SOME COMPLEMENTS OF CAUCHY'S INEQUALITY ON TIME SCALES

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Dedicated to Professor T. Tsuchikura on his 80th birthday

We establish some complements of Cauchy's inequality on time scales. These results extend some inequalities given by Cargo, Diaz, Goldman, Greub, Kantorovich, Makai, Metcalf, Pólya, Rheinboldt, Schweitzer, and Szegö, and so on.

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1. Introduction and preliminaries

To unify the theory of continuous and discrete dynamic systems, in 1990 Hilger [16] proposed the study of dynamic systems on a time scale and developed the calculus for functions on a time scale (i.e., any closed subset of reals). The purpose of this paper is to establish some complements of Cauchy's inequality on time scales, which extend some results of Cargo [6], Diaz, Goldman, and Metcalf [7–11], Goldman [12], Greub and Rheinboldt [13], Kantorovich [17], Schweitzer [29], Pólya and Szegö [26], and so forth. For other related results, we refer to [2, 3, 14, 15, 18–20, 23–27, 30, 31]. To do this, we briefly introduce the time scale calculus as follows.

Definition 1.1. A time scale $\mathbb{T}$ is a closed subset of the set $\mathbb{R}$ of all real numbers. Assume throughout this paper that $\mathbb{T}$ has the topology that it inherits from the standard topology on $\mathbb{R}$. Let $t \in \mathbb{T}$, if $t < \sup\mathbb{T}$, define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{\tau \in \mathbb{T} : \tau > t\} \tag{1.1}$$

and if $t > \inf\mathbb{T}$, define the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) := \sup\{\tau \in \mathbb{T} : \tau < t\}. \tag{1.2}$$

The points $\{t\}$ of a time scale $\mathbb{T}$ can be classified into right-scattered, right-dense, left-scattered, left-dense based on $\sigma(t) > t$, $\sigma(t) = t$, $\rho(t) < t$, and $\rho(t) = t$, respectively. Moreover, define the time scale $\mathbb{T}^\kappa$ as follows:

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T}\setminus(\rho(\sup\mathbb{T}),\ \sup\mathbb{T}) & \text{if } \sup\mathbb{T} < \infty, \\ \mathbb{T} & \text{if } \sup\mathbb{T} = \infty. \end{cases} \tag{1.3}$$

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Throughout this paper, suppose that
(a) $\mathbb{R} = (-\infty, \infty)$;
(b) $\mathbb{T}$ is a time scale;
(c) an interval means the intersection of a real interval with the given time scale.

Definition 1.2. A mapping $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if the following two conditions hold:
(a) f is continuous at each right-dense point or maximal element of $\mathbb{T}$,
(b) the left-sided limit $\lim_{s\to t^-} f(s) = f(t^-)$ exists at each left-dense point $t \in \mathbb{T}$.

Definition 1.3. Assume that $x : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$ (if $t = \sup \mathbb{T}$, assume t is not left-scattered). Then x is called delta-differentiable at $t \in \mathbb{T}$ if there exists a $\theta \in \mathbb{R}$ such that for any given $\varepsilon > 0$, there is a neighborhood U of t such that for all $s \in U$,

$$|x(\sigma(t)) - x(s) - \theta(\sigma(t) - s)| \le \varepsilon |\sigma(t) - s|. \tag{1.4}$$

In this case, θ is called the *delta-derivative* of x at $t \in \mathbb{T}$ and denote it by $\theta = x^{\Delta}(t)$. If x is delta-differentiable at each point of $\mathbb{T}$, say that x is delta-differentiable on $\mathbb{T}$.

It can be shown that if $x : \mathbb{T} \to \mathbb{R}$ is continuous at $t \in \mathbb{T}$, then

$$x^{\Delta}(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t} \quad \text{if } t \text{ is right-scattered,} \tag{1.5}$$

$$x^{\Delta}(t) = \lim_{s\to t} \frac{x(t) - x(s)}{t - s} \quad \text{if } t \text{ is right-dense.} \tag{1.6}$$

In this paper, let

$$C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}) := \{f \mid f : \mathbb{T} \longrightarrow \mathbb{R} \text{ is a rd-continuous function}\}. \tag{1.7}$$

Definition 1.4. Let $f : \mathbb{T}^k \to \mathbb{R}$ be a mapping. Then the mapping $F : \mathbb{T} \to \mathbb{R}$ is an antiderivative of f on $\mathbb{T}$ if it is delta-differentiable on $\mathbb{T}$ and $F^{\Delta}(t) = f(t)$ for $t \in \mathbb{T}^k$.

Definition 1.5. If $f \in C_{\mathrm{rd}}([a,b], \mathbb{R})$ has an antiderivative F, then define the (Cauchy) integral of f by

$$\int_s^t f(r)\Delta r = F(t) - F(s), \tag{1.8}$$

for any $s, t \in [a,b]$.

It follows from Theorem 1.94 of Bohner and Peterson [4] that every rd-continuous function has an antiderivative.

For further information concerning time scales theory, refer to [4, 5, 21].

2. Main results

First, we state the well-known Cauchy inequality on a time scale $\mathbb{T}$, see [1, 4].

THEOREM 2.1 (Cauchy's inequality). *Let $p,f,g \in C_{\mathrm{rd}}([a,b],\mathbb{R})$ with $p \geq 0$ on $[a,b]$. Then*

$$\left(\int_a^b p(x)f^2(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right) \geq \left(\int_a^b p(x)f(x)g(x)\Delta x\right)^2. \tag{R$_1$}$$

Remark 2.2. Cauchy's inequality has the following variants.

