -\operatorname{int} C(\overline{x}), \quad \forall z \in D(\overline{x}). \tag{3.6}$$

That is, the solution set of the GVQEP is nonempty.

Proof. By Theorem 3.1, it is only needed to prove that $\Psi(x,y,z)$ is C_x-0-diagonally quasiconvex for all $y \in Y$. If not, then there exist $y \in Y$ and some finite set $\{z_1,z_2,\dots,z_n\}$ in X, and some point $x \in X$ with $x \in \operatorname{Co}\{z_1,z_2,\dots,z_n\}$, such that for each $j = 1,2,\dots,n$, $\Psi(x,y,z_j) \subseteq -\operatorname{int} C(x)$. Since $P(x,y) = \{z \in X : \Psi(x,y,z) \subseteq -\operatorname{int} C(x)\}$ is a convex set, $x \in P(x,y)$, that is, $\Psi(x,y,x) \subseteq -\operatorname{int} C(x)$, which contradicts the condition (ii). The proof is completed. □

Then, some existence results of a solution for the GVQEP with Φ-condensing mappings are shown as follows.

Theorem 3.3. *Let Z be a topological vector space, let E and F be two locally convex topological vector spaces, let $X \subseteq E$ and $Y \subseteq F$ be nonempty, closed, and convex subsets, let $C : X \to 2^Z$ be a set-valued mapping such that $C(x)$ is a closed pointed and convex cone with $\operatorname{int} C(x) \neq \emptyset$ for each $x \in X$, $D : X \to 2^X$, and $T : X \to 2^Y$ be set-valued mappings with nonempty convex values and compactly open lower sections, and the set $W = \{(x,y) \in X \times Y : x \in D(x)$ and $y \in T(x)\}$ is compactly closed in $X \times Y$. Let $\Psi : X \times Y \times X \to 2^Z$ be a set-valued mapping and $\Phi : 2^E \to L$ be a measure of noncompactness on E. Assume that*

(i) *$M = Z \backslash (-\operatorname{int} C) : X \to 2^Z$ is upper semicontinuous on each compact subset of X;*
(ii) *for each $y \in Y$, $\Psi(x,y,z)$ is C_x-0-diagonally quasiconvex;*

(iii) *for all* $z \in X$, $(x,y) \mapsto \Psi(x,y,z)$ *is upper semicontinuous on each compact subset of* $X \times Y$ *with compact values;*

(iv) *the set-valued map* $D \times T : X \times X \to 2^{X\times Y}$ *defined as* $(D\times T)(x,y) = D(x) \times T(y)$, *for all* $(x,y) \in X \times X$, *is* Φ*-condensing.*

Then, there exists $(\overline{x},\overline{y})$ *in* $X \times Y$ *such that*

$$\overline{x} \in D(\overline{x}), \quad \overline{y} \in T(\overline{x}), \quad \Psi(\overline{x},\overline{y},z) \nsubseteq -\operatorname{int} C(\overline{x}), \quad \forall z \in D(\overline{x}). \tag{3.7}$$

That is, the solution set of the GVQEP is nonempty.

Proof. Let $P : X \times Y \to 2^X$ and $S : X \times Y \to 2^{X\times Y}$ be the same as defined in the proof of Theorem 3.1. Following similar argument in the proof of Theorem 3.1, we have for all $(x,y) \in X \times Y$, $S(x,y)$ is convex, $(x,y) \notin S(x,y)$, and S has compactly open lower sections in $X \times Y$.

By the definition of S, $S(x,y) \subseteq D(x) \times T(x)$ for all $(x,y) \in X \times Y$. Since $D \times T$ is Φ-condensing, so is S. Hence, by Lemma 2.10, $\exists(\overline{x},\overline{y}) \in X \times Y$ such that $S(\overline{x},\overline{y}) = \emptyset$. Since for all $(x,y) \in X \times X$, $D(x)$, and $T(y)$ are nonempty, we have $(\overline{x},\overline{y}) \in W$ and $D(\overline{x}) \cap \operatorname{Co} P(\overline{x},\overline{y}) = \emptyset$. This implies that $(\overline{x},\overline{y}) \in W$ and $D(\overline{x}) \cap P(\overline{x},\overline{y}) = \emptyset$. Therefore,

$$\overline{x} \in D(\overline{x}), \quad \overline{y} \in T(\overline{x}), \quad \Psi(\overline{x},\overline{y},z) \nsubseteq -\operatorname{int} C(\overline{x}), \quad \forall z \in D(\overline{x}). \tag{3.8}$$

That is, the solution set of the GVQEP is nonempty. The proof is completed. □

By Theorem 3.3, and by similar argument to those in the proof of Theorem 3.2, it is easy to obtain the following result.

