CLASSES OF ELLIPTIC MATRICES

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The equivalence between some conditions concerning elliptic matrices is shown, namely, the Cordes condition, a generalized form of Campanato's condition, and a generalized form of a condition of Buică.

1. Introduction

Let Ω be an open bounded set in $\mathbb{R}^n$, $n > 2$, with a sufficiently regular boundary, and let $A(x) = \{a_{ij}(x)\}_{i,j=1,\dots,n}$ be a real matrix, with coefficients $a_{ij} \in L^\infty(\Omega)$. We consider the following problem:

$$\begin{gathered} u \in H^{2,2} \cap H_0^{1,2}(\Omega), \\ \sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u(x) = f(x), \quad \text{a.e. } x \in \Omega. \end{gathered} \tag{1.1}$$

If $f \in L^2(\Omega)$, it is known (see the counterexamples in [6]) that problem (1.1) is not well posed with the only hypothesis of uniform ellipticity on the matrix $A(x)$: there exists a positive constant $\bar{\nu}$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\eta_i\eta_j \geq \bar{\nu}\|\eta\|_n^2, \quad \text{a.e. in } \Omega,\ \forall \eta = (\eta_1,\dots,\eta_n) \in \mathbb{R}^n. \tag{1.2}$$

It is therefore essential, in order to be able to solve Problem (1.1), to assume some hypotheses on $A(x)$ stronger than (1.2). In this paper we consider some of these ones and compare them. More precisely, we will consider the following *conditions* and show that they are equivalent.

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Condition 1.1 (the Cordes condition, see [5, 8]). $\|A(x)\|_{\mathbb{R}^{n^2}} \neq 0$, a.e. in Ω, and there exists $\varepsilon \in (0,1)$ such that

$$\frac{\left(\sum_{i,j=1}^{n} a_{ii}(x)\right)^2}{\sum_{i,j=1}^{n} a_{ij}^2(x)} \geq n-1+\varepsilon, \quad \text{a.e. in } \Omega. \tag{1.3}$$

Condition 1.2 (Condition A_{xp}). There exist four real constants σ, γ, δ, p with $\sigma > 0$, $\gamma > 0$, $\delta \geq 0$, $\gamma + \delta < 1$, $p \geq 1$, and a function $a(x) \in L^\infty(\Omega)$, with $a(x) \geq \sigma$ a.e. in Ω, such that

$$\left| \sum_{i=1}^{n} \xi_{ii} - a(x) \sum_{i,j=1}^{n} a_{ij}(x)\xi_{ij} \right|^p \leq \gamma \|\xi\|_{n^2}^p + \delta \left| \sum_{i=1}^{n} \xi_{ii} \right|^p \tag{1.4}$$

for all $\xi = \{\xi_{ij}\}_{i,j=1,\dots,n} \in \mathbb{R}^{n^2}$, a.e. in Ω.

When $p = 1$, the above *condition* will be simply denoted by *Condition A_x;* it was defined in [10], where it has also been shown to be equivalent to the *Cordes condition.* If $a(x)$ is constant on Ω, *Conditon A_x* is the formulation for linear operators of Campanato's *condition A,* (see [4]), which was defined for nonlinear operators. A particular version of *Condition A_{xp}*, that is, with $p = 2$ and (x) constant, is stated in [7] for nonlinear operators.

Condition 1.3 (Condition B_x). There exist four real positive real constants σ, c_1, c_2, c_3 and a function $\beta \in L^\infty(\Omega)$ such that
 (i) $0 < c_1 - c_2 - c_3 < 1$,
 (ii) $\beta(x) \geq \sigma$ a.e. in Ω,
and moreover

$$\beta(x) \sum_{i,j=1}^{n} a_{ij}(x)\xi_{ij} \sum_{i=1}^{n} \xi_{ii} \geq c_1 \left(\sum_{i=1}^{n} \xi_{ii} \right)^2 - c_2 \left| \sum_{i=1}^{n} \xi_{ii} \right| \|\xi\|_{n^2} - c_3 \|\xi\|_{n^2}^2 \tag{1.5}$$

for all $\xi = \{\xi_{ij}\}_{i,j=1,\cdots,n} \in \mathbb{R}^{n^2}$, a.e. in Ω.