(a) Replacing $f(x)$ and $g(x)$ by $\sqrt{f^2(x)+g^2(x)}$ and $f(x)g(x)/\sqrt{f^2(x)+g^2(x)}$ in (R_1), respectively, we obtain

$$\left[\int_a^b p(x)(f^2(x)+g^2(x))\Delta x\right]\left[\int_a^b \frac{p(x)f^2(x)g^2(x)}{f^2(x)+g^2(x)}\Delta x\right] \geq \left(\int_a^b p(x)f(x)g(x)\Delta x\right)^2. \tag{2.1}$$

(b) Let $fg \geq 0$ and $f \neq 0$ on $[a,b]$. Replacing $f(x)$ and $g(x)$ by $\sqrt{g(x)/f(x)}$ and $\sqrt{f(x)g(x)}$ in (R_1), respectively, then

$$\left(\int_a^b p(x)\frac{g(x)}{f(x)}\Delta x\right)\left(\int_a^b p(x)f(x)g(x)\Delta x\right) \geq \left(\int_a^b p(x)g(x)\Delta x\right)^2. \tag{2.2}$$

(c) Suppose that $g(x) > 0$ on $[a,b]$. Let $f(x)$ and $g(x)$ be replaced by $f(x)/\sqrt{g(x)}$ and $\sqrt{g(x)}$ in (R_1), respectively. Then

$$\left(\int_a^b \frac{p(x)f^2(x)}{g(x)}\Delta x\right)\left(\int_a^b p(x)g(x)\Delta x\right) \geq \left(\int_a^b p(x)f(x)\Delta x\right)^2. \tag{2.3}$$

Remark 2.3. Let $p,f,g \in C_{\mathrm{rd}}([a,b],[0,\infty))$ and $I_n = \int_a^b p(x)(f(x))^n g(x)\Delta x$. Then it follows from Cauchy's inequality (R_1) that

$$I_{n-1}^2 \leq I_n I_{n-2} \tag{2.4}$$

for any integer $n \geq 2$.

Next, we state and prove some complements of Cauchy's inequality on time scales.

THEOREM 2.4. *Suppose that $p,f,g \in C_{\mathrm{rd}}([a,b],\mathbb{R})$, $p(x) \geq 0$, and $f(x) \neq 0$ on $[a,b]$ with $\int_a^b p(x)f(x)g(x)\Delta x \neq 0$. If $m,M \in \mathbb{R}$ are such that*

$$m \leq \frac{g(x)}{f(x)} \leq M \tag{2.5}$$

for $x \in [a,b]$, then the following two statements hold:

$$\begin{aligned}
&\int_a^b p(x)g^2(x)\Delta x + Mm\int_a^b p(x)f^2(x)\Delta x \\
&\quad\leq (M+m)\int_a^b p(x)f(x)g(x)\Delta x \qquad\qquad (\mathrm{R}_2)\\
&\quad\leq |M+m|\sqrt{\left(\int_a^b p(x)f^2(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right)}
\end{aligned}$$

with equality in the first inequality sign "≤" if and only if $g(x) = mf(x)$ *or* $g(x) = Mf(x)$ *on* $[a,b]$.

$$\frac{1}{4}\left(\sqrt{\frac{m}{M}}+\sqrt{\frac{M}{m}}\right)^2 = \frac{(M+m)^2}{4Mm} \geq \frac{\left(\int_a^b p(x)f^2(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right)}{\left(\int_a^b p(x)f(x)g(x)\Delta x\right)^2} \geq 1, \qquad (\mathrm{R}_3)$$

if $Mm > 0$, *that is,*

$$\frac{1}{4}\left(\sqrt{\frac{m}{M}}-\sqrt{\frac{M}{m}}\right)^2 \geq \frac{\left(\int_a^b p(x)f^2(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right)-\left(\int_a^b p(x)f(x)g(x)\Delta x\right)^2}{\left(\int_a^b p(x)f(x)g(x)\Delta x\right)^2} \geq 0 \tag{2.6}$$

if $Mm > 0$.

Proof. It follows from (2.5) that

$$p(x)\left[\frac{g(x)}{f(x)}-m\right]\left[M-\frac{g(x)}{f(x)}\right]f^2(x) \geq 0 \quad \text{on } [a,b]. \tag{2.7}$$

Thus,

$$\begin{aligned}\int_a^b Mp(x)f(x)g(x)\Delta x - \int_a^b p(x)g^2(x)\Delta x - Mm\int_a^b p(x)f^2(x)\Delta x\\ + m\int_a^b p(x)f(x)g(x)\Delta x \geq 0.\end{aligned} \tag{2.8}$$

This inequality and (R_1) imply that (R_2) holds.

On the other hand, it follows from $Mm > 0$ and

$$\left[\left(\int_a^b p(x)g^2(x)\Delta x\right)^{1/2} - \left(Mm\int_a^b p(x)f^2(x)\Delta x\right)^{1/2}\right]^2 \geq 0 \tag{2.9}$$

that

$$(M+m)^2\left(\int_a^b p(x)f(x)g(x)\Delta x\right)^2 \geq 4Mm\int_a^b p(x)f^2(x)\Delta x\int_a^b p(x)g^2(x)\Delta x. \tag{2.10}$$

This and (R_2) imply that (R_3) holds. This completes the proof. □

Remark 2.5. Clearly, (R_2) implies (R_3) if $Mm > 0$. Hence (R_2) and (R_3) are equivalent.

Remark 2.6. Under the conditions of Theorem 2.4, if $\lambda \in (0,1)$ and $Mm > 0$, then it follows from (R_2) and the arithmetic-geometric mean inequality that

$$\begin{aligned}
\left(\frac{1}{1-\lambda}\int_a^b p(x)g^2(x)\Delta x\right)^{1-\lambda}&\left(\frac{Mm}{\lambda}\int_a^b p(x)f^2(x)\Delta x\right)^{\lambda} \\
&\leq \int_a^b p(x)g^2(x)\Delta x + Mm\int_a^b p(x)f^2(x)\Delta x \\
&\leq (M+m)\int_a^b p(x)f(x)g(x)\Delta x,
\end{aligned} \tag{2.11}$$

which implies that

$$\left(\int_a^b p(x)g^2(x)\Delta x\right)^{1-\lambda}\left(\int_a^b p(x)f^2(x)\Delta x\right)^{\lambda} \leq \lambda^{\lambda}(1-\lambda)^{1-\lambda}\frac{M+m}{(Mm)^{\lambda}}\int_a^b p(x)f(x)g(x)\Delta x. \tag{r_0}$$

Letting $\lambda \to 0^+$ in inequality (r_0), we get

$$\int_a^b p(x)g^2(x)\Delta x \leq (M+m)\int_a^b p(x)f(x)g(x)\Delta x. \tag{r_1}$$