Theorem 3.4. *Let* Z *be a topological vector space, let* E *and* F *be two locally convex topological vector spaces, let* $X \subseteq E$ *and* $Y \subseteq F$ *be nonempty, closed, and convex subsets, let* $C : X \to 2^Z$ *be a set-valued mapping such that* $C(x)$ *is a closed pointed and convex cone with* $\operatorname{int} C(x) \neq \emptyset$ *for each* $x \in X$, *let* $D : X \to 2^X$ *and* $T : X \to 2^Y$ *be set-valued mappings with nonempty convex values and compactly open lower sections, and the set* $W = \{(x,y) \in X \times Y : x \in D(x)$ *and* $y \in T(x)\}$ *is compactly closed in* $X \times Y$. *Let* $\Psi : X \times Y \times X \to 2^Z$ *be a set-valued mapping and let* $\Phi : 2^E \to L$ *be a measure of noncompactness on* E. *Assume that*

(i) $M = Z\backslash(-\operatorname{int} C) : X \to 2^Z$ *is upper semicontinuous on each compact subset of* X;

(ii) *for all* $x \in X$, *for all* $y \in Y$, $\Psi(x,y,x) \nsubseteq -\operatorname{int} C(x)$;

(iii) *for each* $(x,y) \in X \times Y$, *the set* $P(x,y) = \{z \in X : \Psi(x,y,z) \subseteq -\operatorname{int} C(x)\}$ *is a convex set;*

(iv) *for all* $z \in X$, $(x,y) \mapsto \Psi(x,y,z)$ *is upper semicontinuous on each compact subset of* $X \times Y$ *with compact values;*

(v) *the set-valued map* $D \times T : X \times X \to 2^{X\times Y}$ *defined as* $(D\times T)(x,y) = D(x) \times T(y)$, *for all* $(x,y) \in X \times X$, *is* Φ*-condensing.*

Then, there exists $(\overline{x},\overline{y})$ *in* $X \times Y$ *such that*

$$\overline{x} \in D(\overline{x}), \quad \overline{y} \in T(\overline{x}), \quad \Psi(\overline{x},\overline{y},z) \nsubseteq -\operatorname{int} C(\overline{x}), \quad \forall z \in D(\overline{x}). \tag{3.9}$$

That is, the solution set of the GVQEP is nonempty.

Remark 3.5. If for each $y \in Y$, $\Psi(x,y,z)$ is C_x-convex-like, then the condition (iii) in both Theorems 3.2 and 3.4 holds. In fact, for any $z_1,z_2 \in P(x,y)$, that is, $z_1,z_2 \in X$, $\Psi(x,y,z_1) \subseteq -\operatorname{int} C(x)$ and $\Psi(x,y,z_2) \subseteq -\operatorname{int} C(x)$. Then, for all $\lambda \in [0,1]$, $\lambda z_1 + (1-\lambda)z_2 \in X$ since X is convex. And since $\Psi(x,y,z)$ is C_x-quasiconvex-like for all $y \in Y$, we have either

$$\Psi(x,y,\lambda z_1 + (1-\lambda)z_2) \subseteq \Psi(x,y,z_1) - C(x) \subseteq -C(x) - \operatorname{int} C(x) \subseteq -\operatorname{int} C(x), \tag{3.10}$$

or

$$\Psi(x,y,\lambda z_1 + (1-\lambda)z_2) \subseteq \Psi(x,y,z_2) - C(x) \subseteq -C(x) - \operatorname{int} C(x) \subseteq -\operatorname{int} C(x). \tag{3.11}$$

In both cases, we get $\Psi(x,y,\lambda z_1 + (1-\lambda)z_2) \subseteq -\operatorname{int} C(x)$. Hence, $\lambda z_1 + (1-\lambda)z_2 \in P(x,y)$ for all $(x,y) \in X \times Y$, and therefore $P(x,y)$ is convex.