If $\beta(x)$ is constant on Ω, we will denote this *condition* as *Condition B;* it has been defined by Buică in [2].

The importance of *Conditions A_{xp}* or B_x is in the fact that they allow to show in a relatively simple manner, by means of *near operators theory* (see [4, 9]) or *weakly near operators theory* (see [1–3]), that problem (1.1) is well posed. The usefulness of showing the equivalence among these *conditions* is due to the fact that to verify whether a matrix satisfies *Condition A_{xp}* or B_x is very complicated, even if $n = 2$, while to verify whether it satisfies the *Cordes condition* is much simpler.

2. A procedure of decomposition for matrices

In this section we consider a short procedure of decomposition of the matrices A and I which has been developed in [10]. We set

$$\begin{gathered}\Omega_0 = \{x \in \Omega : \text{there exists } b(x) \in \mathbb{R} \text{ such that } b(x)A(x) = I\};\\ \Omega_1 = \Omega \backslash \Omega_0.\end{gathered} \tag{2.1}$$

Remark 2.1. Set $M = \sup_\Omega \|A(x)\|$, $\bar{\nu} = \inf_\Omega \|A(x)\|$, accordingly $n\bar{\nu} \le (A(x) \mid I) \le nM$. Then, for each $x \in \Omega_0$, we obtain $1/M \le b(x) \le 1/\bar{\nu}$.

We can assume $\operatorname{meas}\Omega_1 > 0$, since otherwise as we will see in the following it is easy to show the equivalence between the above *conditions.* We set for all $x \in \Omega_1$: $W(x) = \{B(x) : B(x) = sI + rA(x),\ s,r \in \mathbb{R}\}$; $\Sigma_x = W(x) \cap S(I,1)$ (where $S(I,1) = \{B : \|B - I\|_{\mathbb{R}^{n^2}} < 1\}$).

Let $v_1, w_2 \in W(x)$ be the projections of I on the lines through the zero vector of $\mathbb{R}^{n^2}$ and tangent to Σ_x. Moreover let v_2 be the projection of I on the line through the zero vector of $\mathbb{R}^{n^2}$ and perpendicular to v_1, and let w_1 be the projection of I on the line through the zero vector of $\mathbb{R}^{n^2}$ and perpendicular to w_2. In this manner we find two systems of orthogonal vectors $\{v_1, v_2\}$, $\{w_1, w_2\}$, with $v_i = v_i(x)$, $w_i = w_i(x)$, $i = 1,2$. Each of them is a basis in the plane $W(x)$. Then $I = v_1 + v_2 = w_1 + w_2$, and there are L^∞ functions $a_i = a_i(x)$ and $b_i = b_i(x)$, $i = 1,2$, such that

$A(x) = a_1(x)v_1(x) + a_2(x)v_2(x) = b_1(x)w_1(x) + b_2(x)w_2(x)$. (As $\|v_1\| = \|w_2\| = \sqrt{n-1}$ and $\|v_2\| = \|w_1\| = 1$, then for $i = 1,2$, $a_i^2 \le a_1^2(n-1) + a_2^2 = (a_1v_1 + a_2v_2 \mid a_1v_1 + a_2v_2) = (A(x) \mid A(x)) = \|A(x)\|^2$; here if $B = \{b_{ij}\}_{i,j=1,\cdots,n}$ and $C = \{c_{ij}\}_{i,j=1,\cdots,n}$, we set $(B \mid C) = \sum_{i,j=1}^n b_{ij}c_{ij}$.) Set