Obviously, (r_1) is weaker than the inequality

$$\int_a^b p(x)g^2(x)\Delta x \leq \int_a^b Mp(x)f(x)g(x)\Delta x \quad \text{if } m < 0, \tag{2.12}$$

and (r_1) is also weaker than the inequality

$$\int_a^b p(x)g^2(x)\Delta x \leq \int_a^b mp(x)f(x)g(x)\Delta x \quad \text{if } M < 0. \tag{2.13}$$

Letting $\lambda \to 1^-$ in inequality (r_0), we get

$$\int_a^b p(x)f^2(x)\Delta x \leq \left(\frac{1}{M}+\frac{1}{m}\right)\int_a^b p(x)f(x)g(x)\Delta x \tag{2.14}$$

$$= \int_a^b p(x)f(x)\frac{g(x)}{m}\Delta x + \int_a^b p(x)f(x)\frac{g(x)}{M}\Delta x. \tag{2.15}$$

Evidently, it follows from (2.5) that

$$\begin{aligned}
\int_a^b p(x)f^2(x)\Delta x &\leq \int_a^b p(x)f(x)\frac{g(x)}{m}\Delta x \quad \text{if } m > 0, \\
\int_a^b p(x)f^2(x)\Delta x &\leq \int_a^b p(x)f(x)\frac{g(x)}{M}\Delta x \quad \text{if } M < 0.
\end{aligned} \tag{2.16}$$

Remark 2.7. The inequality (R_2) extends [6, Theorems 1 and 2] and inequality (3.3) in Makai [22].

Remark 2.8. Let $F \in C_{\mathrm{rd}}([a,b],(0,\infty))$. If $f = F^{-1/2}$, $g = F^{1/2}$, then (R_2) is reduced to

$$\int_a^b p(x)F(x)\Delta x + Mm\int_a^b \frac{p(x)}{F(x)}\Delta x \le (M+m)\int_a^b p(x)\Delta x, \tag{R_2^*}$$

which extends Rennie's result [28].

Conversely, if we take $F = g/f$, $p = pfg$, then (R_2^*) is reduced to (R_2). Thus, (R_2) and (R_2^*) are equivalent if $F \in C_{\mathrm{rd}}([a,b],(0,\infty))$.

Remark 2.9. Let $F \in C_{\mathrm{rd}}([a,b],(0,\infty))$. If $f = F^{-1/2}$, $g = F^{1/2}$, then (R_3) is reduced to

$$\frac{(M+m)^2}{4Mm} \ge \frac{\int_a^b p(x)F(x)\Delta x \int_a^b (p(x)/F(x))\Delta x}{\left(\int_a^b p(x)\Delta x\right)^2} \ge 1, \tag{R_3^*}$$

which generalizes some results in [17, 29, 31]. Conversely, if $F = g/f$, $p = pfg$, then (R_3^*) is reduced to (R_3). Hence, (R_3) and (R_3^*) are equivalent.

Moreover, if $p(x) = 1$, then (R_3) is reduced to

$$\frac{(M+m)^2}{4Mm} \ge \frac{\int_a^b f^2(x)\Delta x \int_a^b g^2(x)\Delta x}{\left(\int_a^b f(x)g(x)\Delta x\right)^2} \ge 1, \tag{R_3^{**}}$$

which extends a result in [26]. Obviously, (R_3) and (R_3^{**}) are also equivalent if f and g are replaced by $\sqrt{p}f$ and $\sqrt{p}g$, respectively, in (R_3^{**}).

Remark 2.10. Let $p(x) > 0$ on $[a,b]$. If $g(x)$ is replaced by $f(x)/p(x)$, then (R_3) is reduced to

$$\frac{(M+m)^2}{4Mm} \ge \frac{\left(\int_a^b p(x)f^2(x)\Delta x\right)\left(\int_a^b (f^2(x)/p(x))\Delta x\right)}{\left(\int_a^b f^2(x)\Delta x\right)^2} \ge 1. \tag{2.17}$$

Similarly, we can prove the following.

THEOREM 2.11. *Let $p,f,g \in C_{\mathrm{rd}}([a,b],\mathbb{R})$ with $p(x) \ge 0$ on $[a,b]$. Suppose that there exist four constants $h,H,m,M \in \mathbb{R}$ such that*

$$\big(Mf(g) - hg(x)\big)\big(Hg(x) - mf(x)\big) \ge 0 \tag{2.18}$$

on $[a,b]$. Then

$$\begin{aligned} Mm\int_a^b p(x)f^2(x)\Delta x + Hh\int_a^b p(x)g^2(x)\Delta x &\\ \le (HM+hm)\int_a^b p(x)f(x)g(x)\Delta x &\\ \le |HM+hm|\left(\int_a^b p(x)f^2(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right). & \end{aligned} \tag{2.19}$$

Moreover, if $HMhm > 0$, then

$$\left(\int_a^b p(x)f^2(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right) \le \left(\frac{HM+hm}{4HMhm}\right)^2\left(\int_a^b p(x)f(x)g(x)\Delta x\right)^2. \tag{2.20}$$

Hence

$$\begin{aligned}\sqrt{\frac{Mm}{Hh}}\left(\int_a^b p(x)f^2(x)\Delta x\right)+\sqrt{\frac{Hh}{Mm}}\left(\int_a^b p(x)g^2(x)\Delta x\right)\\ \le\left(\sqrt{\frac{HM}{hm}}+\sqrt{\frac{hm}{HM}}\right)\left(\int_a^b p(x)f(x)g(x)\Delta x\right).\end{aligned} \tag{2.21}$$

Remark 2.12. Theorem 2.11 extends a result in [20, page 18].

The following is an extension of [24, Theorem 1, page 122].