Remark 3.6. Theorems 3.1, 3.2, 3.3, and 3.4, respectively, generalize those results in [5–7, 12, 13, 19] from scalar or vector-valued case to set-valued case with noncompact and nonmonotone conditions.

By [10, Lemma 2], we know that if $\bar{x}$ is a solution of GVEP3, then it is also is a solution of GVEP1. Fu and Wan [10] got some existence results of a solution for GVEP1. Let $f = -\Psi$ and $D(x) = X$ for all $x \in X$, by Theorems 3.1 and 3.3, respectively, we can obtain the existence results of a solution for GVEP3 as follows.

Corollary 3.7. *Let Z be a topological vector space, let E and F be two Hausdorff topological vector spaces, let $X \subseteq E$ and $Y \subseteq F$ be nonempty and convex subsets, let $C : X \to 2^Z$ be a set-valued mapping such that $C(x)$ is a closed pointed and convex cone with $\operatorname{int} C(x) \neq \emptyset$ for each $x \in X$, let $T : X \to 2^Y$ be a set-valued mapping with nonempty convex values and open lower sections, and the set $W = \{(x,y) \in X \times Y : y \in T(x)\}$ is closed in $X \times Y$. Let $f : X \times Y \times X \to 2^Z$ be a set-valued mapping. Assume that*

(i) *$M = Z \setminus (-\operatorname{int} C) : X \to 2^Z$ is upper semicontinuous;*

(ii) *for each $y \in Y$, $-f(x,y,z)$ is C_x-0-diagonally quasiconvex;*

(iii) *for all $z \in X$, $(x,y) \mapsto -f(x,y,z)$ is upper semicontinuous on $X \times Y$ with compact values;*

(iv) *there exist nonempty and compact subsets $N \subseteq X$, $K \subseteq Y$ and nonempty, compact, and convex subsets $B \subseteq X$, $A \subseteq Y$ such that for all $(x,y) \in X \times Y \setminus N \times K$ $\exists \bar{u} \in B$, $\bar{v} \in A$ satisfying $\bar{v} \in T(x)$ and $f(x,y,\bar{u}) \subseteq \operatorname{int} C(x)$.*

Then, there exists $\bar{x}$ in X and $\bar{y} \in T(\bar{x})$ such that $f(\bar{x},\bar{y},z) \nsubseteq \operatorname{int} C(\bar{x})$, for all $z \in X$. That is, the solution set of the GVEP3 is nonempty.

Corollary 3.8. *Let Z be a topological vector space, let E and F be two locally convex topological vector spaces, let $X \subseteq E$ and $Y \subseteq F$ be nonempty, closed and convex subsets, let $C : X \to 2^Z$ be a set-valued mapping such that $C(x)$ is a closed pointed and convex cone with $\operatorname{int} C(x) \neq \emptyset$ for each $x \in X$, let $T : X \to 2^Y$ be set-valued mapping with nonempty convex values and compactly open lower sections, and the set $W = \{(x,y) \in X \times Y : y \in T(x)\}$ is compactly closed in $X \times Y$. Let $f : X \times Y \times X \to 2^Z$ be a set-valued mapping and let $\Phi : 2^E \to L$ be a measure of noncompactness on E. Assume that*

(i) *$M = Z \setminus (-\operatorname{int} C) : X \to 2^Z$ is upper semicontinuous on each compact subset of X;*

(ii) *for each $y \in Y$, $-f(x,y,z)$ is C_x-0-diagonally quasiconvex;*

(iii) *for all* $z \in X$, $(x,y) \mapsto -f(x,y,z)$ *is upper semicontinuous on each compact subset of* $X \times Y$ *with compact values;*

(iv) *the set-valued map* $T : X \to 2^Y$ *is* Φ*-condensing.*

Then, there exists $\overline{x}$ *in* X *and* $\overline{y} \in T(\overline{x})$ *such that* $f(\overline{x},\overline{y},z) \notin \operatorname{int} C(\overline{x})$, *for all* $z \in X$. *That is, the solution set of the GVEP3 is nonempty.*

Remark 3.9. The condition (iii) of both Corollaries 3.7 and 3.8 can be replaced by the following conditions:

(a) for all $x \in X$, for all $y \in Y$, $f(x,y,x) \notin \operatorname{int} C(x)$;

(b) for each $(x,y) \in X \times Y$, the set $P(x,y) = \{z \in X : f(x,y,z) \subseteq \operatorname{int} C(x)\}$ is a convex set.