$$\begin{aligned}
Q_v(x,\nu,\tau) &= \{\xi \in \mathbb{R}^{n^2} : \xi = sv_1 + tv_2,\ 0 < \nu \le s,\ t \le \tau\},\\
Q_w(x,\nu,\tau) &= \{\xi \in \mathbb{R}^{n^2} : \xi = sw_1 + tw_2,\ 0 < \nu \le s,\ t \le \tau\},\\
R(x,\nu_0,\tau_0) &= \{\xi \in \mathbb{R}^{n^2} : \xi = sw_2 + tv_1,\ 0 < \nu_0 \le s,\ t \le \tau_0\},\\
C(\Sigma_x) &= \{v : v \in W(x) \text{ such that } \exists z \in \Sigma_x,\ \exists t > 0 \text{ for which } v = tz\},\\
C_\rho(x) &= \{v : v \in C(\Sigma_x) : \exists t > 0 \text{ such that } \|I - tv\| < \rho\}, \quad 0 < \rho < 1.
\end{aligned} \tag{2.2}$$

The following propositions are proved in [10].

Proposition 2.2. *For all $\tau, \nu > 0$ with $\nu \le \tau$, $\exists \tau_0$, ν_0, $0 < \tau_0 < \nu_0$, such that for all $x \in \Omega_1$,*

$$Q_v(x,\nu,\tau) \cap Q_w(x,\nu,\tau) \subset R(x,\nu_0,\tau_0). \tag{2.3}$$

Proposition 2.3. *For all τ_0, ν_0, $0 < \tau_0 < \nu_0$, there exists $\rho \in (0,1)$ such that for all $x \in \Omega_1$,*

$$R(x,\nu_0,\tau_0) \subset C_\rho(x). \tag{2.4}$$

3. Condition B_x

Proposition 3.1. *Condition A_x and Condition B_x are equivalent.*

Proof. We assume that A satisfies *Condition* A_x. It follows (from (1.4) with $p = 1$) by squaring both members

$$(I \mid \xi)^2 - 2a(x)(A \mid \xi)(I \mid \xi) \leq \gamma^2\|\xi\|^2 + 2\gamma\delta|(I \mid \xi)|\|\xi\| + \delta^2(I \mid \xi)^2 \tag{3.1}$$

then

$$2a(x)(A \mid \xi)(I \mid \xi) \geq (1-\delta^2)(I \mid \xi)^2 - 2\gamma\delta|(I \mid \xi)|\|\xi\| - \gamma^2\|\xi\|^2. \tag{3.2}$$

This is *Condition* B_x with $b(x) = 2a(x)$, $c_1 = 1 - \delta^2$, $c_2 = 2\gamma\delta$, $c_3 = \gamma^2$. □

Conversely, we set $\mathbf{A}(x) = \beta(x)A(x)$ and assume that *Condition B* holds for $\mathbf{A}$, then we will show that $\mathbf{A}$ also satisfies *Condition* A_x. To this purpose we write *Condition B* in the following form: there exist four real positive constants M, c_1, c_2, c_3 with $0 < c_1 - c_2 - c_3 < 1$, $\sup_{x\in\Omega}\|\mathbf{A}(x)\| \leq M$ such that

$$(\mathbf{A}(x) \mid \xi)(I \mid \xi) \geq c_1(I \mid \xi)^2 - c_2|(I \mid \xi)|\|\xi\| - c_3\|\xi\|^2, \tag{3.3}$$

for all $\xi \in \mathbb{R}^{n^2}$, a.e. in Ω. Then we obtain the thesis by using the decomposition of $\mathbf{A}$ and I stated in Section 2. For this we distinguish two cases: $x \in \Omega_0$ and $x \in \Omega_1$.

If $x \in \Omega_0$, that is, there exists $b(x)$ such that $b(x)\mathbf{A}(x) = I$, then *Condition* A_x is trivially true (take in (1.4) $a(x) = b(x)$).