Theorem 2.13. *Let $p,f,g \in C_{\mathrm{rd}}([a,b],[0,\infty))$. Suppose that there exist six constants α,β,h, $H,m,M \in (0,\infty)$ such that $h \le f(x) \le H$, $m \le g(x) \le M$ on $[a,b]$, $1 > \alpha \ge \beta > 0$ and $\alpha+\beta = 1$. Then the following two inequalities hold:*

$$\left(\int_a^b p(x)f(x)\Delta x\right)^\alpha\left(\int_a^b \frac{p(x)}{f(x)}\Delta x\right)^\beta \le \frac{\alpha H+\beta h}{(Hh)^\beta}\left(\int_a^b p(x)\Delta x\right), \tag{R_4}$$

$$\left(\int_a^b p(x)f^2(x)\Delta x\right)^\alpha\left(\int_a^b p(x)g^2(x)\Delta x\right)^\beta \le \frac{\alpha HM+\beta hm}{(Hh)^\beta(Mm)^\alpha}\left(\int_a^b p(x)f(x)g(x)\Delta x\right). \tag{R_5}$$

Proof. Since $(\alpha f(x)-\beta h)(f(x)-H) \le 0$ on $[a,b]$, we have

$$\alpha f^2(x)-(\alpha H+\beta h)f(x)+\beta Hh \le 0. \tag{2.22}$$

Thus,

$$\alpha p(x)f(x)+\beta Hh\frac{p(x)}{f(x)} \le (\alpha H+\beta h)p(x). \tag{R_6}$$

A direct consequence of the foregoing inequality with appeal to the arithmetic-geometric mean inequality leads to

$$\begin{aligned}&\left(\int_a^b p(x)f(x)\Delta x\right)^\alpha\left(\int_a^b \frac{p(x)}{f(x)}\Delta x\right)^\beta\\ &\quad=\frac{1}{(Hh)^\beta}\left(\int_a^b p(x)f(x)\Delta x\right)^\alpha\left(Hh\int_a^b \frac{p(x)}{f(x)}\right)^\beta\\ &\quad\le\frac{1}{(Hh)^\beta}\left(\alpha\int_a^b p(x)f(x)\Delta x+\beta Hh\int_a^b \frac{p(x)}{f(x)}\Delta x\right)\\ &\quad\le\frac{\alpha H+\beta h}{(Hh)^\beta}\left(\int_a^b p(x)\Delta x\right).\end{aligned} \tag{2.23}$$

Thus, (R_4) holds.

Replacing $p(x)$ and $f(x)$ by $p(x)f(x)g(x)$ and $f(x)/g(x)$ in (R_4), respectively, and using $h/M \le f(x)/g(x) \le H/m$, we obtain

$$\left(\int_a^b p(x)f^2(x)\Delta x\right)^{\alpha}\left(\int_a^b p(x)g^2(x)\Delta x\right)^{\beta} \le \frac{\alpha HM+\beta hm}{(Hh)^{\beta}(Mm)^{\alpha}}\left(\int_a^b p(x)f(x)g(x)\Delta x\right). \tag{2.24}$$

Hence, (R_5) holds. □

Remark 2.14. Let $p,f,g \in C_{\mathrm{rd}}([a,b],\mathbb{R})$ with $p(x) \ge 0$ on $[a,b]$. Suppose that there are $\alpha,\beta,m,M \in \mathbb{R}$ such that $0<\beta\le\alpha<1$, $\alpha+\beta=1$, and

$$m \le \frac{g(x)}{f(x)} \le M \quad \text{on } [a,b]. \tag{2.25}$$

Replacing h, H and $f(x)$ by m, M and $g(x)/f(x)$ in (R_6), respectively, and then integrating the resulting inequality from a to b, we obtain

$$\alpha\int_a^b p(x)g^2(x)\Delta x+\beta Mm\int_a^b p(x)f^2(x)\Delta x \le (\beta m+\alpha M)\int_a^b p(x)f(x)g(x)\Delta x, \tag{2.26}$$

which is an extension of inequality (R_2).

Remark 2.15. Clearly, (R_4) and (R_5) are equivalent. In fact, let $f^2(x)=F(x)$, $g^2(x)=1/F(x)$, where $F \in C_{\mathrm{rd}}([a,b],(0,\infty))$. If $0<k\le F(x)\le K$ on $[a,b]$, then

$$\begin{aligned} h &:= \sqrt{k} \le f(x) \le \sqrt{K} := H, \\ m &:= \frac{1}{\sqrt{K}} \le g(x) \le \frac{1}{\sqrt{k}} := M. \end{aligned} \tag{2.27}$$

Thus, (R_5) is reduced to (R_4). Similarly, (R_4) is reduced to (R_5). Hence, (R_4) and (R_5) are equivalent.

Remark 2.16. Let $\alpha=\beta=1/2$. Then (R_5) is reduced to

$$\frac{(HM+hm)^2}{4HMhm} \ge \frac{\left(\int_a^b p(x)f^2(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right)}{\left(\int_a^b p(x)f(x)g(x)\Delta x\right)^2} \ge 1, \tag{R_3^{***}}$$

which is an extension of a result in Greub and Rheinboldt [13], see also [3, 23, 24]. In fact, (R_3) and (R_3^{***}) are equivalent.

The following is an example of the presented theory with $\mathbb{T}=\mathbb{Z}$.

Remark 2.17. Let $p_i \geq 0$, $0 < h \leq a_i \leq H$, $0 < m \leq b_i \leq M$ for $i = 1,2,\dots,n$. If $1 > \alpha \geq \beta > 0$ with $\alpha + \beta = 1$, then (R_3), (R_4) and (R_3^{***}) are reduced to

$$\left(\sum_{i=1}^{n} p_i a_i\right)^{\alpha}\left(\sum_{i=1}^{n} \frac{p_i}{a_i}\right)^{\beta} \leq \frac{\alpha H + \beta h}{(Hh)^{\beta}}\left(\sum_{i=1}^{n} p_i\right), \tag{a}$$

$$\left(\sum_{i=1}^{n} p_i a_i^2\right)^{\alpha}\left(\sum_{i=1}^{n} p_i b_i^2\right)^{\beta} \leq \frac{\alpha HM + \beta hm}{(Hh)^{\beta}(Mm)^{\alpha}}\left(\sum_{i=1}^{n} p_i a_i b_i\right), \tag{b}$$

$$\frac{1}{4}\left(\sqrt{\frac{HM}{hm}} + \sqrt{\frac{hm}{HM}}\right)^2 = \frac{(HM+hm)^2}{4HMhm} \geq \frac{\left(\sum_{i=1}^{n} p_i a_i^2\right)\left(\sum_{i=1}^{n} p_i b_i^2\right)}{\left(\sum_{i=1}^{n} p_i a_i b_i\right)^2}, \tag{c}$$

respectively (see [24, pages 121 and 122]). Inequality (c) generalizes some results of [13, 16].