If $C(x) = C$ for all $x \in X$ and Ψ is replaced by a single-valued mapping f, then by Theorems 3.1 and 3.3, respectively, we have the following two results which are generalizations of [7, Theorems 3.2 and 3.5] and [19, Theorems 6 and 7].

Corollary 3.10. *Let* Z *be a topological vector space, let* E *and* F *be two Hausdorff topological vector spaces, let* $X \subseteq E$ *and* $Y \subseteq F$ *be nonempty and convex subsets, let* $C \subseteq Z$ *be a closed pointed and convex cone with* $\operatorname{int} C \neq \emptyset$, *let* $D : X \to 2^X$ *and* $T : X \to 2^Y$ *be set-valued mappings with nonempty convex values and open lower sections, and the set* $W = \{(x,y) \in X \times Y : x \in D(x)$ *and* $y \in T(x)\}$ *is closed in* $X \times Y$. *Let* $f : X \times Y \times X \to Z$ *be a single-valued mapping. Assume that*

(i) *for each* $y \in Y$, $f(x,y,z)$ *is* C*-0-diagonally quasiconvex;*

(ii) *for all* $z \in X$, $(x,y) \mapsto f(x,y,z)$ *is continuous on* $X \times Y$;

(iii) *there exist nonempty and compact subset* $N \subseteq X$ *and* $K \subseteq Y$ *and nonempty, compact, and convex subset* $B \subseteq X$, $A \subseteq Y$ *such that for all* $(x,y) \in X \times Y \backslash N \times K$ $\exists \overline{u} \in B$, $\overline{v} \in A$ *satisfying* $\overline{u} \in D(x)$, $\overline{v} \in T(x)$ *and* $f(x,y,\overline{u}) \in -\operatorname{int} C$.

Then, there exists $(\overline{x},\overline{y})$ *in* $X \times Y$ *such that*

$$\overline{x} \in D(\overline{x}), \quad \overline{y} \in T(\overline{x}) \quad f(\overline{x},\overline{y},z) \notin -\operatorname{int} C, \quad \forall z \in D(\overline{x}). \tag{3.12}$$

Corollary 3.11. *Let* Z *be a topological vector space, let* E *and* F *be two locally convex topological vector spaces, let* $X \subseteq E$ *and* $Y \subseteq F$ *be nonempty, closed, and convex subsets, let* $C \subseteq Z$ *be a closed pointed and convex cone with* $\operatorname{int} C \neq \emptyset$, *let* $D : X \to 2^X$ *and* $T : X \to 2^Y$ *be set-valued mappings with nonempty convex values and compactly open lower sections, and the set* $W = \{(x,y) \in X \times Y : x \in D(x)$ *and* $y \in T(x)\}$ *is compactly closed in* $X \times Y$. *Let* $f : X \times Y \times X \to Z$ *be a single-valued mapping and let* $\Phi : 2^E \to L$ *be a measure of noncompactness on* E. *Assume that*

(i) *for each* $y \in Y$, $f(x,y,z)$ *is* C*-0-diagonally quasiconvex;*

(ii) *for all* $z \in X$, $(x,y) \mapsto f(x,y,z)$ *is continuous on each compact subset of* $X \times Y$ *with compact values;*

(iii) *the set-valued map* $D \times T : X \times X \to 2^{X \times Y}$ *defined as* $(D \times T)(x,y) = D(x) \times T(y)$, *for all* $(x,y) \in X \times X$, *is* Φ*-condensing.*

Then, there exists $(\overline{x},\overline{y})$ *in* $X \times Y$ *such that*

$$\overline{x} \in D(\overline{x}), \quad \overline{y} \in T(\overline{x}), \quad f(\overline{x},\overline{y},z) \notin -\operatorname{int} C, \quad \forall z \in D(\overline{x}). \tag{3.13}$$

Remark 3.12. The condition (ii) in both Corollaries 3.10 and 3.11 can be replaced by the following conditions:

(a) for all $x \in X$, for all $y \in Y$, $f(x,y,x) \notin -\operatorname{int} C$;

(b) for each $(x,y) \in X \times Y$, the set $P(x,y) = \{z \in X : f(x,y,z) \in -\operatorname{int} C\}$ is a convex set.