Instead, if $x \in \Omega_1$, with $\operatorname{meas}\Omega_1 > 0$, we observe that (3.3) holds in particulcular for $\xi \in W(x)$. So we can write ξ as a linear combination of the basis $\{v_1(x), v_2(x)\}$. Now, let $t_1, t_2 \in \mathbb{R}$ be such that $\xi = t_1v_1(x) + t_2v_2(x)$, accordingly $\|\xi\|^2 = (\xi \mid \xi) = t_1^2(n-1) + t_2^2$, then

$$\begin{aligned}(\mathbf{A} \mid \xi) &= (a_1(x)v_1 + a_2(x)v_2 \mid t_1v_1 + t_2v_2) = a_1t_1(n-1) + a_2t_2,\\ (I \mid \xi) &= (v_1 + v_2 \mid t_1v_1 + t_2v_2) = t_1(n-1) + t_2.\end{aligned} \tag{3.4}$$

Now, (3.4) and the above remarks yield the following form of *Condition B*: for each $\xi \in W(x)$,

$$\begin{aligned}(\mathbf{A} \mid \xi)(I \mid \xi) &= [a_1t_1(n-1) + a_2t_2][t_1(n-1) + t_2]\\ &\geq c_1[t_1(n-1) + t_2]^2 - c_2[t_1(n-1) + t_2]\sqrt{t_1^2(n-1) + t_2^2} - c_3[t_1^2(n-1) + t_2^2].\end{aligned} \tag{3.5}$$

Put

$$\begin{aligned}F(t_1, t_2) = {}&[a_1t_1(n-1) + a_2t_2][t_1(n-1) + t_2] - c_1[t_1(n-1) + t_2]^2\\ &+ c_2[t_1(n-1) + t_2]\sqrt{t_1^2(n-1) + t_2^2} + c_3[t_1^2(n-1) + t_2^2].\end{aligned} \tag{3.6}$$

Remark that

$$F(t_1, t_2) \geq 0, \quad \forall (t_1, t_2) \in \mathbb{R}^2 \text{ (by (3.5))}. \tag{3.7}$$

In particular

$$F\left(\frac{1}{\sqrt{n-1}},0\right) = a_1(n-1) - c_1(n-1) + c_2\sqrt{n-1} + c_3 \geq 0 \tag{3.8}$$

from which

$$a_1(x) \geq c_1 - \frac{c_2}{\sqrt{n-1}} - \frac{c_3}{n-1} \geq c_1 - c_2 - c_3 > 0. \tag{3.9}$$

While the inequality $F(0,1) = a_2(x) - c_1 + c_2 + c_3 \geq 0$ implies $a_2(x) \geq c_1 - c_2 - c_3 > 0$.

In the same way, by taking the system of orthogonal vectors $\{w_1, w_2\}$ as basis of $W(x)$, it follows that

$$b_i(x) \geq c_1 - c_2 - c_3 > 0, \quad i = 1,2,\ x \in \Omega_1. \tag{3.10}$$

So we have shown (see Section 2) that $\mathbf{A}(x) \in Q_v(x,\nu,\tau) \cap Q_w(x,\nu,\tau)$. This implies, by Proposition 2.2, $\mathbf{A}(x) \in R(x,\nu_0,\tau_0)$, then by Proposition 2.3, $\mathbf{A}(x) \in C_\rho(x)$, which is equivalent to say that *Condition* A_x is valid with $\delta = 0$.

Taking into account this proposition and the equivalence between the *Cordes condition* and *Condition* A_x, shown in [10], we have the following.

Corollary 3.2. *Condition* B_x *and the Cordes condition are equivalent.*

The following example states that *Condition B* is stronger than *Condition* A_x and therefore is also stronger than the *Cordes condition*.