If $\alpha = \beta = 1/2$, then (a), (b) are reduced to

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} \frac{1}{a_i}\right) \leq \frac{(H+h)^2}{4Hh} n^2 \quad \text{(Schweitzer [29])}, \tag{a*}$$

$$\left(\sum_{i=1}^{n} p_i a_i\right)\left(\sum_{i=1}^{n} \frac{p_i}{a_i}\right) \leq \frac{(H+h)^2}{4Hh}\left(\sum_{i=1}^{n} p_i\right) \quad \text{(Kantorovich [17])}, \tag{a**}$$

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leq \frac{HM+hm}{4HMhm}\left(\sum_{i=1}^{n} a_i b_i\right)^2 \quad \text{(Pólya and Szegö [26])}. \tag{b*}$$

3. More results

In this section, we give some inequalities on time scales which extend some results in [23, 27, 31]. To do this, let $f,g \in C_{\mathrm{rd}}([a,b],\mathbb{R})$ and $p \in C_{\mathrm{rd}}([a,b],[0,\infty))$, we define the operator $T_p(f,g)$ as follows:

$$T_p(f,g) := \frac{1}{2}\int_a^b \int_a^b p(x)p(t)\big(f(x) - f(t)\big)\big(g(x) - g(t)\big)\Delta x \Delta t. \tag{3.1}$$

In fact,

$$T_p(f,g) = \left[\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)f(x)g(x)\Delta x\right) - \left(\int_a^b p(x)f(x)\Delta x\right)\left(\int_a^b p(x)g(x)\Delta x\right)\right] \tag{3.2}$$

and $T_p(f,f) \geq 0$. Invoking (3.1) and Cauchy's inequality (R_1) yields the following inequality

$$\left[T_p(f,g)\right]^2 \leq T_p(f,f)T_p(g,g). \tag{3.3}$$

Remark 3.1. It follows from (3.3) that

$$\begin{aligned}
&\left[\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)f(x)g(x)\Delta x\right)-\left(\int_a^b p(x)f(x)\Delta x\right)\left(\int_a^b p(x)g(x)\Delta x\right)\right]^2\\
&\quad\le\left[\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)f^2(x)\Delta x\right)-\left(\int_a^b p(x)f(x)\Delta x\right)^2\right]\\
&\quad\quad\times\left[\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right)-\left(\int_a^b p(x)g(x)\Delta x\right)^2\right],
\end{aligned}\tag{3.4}$$

that is,

$$\begin{aligned}
&\left(\int_a^b p(x)\Delta x\right)^2\left(\int_a^b p(x)f(x)g(x)\Delta x\right)^2\\
&\quad-2\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)f(x)\Delta x\right)\left(\int_a^b p(x)g(x)\Delta x\right)\left(\int_a^b p(x)f(x)g(x)\Delta x\right)\\
&\quad+\left(\int_a^b p(x)f(x)\Delta x\right)^2\left(\int_a^b p(x)g(x)\Delta x\right)^2\\
&\le\left(\int_a^b p(x)\Delta x\right)^2\left(\int_a^b p(x)f^2(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right)\\
&\quad-\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)f^2(x)\Delta x\right)\left(\int_a^b p(x)g(x)\Delta x\right)^2\\
&\quad-\left(\int_a^b p(x)f(x)\Delta x\right)^2\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right)\\
&\quad+\left(\int_a^b p(x)f(x)\Delta x\right)^2\left(\int_a^b p(x)g(x)\Delta x\right)^2,
\end{aligned}\tag{3.5}$$

which implies that

$$\begin{aligned}
&\left(\int_a^b p(x)f^2(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right)-\left(\int_a^b p(x)f(x)g(x)\Delta x\right)^2\\
&\quad\ge\frac{1}{\int_a^b p(x)\Delta x}\left[\left(\int_a^b p(x)f^2(x)\Delta x\right)\left(\int_a^b p(x)f(x)g(x)\Delta x\right)^2\right.\\
&\qquad\qquad+\left(\int_a^b p(x)g^2(x)\Delta x\right)\left(\int_a^b p(x)f(x)\Delta x\right)^2\\
&\qquad\qquad\left.-2\left(\int_a^b p(x)f(x)\Delta x\right)\left(\int_a^b p(x)g(x)\Delta x\right)\left(\int_a^b p(x)f(x)g(x)\Delta x\right)\right].
\end{aligned}\tag{R_7}$$

It follows from the arithmetic-geometric mean inequality that the right-hand side of inequality (R_7) is greater than or equal to

$$\begin{aligned}
&\frac{1}{\int_a^b p(x)\Delta x}\Bigg[2\left(\int_a^b p(x)f(x)\Delta x\right)\left(\int_a^b p(x)g(x)\Delta x\right)\\
&\qquad\times\sqrt{\left(\int_a^b p(x)f^2(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right)}\\
&\qquad-2\left(\int_a^b p(x)f(x)g(x)\Delta x\right)\left(\int_a^b p(x)f(x)\Delta x\right)\left(\int_a^b p(x)g(x)\Delta x\right)\Bigg] \qquad (3.6)\\
&\quad=\frac{2}{\int_a^b p(x)\Delta x}\left(\int_a^b p(x)f(x)\Delta x\right)\left(\int_a^b p(x)g(x)\Delta x\right)\\
&\quad\times\left[\left(\int_a^b p(x)f(x)g(x)\Delta x\right)-\left(\int_a^b p(x)f(x)g(x)\Delta x\right)\right]=0.
\end{aligned}$$

This means (R_7) is stronger than inequality (R_1).