Let $Y = \{\overline{y}\}$. Define a set-valued mapping $T : X \to 2^Y$ as $T(x) = \{\overline{y}\}$ for all $x \in X$ and define another set-valued mapping $\Psi : X \times Y \times X$ as $\Psi(x,\overline{y},z) = \varphi(x,z)$, for all $(x,\overline{y},z) \in X \times Y \times X$. Then by Theorems 3.1 and 3.3, respectively, we have following results which are generalizations of [1, Corollary 3.1].

Corollary 3.13. *Let Z be a topological vector space, let E be a Hausdorff topological vector space, let $X \subseteq E$ be a nonempty and convex subset, let $C : X \to 2^Z$ be a set-valued mapping such that $C(x)$ is a closed pointed and convex cone with $\operatorname{int} C(x) \neq \emptyset$ for each $x \in X$, let $D : X \to 2^X$ be a set-valued mapping with nonempty convex values and open lower sections, and the set $W = \{x \in X : x \in D(x)\}$ is closed in X. Let $\varphi : X \times X \to 2^Z$ be a set-valued mapping. Assume that*

(i) *$M = Z \backslash (-\operatorname{int} C) : X \to 2^Z$ is upper semicontinuous;*

(ii) *$\varphi(x,z)$ is C_x-0-diagonally quasiconvex;*

(iii) *for all $z \in X$, $x \mapsto \varphi(x,z)$ is upper semicontinuous on X with compact values;*

(iv) *there exist nonempty and compact subset $N \subseteq X$ and nonempty, compact, and convex subset $B \subseteq X$ such that for all $x \in X \backslash N$ $\exists \overline{u} \in B$ satisfying $\overline{u} \in D(x)$ and $\varphi(x,\overline{u}) \subseteq -\operatorname{int} C(x)$.*

Then, there exists $\overline{x}$ in X such that

$$\overline{x} \in D(\overline{x}), \quad \varphi(\overline{x},z) \nsubseteq -\operatorname{int} C(\overline{x}), \quad \forall z \in D(\overline{x}). \tag{3.14}$$

Corollary 3.14. *Let Z be a topological vector space, let E be a locally convex topological vector space, let $X \subseteq E$ be nonempty, closed, and convex subset, let $C : X \to 2^Z$ be a set-valued mapping such that $C(x)$ is a closed pointed and convex cone with $\operatorname{int} C(x) \neq \emptyset$ for each $x \in X$, let $D : X \to 2^X$ be set-valued mapping with nonempty convex values and compactly open lower sections, and the set $W = \{x \in X : x \in D(x)\}$ is compactly closed in X. Let $\varphi : X \times X \to 2^Z$ be a set-valued mapping, and let $\Phi : 2^E \to L$ be a measure of noncompactness on E. Assume that*

(i) *$M = Z \backslash (-\operatorname{int} C) : X \to 2^Z$ is upper semicontinuous on each compact subset of X;*

(ii) *$\varphi(x,z)$ is C_x-0-diagonally quasiconvex;*

(iii) *for all $z \in X$, $x \mapsto \varphi(x,z)$ is upper semicontinuous on each compact subset of X with compact values;*

(iv) *the set-valued map $D : X \to 2^X$ is Φ-condensing.*

Then, there exists $\overline{x}$ in X such that

$$\overline{x} \in D(\overline{x}), \quad \varphi(\overline{x},z) \nsubseteq -\operatorname{int} C(\overline{x}), \quad \forall z \in D(\overline{x}). \tag{3.15}$$

Remark 3.15. The condition (ii) in both Corollaries 3.13 and 3.14 can be replaced by the following conditions:

(a) for all $x \in X$, $\varphi(x,x) \nsubseteq -\operatorname{int} C(x)$;

(b) for each $x \in X$, the set $P(x) = \{z \in X : \varphi(x,z) \subseteq -\operatorname{int} C(x)\}$ is a convex set.

Acknowledgments

This paper was supported by the National Natural Science Foundation of China (Grant no. 10171118), Education Committee Project, Research Foundation of Chongqing (Grant no. 030801), and Natural Science Foundation of Chongqing (no. 8409).

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