Example 3.3. Let $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 = \{(x_1,x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, 0 < x_2 \leq 1\}$ and $\Omega_2 = \{(x_1,x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, 1 < x_2 < 2\}$, moreover

$$A(x) = \begin{cases} A_1, & \text{if } x \in \Omega_1, \\ A_2, & \text{if } x \in \Omega_2, \end{cases} \qquad A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 200 & -150 \\ -150 & 200 \end{pmatrix}. \tag{3.11}$$

A is uniformly elliptic on Ω and, since $n = 2$, this implies the *Cordes condition* and therefore also *Condition* A_x (see [10]). Nevertheless A does not satisfy *Condition B*. Indeed, we consider $x \in \Omega_1$, then $A(x) = A_1$. We observe that if A_1 satisfied *Condition B,* it would be

$$(A_1 \mid \xi)(I \mid \xi) \geq c_1 (I \mid \xi)^2 - c_2 \left| (I \mid \xi) \right| \|\xi\| - c_3 \|\xi\|^2 \tag{3.12}$$

for each $\xi \in \mathbb{R}^4$, that is,

$$(1 - c_1)(I \mid \xi)^2 + c_2 \left| (I \mid \xi) \right| \|\xi\| + c_3 \|\xi\|^2 \geq 0. \tag{3.13}$$

The bilinear form $\Phi(X,Y) = (1-c_1)X^2 + c_2XY + c_3Y^2$, where $(X,Y) \in \mathbb{R}^2$, is nonnegative if $(1-c_1)c_3 \geq c_2^2/4$. In particular it must hold $c_1 < 1$. Otherwise if $A(x)$ satisfied *Condition B* on Ω_2 it would be

$$(A_2 \mid \xi)(I \mid \xi) \geq c_1 (I \mid \xi)^2 - c_2 \left| (I \mid \xi) \right| \|\xi\| - c_3 \|\xi\|^2, \tag{3.14}$$

where c_1, c_2, c_3 are the above determined constants for the matrix A_1. Now we consider the matrix

$$\xi = \begin{pmatrix} -1 & 0 \\ -2 & 0 \end{pmatrix}, \tag{3.15}$$

by replacing it in (3.14), we obtain $-100 \geq c_1 - c_2\sqrt{5} - 5c_3$, that is, $c_2(\sqrt{5}-1) + 4c_3 \geq c_1 - c_2 - c_3 + 100$; that implies (because by hypothesis it holds $c_1 > c_2 + c_3$) $4c_1 > 4(c_2 + c_3) \geq 100$, then $c_1 \geq 25$. This contradicts what we have obtained for A_1, that is, $c_1 < 1$.

4. Condition A_{xp}

We prove equivalence between the *Cordes condition* and *Condition* A_{xp} in the same way used in [10] for the proof of equivalence between *Condition A* and the *Cordes condition.* The first step is following.

Lemma 4.1. *Condition A_{xp} with $\delta = 0$ is equivalent to Cordes Condition.*

Proof (see also [10]). We can write *Condition* A_{xp}, if $\delta = 0$, as follows:

$$\left| (I - a(x)A(x) \mid \xi) \right| \leq \gamma^{1/p} \|\xi\| \tag{4.1}$$

for all $\xi \in \mathbb{R}^{n^2}$, and $p \geq 1$. This is just *Condition* A_x with $\delta = 0$ and, accordingly to what proved in [10], this is equivalent to the *Cordes condition.* □

The second step for the achievement of our goal is following.

Lemma 4.2. *If $A(x)$ satisfies Condition A_{xp} for some function $a(x)$ and some constants σ, γ, δ, then it satisfies the same condition with $\delta = 0$ and possibly different σ, γ, $a(x)$.*

Proof. We proceed on the line of the proof of [10, Lemma 3.3]. We follow the notations of Section 2. *Condition* A_{xp}, with $\delta \neq 0$, yields *Condition* A_{xp} with $\delta = 0$, by replacing the coefficient $a(x)$ of the first *condition* with a new coefficient $\bar{a}(x)$, defined by

$$\bar{a}(x) = \begin{cases} b(x), & \text{if } x \in \Omega_0, \\ c(x), & \text{if } x \in \Omega_1. \end{cases} \tag{4.2}$$