THEOREM 3.2. *Let $p, f, g \in C_{\rm rd}([a,b],[0,\infty))$.*

(a) *If there exist four constants $H,h,M,m \in \mathbb{R}$ such that $[Hg(x)-mf(x)][Mf(x)-hg(x)] \ge 0$ on $[a,b]$, then*

$$(HM+hm)\int_a^b p(x)f(x)g(x)\Delta x \ge Hh\int_a^b p(x)g^2(x)\Delta x + Mm\int_a^b p(x)f^2(x)\Delta x. \qquad (3.7)$$

(b) *If there exist four constants $H,h,M,m \in \mathbb{R}$ such that for all $x,t \in [a,b]$ with $[Hg(x)-mf(t)][Mf(t)-hg(x)] \ge 0$, then*

$$\begin{aligned}
&(HM+hm)\left(\int_a^b p(x)f(x)\Delta x\right)\left(\int_a^b p(x)g(x)\Delta x\right)\\
&\quad\ge Hh\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right) \qquad (3.8)\\
&\qquad+Mm\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)f^2(x)\Delta x\right).
\end{aligned}$$

(c) *If $Hh>0$ and $Mm>0$, then*

$$\begin{aligned}
&(HM+hm)\left(\int_a^b p(x)f(x)\Delta x\right)\left(\int_a^b p(x)g(x)\Delta x\right)\\
&\quad\ge Hh\left(\int_a^b p(x)g(x)\Delta x\right)^2+Mm\left(\int_a^b p(x)f(x)\Delta x\right)^2.
\end{aligned} \qquad (3.9)$$

(d) *If $Hh>0$ and $Mm>0$, then*

$$\begin{aligned}
&(HM+hm)\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)f(x)g(x)\Delta x\right)\\
&\quad\ge Hh\left(\int_a^b p(x)g(x)\Delta x\right)^2+Mm\left(\int_a^b p(x)f(x)\Delta x\right)^2.
\end{aligned} \qquad (3.10)$$

Proof

Case (a). It follows from the assumption that

$$p(x)\big(Hg(x)-mf(x)\big)\big(Mf(x)-hg(x)\big)\geq 0 \quad \text{on } [a,b], \tag{3.11}$$

which implies that

$$(HM+hm)p(x)f(x)g(x)\geq Hhp(x)g^2(x)+Mmp(x)f^2(x) \quad \text{on } [a,b]. \tag{3.12}$$

Thus,

$$(HM+hm)\int_a^b p(x)f(x)g(x)\Delta x\geq Hh\int_a^b p(x)g^2(x)\Delta x+Mm\int_a^b p(x)f^2(x)\Delta x. \tag{3.13}$$

Case (b). It follows from the assumption that for $x,t\in[a,b]$,

$$p(x)p(t)\big(Hg(x)-mf(t)\big)\big(Mf(t)-hg(x)\big)\geq 0, \tag{3.14}$$

which implies that

$$\begin{aligned} &HMp(x)p(t)f(t)g(x)+hmp(x)p(t)f(t)g(x)\\ &\quad\geq Hhp(x)p(t)g^2(x)+Mmp(x)p(t)f^2(t). \end{aligned} \tag{3.15}$$

Therefore,

$$\begin{aligned} &(HM+hm)\left(\int_a^b p(x)g(x)\Delta x\right)\left(\int_a^b p(t)f(t)\Delta t\right)\\ &\quad\geq Hh\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right)+Mm\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)f^2(x)\Delta x\right). \end{aligned} \tag{3.16}$$

Cases (c) and (d). It follows from Cauchy's inequality (R_1) that

$$\begin{aligned} \left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)f^2(x)\Delta x\right)\geq\left(\int_a^b p(x)f(x)\Delta x\right)^2,\\ \left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right)\geq\left(\int_a^b p(x)g(x)\Delta x\right)^2. \end{aligned} \tag{3.17}$$

Combining (a), (b), and the preceding two inequalities, we see that

$$\begin{aligned} &(HM+hm)\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)f(x)g(x)\Delta x\right)\\ &\quad\geq Hh\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right)+Mm\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)f^2(x)\Delta x\right)\\ &\quad\geq Hh\left(\int_a^b p(x)g(x)\Delta x\right)^2+Mm\left(\int_a^b p(x)f(x)\Delta x\right)^2. \end{aligned} \tag{3.18}$$

This completes the proof of (c).

Furthermore,

$$\begin{aligned}(HM+hm)&\left(\int_a^b p(x)f(x)\Delta x\right)\left(\int_a^b p(x)g(x)\Delta x\right)\\ &\geq Hh\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right)+Mm\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)f^2(x)\Delta x\right)\\ &\geq Hh\left(\int_a^b p(x)g(x)\Delta x\right)^2+Mm\left(\int_a^b p(x)f(x)\Delta x\right)^2.\end{aligned} \tag{3.19}$$

Hence, (d) holds. □

Remark 3.3. It follows from (a) of Theorem 3.2 that

$$\begin{aligned}&-(HM+hm)\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)f(x)g(x)\Delta x\right)\\ &\quad+Hh\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right)\\ &\quad+Mm\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)f^2(x)\Delta x\right)\leq 0,\end{aligned} \tag{3.20}$$

hence

$$\begin{aligned}&-(HM+hm)T_p(f,g)+Hh\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right)\\ &\quad+Mm\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)f^2(x)\Delta x\right)\\ &\quad\leq (HM+hm)\left(\int_a^b p(x)f(x)\Delta x\right)\left(\int_a^b p(x)g(x)\Delta x\right),\end{aligned} \tag{3.21}$$

where $T_p(f,g)$ is defined as in (3.1). The foregoing inequality is stronger than Theorem 3.2(b). Hence, by Cauchy's inequality,

$$\begin{aligned}&-(HM+hm)T_p(f,g)+Hh\left(\int_a^b p(x)g(x)\Delta x\right)^2+Mm\left(\int_a^b p(x)f(x)\Delta x\right)^2\\ &\quad\leq (HM+hm)\left(\int_a^b p(x)f(x)\Delta x\right)\left(\int_a^b p(x)g(x)\Delta x\right).\end{aligned} \tag{R$_8$}$$

Remark 3.4. If $HMhm>0$, then it follows from (a) of Theorem 3.2 that

$$(HM+hm)\left(\int_a^b p(x)f(x)g(x)\Delta x\right)\geq 2\sqrt{HMhm\left(\int_a^b p(x)f^2(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right)}. \tag{3.22}$$

Moreover, if $\int_a^b p(x)f(x)g(x)\Delta x > 0$, then

$$\frac{(HM+hm)^2}{4HMhm} = \frac{1}{4}\left(\sqrt{\frac{HM}{hm}}+\sqrt{\frac{hm}{HM}}\right)^2 \geq \frac{\left(\int_a^b p(x)f^2(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right)}{\left(\int_a^b p(x)f(x)g(x)\Delta x\right)^2}. \tag{3.23}$$

This is a generalized Cauchy's inequality.

Remark 3.5. It follows from (b) of Theorem 3.2 that

$$\begin{aligned}
&(HM+hm)^2\left(\int_a^b p(x)f(x)\Delta x\right)^2\left(\int_a^b p(x)g(x)\Delta x\right)^2\\
&\quad\geq H^2h^2\left(\int_a^b p(x)\Delta x\right)^2\left(\int_a^b p(x)g^2(x)\Delta x\right)^2\\
&\qquad+M^2m^2\left(\int_a^b p(x)\Delta x\right)^2\left(\int_a^b p(x)f^2(x)\Delta x\right)^2\\
&\qquad+2HMhm\left(\int_a^b p(x)\Delta x\right)^2\left(\int_a^b p(x)f^2(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right)\\
&\quad\geq 4HMhm\left(\int_a^b p(x)\Delta x\right)^2\left(\int_a^b p(x)f^2(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right).
\end{aligned}\tag{3.24}$$

Hence, if $HMhm > 0$, then

$$\begin{aligned}
&\left(\int_a^b p(x)f^2(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right)\\
&\qquad\leq \frac{(HM+hm)^2}{4HMhm}\left(\frac{\int_a^b p(x)f(x)\Delta x\int_a^b p(x)g(x)\Delta x}{\int_a^b p(x)\Delta x}\right)^2.
\end{aligned}\tag{3.25}$$

Remark 3.6. It follows from (b) of Theorem 3.2 that

$$\begin{aligned}
&-(HM+hm)\left(\int_a^b p(x)f(x)\Delta x\right)\left(\int_a^b p(x)g(x)\Delta x\right)\\
&\quad+Hh\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right)\\
&\quad+Mm\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)f^2(x)\Delta x\right)\leq 0,
\end{aligned}\tag{3.26}$$

hence

$$\begin{aligned}(HM+hm)\bigg[\bigg(\int_a^b p(x)\Delta x\bigg)\bigg(\int_a^b p(x)f(x)g(x)\Delta x\bigg)\\ -\bigg(\int_a^b p(x)f(x)\Delta x\bigg)\bigg(\int_a^b p(x)g(x)\Delta x\bigg)\bigg]\\ +Hh\bigg(\int_a^b p(x)\Delta x\bigg)\bigg(\int_a^b p(x)g^2(x)\Delta x\bigg)+Mm\bigg(\int_a^b p(x)\Delta x\bigg)\bigg(\int_a^b p(x)f^2(x)\Delta x\bigg)\\ \le (HM+hm)\bigg(\int_a^b p(x)\Delta x\bigg)\bigg(\int_a^b p(x)f(x)g(x)\Delta x\bigg).\end{aligned} \tag{3.27}$$

Then by Cauchy's inequality (R_1),

$$\begin{aligned}(HM+hm)T_p(f,g)+Hh\bigg(\int_a^b p(x)g(x)\Delta x\bigg)^2+Mm\bigg(\int_a^b p(x)f(x)\Delta x\bigg)^2\\ \le (HM+hm)\bigg(\int_a^b p(x)\Delta x\bigg)\bigg(\int_a^b p(x)f(x)g(x)\Delta x\bigg),\end{aligned} \tag{R$_9$}$$

where $T_p(f,g)$ is defined as (3.1).

Remark 3.7. If $HMhm>0$, then it follows from (c) of Theorem 3.2 that

$$\begin{aligned}(HM+hm)\bigg(\int_a^b p(x)\Delta x\bigg)\bigg(\int_a^b p(x)f(x)g(x)\Delta x\bigg)\\ \ge 2\sqrt{HMhm\bigg(\int_a^b p(x)g(x)\Delta x\bigg)^2\bigg(\int_a^b p(x)f(x)\Delta x\bigg)^2}.\end{aligned} \tag{3.28}$$

Hence

$$\frac{1}{2}\left(\sqrt{\frac{HM}{hm}}+\sqrt{\frac{hm}{HM}}\right)=\frac{HM+hm}{2\sqrt{HMhm}}\ge\frac{\big(\int_a^b p(x)f(x)\Delta x\big)\big(\int_a^b p(x)g(x)\Delta x\big)}{\big(\int_a^b p(x)\Delta x\big)\big(\int_a^b p(x)f(x)g(x)\Delta x\big)}. \tag{3.29}$$

Theorem 3.8. *Let $p,f,g\in C_{\mathrm{rd}}([a,b],[0,\infty))$. Then*

(a)

$$\begin{aligned}\bigg[\bigg(\int_a^b p(x)\Delta x\bigg)\bigg(\int_a^b p(x)f^2(x)\Delta x\bigg)+\bigg(\int_a^b p(x)f(x)\Delta x\bigg)^2\bigg]\\ \times\bigg[\bigg(\int_a^b p(x)\Delta x\bigg)\bigg(\int_a^b p(x)g^2(x)\Delta x\bigg)+\bigg(\int_a^b p(x)g(x)\Delta x\bigg)^2\bigg]\\ \ge\bigg[\bigg(\int_a^b p(x)\Delta x\bigg)\bigg(\int_a^b p(x)f(x)g(x)\Delta x\bigg)+\bigg(\int_a^b p(x)f(x)\Delta x\bigg)\bigg(\int_a^b p(x)g(x)\Delta x\bigg)\bigg]^2.\end{aligned} \tag{3.30}$$

Moreover, under the assumption of (a) and (b) in Theorem 3.2, then the following two inequalities hold:

(b)

$$
\begin{aligned}
&(HK+hm)^2\bigg[\bigg(\int_a^b p(x)\Delta x\bigg)\bigg(\int_a^b p(x)f(x)g(x)\Delta x\bigg)\\
&\qquad+\bigg(\int_a^b p(x)f(x)\Delta x\bigg)\bigg(\int_a^b p(x)g(x)\Delta x\bigg)\bigg]^2\\
&\quad\geq 4HMhm\bigg[\bigg(\int_a^b p(x)\Delta x\bigg)\bigg(\int_a^b p(x)g^2(x)\Delta x\bigg)+\bigg(\int_a^b p(x)g(x)\Delta x\bigg)^2\bigg]\\
&\qquad\times\bigg[\bigg(\int_a^b p(x)\Delta x\bigg)\bigg(\int_a^b p(x)f^2(x)\Delta x\bigg)+\bigg(\int_a^b p(x)f(x)\Delta x\bigg)^2\bigg]^2.
\end{aligned}
\tag{3.31}
$$

(c)

$$
\begin{aligned}
1&\leq\bigg[\bigg(\int_a^b p(x)\Delta x\bigg)\bigg(\int_a^b p(x)f^2(x)\Delta x\bigg)+\bigg(\int_a^b p(x)f(x)\Delta x\bigg)^2\bigg]\\
&\quad\times\frac{\big[(\int_a^b p(x)\Delta x)(\int_a^b p(x)g^2(x)\Delta x)+(\int_a^b p(x)g(x)\Delta x)^2\big]}{\big[(\int_a^b p(x)\Delta x)(\int_a^b p(x)f(x)g(x)\Delta x)+(\int_a^b p(x)f(x)\Delta x)(\int_a^b p(x)g(x)\Delta)\big]^2}\\
&\leq\frac{(HM+hm)^2}{4HMhm}.
\end{aligned}
\tag{3.32}
$$

Proof

Case (a). A straightforward calculation shows that

$$
\begin{aligned}
&\bigg[\bigg(\int_a^b p(x)\Delta x\bigg)\bigg(\int_a^b p(x)f^2(x)\Delta x\bigg)+\bigg(\int_a^b p(x)f(x)\Delta x\bigg)^2\bigg]\\
&\qquad\times\bigg[\bigg(\int_a^b p(x)\Delta x\bigg)\bigg(\int_a^b p(x)g^2(x)\Delta x\bigg)+\bigg(\int_a^b p(x)g(x)\Delta x\bigg)^2\bigg]\\
&\quad=\bigg(\int_a^b p(x)\Delta x\bigg)^2\bigg(\int_a^b p(x)f^2(x)\Delta x\bigg)\bigg(\int_a^b p(x)g^2(x)\Delta x\bigg)\\
&\qquad+\bigg(\int_a^b p(x)\Delta x\bigg)\bigg(\int_a^b p(x)f^2(x)\Delta x\bigg)\bigg(\int_a^b p(x)g(x)\Delta x\bigg)^2\\
&\qquad+\bigg(\int_a^b p(x)f(x)\Delta x\bigg)^2\bigg(\int_a^b p(x)\Delta x\bigg)\bigg(\int_a^b p(x)g^2(x)\Delta x\bigg)\\
&\qquad+\bigg(\int_a^b p(x)f(x)\Delta x\bigg)^2\bigg(\int_a^b p(x)g(x)\Delta x\bigg)^2\\
&\quad\geq\bigg[\bigg(\int_a^b p(x)\Delta x\bigg)\bigg(\int_a^b p(x)f(x)g(x)\Delta x\bigg)\bigg]^2+2\bigg(\int_a^b p(x)\Delta x\bigg)\bigg(\int_a^b p(x)f(x)\Delta x\bigg)\\
&\qquad\times\bigg(\int_a^b p(x)g(x)\Delta x\bigg)\sqrt{\bigg(\int_a^b p(x)f^2(x)\Delta x\bigg)\bigg(\int_a^b p(x)g^2(x)\Delta x\bigg)}\\
&\qquad+\bigg(\int_a^b p(x)f(x)\Delta x\bigg)^2\bigg(\int_a^b p(x)g(x)\Delta x\bigg)^2\quad\text{(by Cauchy's inequality)}
\end{aligned}
$$

$$\begin{aligned}
&\geq \left[\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)f(x)g(x)\Delta x\right)\right]^2 \\
&\quad + 2\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)f(x)\Delta x\right)\left(\int_a^b p(x)g(x)\Delta x\right)\left(\int_a^b p(x)f(x)g(x)\Delta x\right) \\
&\quad + \left[\left(\int_a^b p(x)f(x)\Delta x\right)\left(\int_a^b p(x)g(x)\Delta x\right)\right]^2 \\
&= \left[\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)f(x)g(x)\Delta x\right) + \left(\int_a^b p(x)f(x)\Delta x\right)\left(\int_a^b p(x)g(x)\Delta x\right)\right]^2.
\end{aligned} \tag{3.33}$$

Case (b). It is follows from (R_8) and (R_9) that

$$\begin{aligned}
&(HM+hm)^2\left[\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)f(x)g(x)\Delta x\right)\right. \\
&\qquad\qquad \left. + \left(\int_a^b p(x)f(x)\Delta x\right)\left(\int_a^b p(x)g(x)\Delta x\right)\right]^2 \\
&\quad \geq \left\{Hh\left[\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right) + \left(\int_a^b p(x)g(x)\Delta x\right)^2\right]\right. \\
&\qquad \left. + Mm\left[\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)f^2(x)\Delta x\right) + \left(\int_a^b p(x)f(x)\Delta x\right)^2\right]\right\}^2 \\
&\quad \geq 4HMhm\left[\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)g^2(x)\Delta x\right) + \left(\int_a^b p(x)g(x)\Delta x\right)^2\right] \\
&\qquad \times \left[\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)f^2(x)\Delta x\right) + \left(\int_a^b p(x)f(x)\Delta x\right)^2\right]^2.
\end{aligned} \tag{3.34}$$

This completes the proof of (b).

Case (c). Clearly, (c) follows from (a) and (b).

□

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