If $x \in \Omega_0$, then *Condition* A_{xp} with $\delta = 0$ is trivially satisfied. Moreover, by Remark 2.1, $1/M \leq b(x) \leq 1/\bar{\nu}$. Now let $x \in \Omega_1$. We prove the existence of a function $c(x)$ by means of the decomposition of matrices $A(x)$, I stated in Section 2 and replacing the expressions obtained in *Condition* A_{xp}:

$$\begin{aligned} \left| (I - a(x)A(x) \mid \xi) \right|^p &= \left| (v_1 + v_2 - a(x)(a_1 v_1 + a_2 v_2) \mid \xi) \right|^p \\ &= (\text{take } \xi = v_i,\ i = 1,2) \\ &= \left| (v_1 + v_2 - a(x)(a_1 v_1 + a_2 v_2) \mid v_i) \right|^p = \left| \|v_i\|^2 - a(x)a_i \|v_i\|^2 \right|^p \\ &= \left| 1 - a(x)a_i \right|^p \|v_i\|^{2p} \leq \gamma \|v_i\|^p + \delta (v_1 + v_2 \mid v_i)^p = \gamma \|v_i\|^p + \delta \|v_i\|^{2p}. \end{aligned} \tag{4.3}$$

From this

$$\frac{1}{a(x)}\left(1-\frac{\sqrt[p]{\gamma+\delta\|v_i\|^p}}{\|v_i\|}\right)\leq a_i\leq\frac{1}{a(x)}\left(1+\frac{\sqrt[p]{\gamma+\delta\|v_i\|^p}}{\|v_i\|}\right). \tag{4.4}$$

We observe that

$$1-(\gamma+\delta)^{1/p}\leq 1-\frac{\sqrt[p]{\gamma+\delta\|v_i\|^p}}{\|v_i\|},\qquad 1+\frac{\sqrt[p]{\gamma+\delta\|v_i\|^p}}{\|v_i\|}\leq 1+(\gamma+\delta)^{1/p}. \tag{4.5}$$

Using $\|v_1\|=\sqrt{n-1}$, $v_2=1$, we can write

$$\frac{\gamma+\delta\|v_i\|^p}{\|v_i\|^p}\leq\gamma+\delta,\quad i=1,2. \tag{4.6}$$

We conclude, from (4.4), by setting

$$M_1=\sup_{\Omega}a(x),\qquad \nu=\frac{1}{M_1}\left[1-(\gamma+\delta)^{\frac{1}{p}}\right],\qquad \tau=\frac{1}{\sigma}\left[1+(\gamma+\delta)^{1/p}\right] \tag{4.7}$$

for all $x\in\Omega_1$, $A(x)\in Q_v(x,\nu,\tau)$. Then by taking $\xi=w_i$ $(i=1,2)$ in *Condition* A_{xp}, with similar calculations, we obtain for all $x\in\Omega_1$, $A(x)\in Q_w(x,\nu,\tau)$. Then for all $x\in\Omega_1$, $A(x)\in Q_v(x,\nu,\tau)\cap Q_w(x,\nu,\tau)$. From Proposition 2.2 it follows that there exist ν_0,τ_0, with $0<\nu_0<\tau_0$, such that $A(x)\in R(x,\nu_0,\tau_0)$. By Proposition 2.3 there exists $\rho\in(0,1)$ such that $A(x)\in C_\rho(x)$, that is, there exist $c(x)>0$ and $\rho\in(0,1)$ such that

$$\|I-c(x)A(x)\|\leq\rho. \tag{4.8}$$

(This inequality also implies $(\sqrt{n}-1)/M<c(x)<(\sqrt{n}+1)/\bar{\nu}$, $x\in\Omega_1$.) □

From Lemmas 4.1 and 4.2 we have the following.

THEOREM 4.3. *The Cordes condition and Condition* A_{xp} *are equivalent.*

This theorem and Corollary 3.2 imply the following.

COROLLARY 4.4. *Condition* B_x *and Condition* A_{xp} *are equivalent.*

Theorem 4.3 and Corollary 3.2, by the results proved in [10], imply the following.

COROLLARY 4.5. *Let* $n=2$. *Then every uniformly elliptic symmetric matrix satisfies Condition* A_{xp} *and Condition* B_x.

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