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Research Article

Discontinuous Variational-Hemivariational Inequalities Involving the p-Laplacian

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We deal with discontinuous quasilinear elliptic variational-hemivariational inequalities. By using the method of sub- and supersolutions and based on the results of S. Carl, we extend the theory for discontinuous problems. The proof of the existence of extremal solutions within a given order interval of sub- and supersolutions is the main goal of this paper. In the last part, we give an example of the construction of sub- and supersolutions.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain with Lipschitz boundary $\partial\Omega$. As $V = W^{1,p}(\Omega)$ and $V_0 = W_0^{1,p}(\Omega)$, $1 < p < \infty$, we denote the usual Sobolev spaces with their dual spaces $V^* = (W^{1,p}(\Omega))^*$ and $V_0^* = W^{-1,q}(\Omega)$, respectively (q is the Hölder conjugate of p). In this paper, we consider the following elliptic variational-hemivariational inequality

$$u \in K : \langle -\Delta_p u + F(u), v - u\rangle + \int_\Omega j^0(u; v-u)\,dx \geq 0, \quad \forall v \in K, \tag{1.1}$$

where $j^0(s;r)$ denotes the generalized directional derivative of the locally Lipschitz function $j : \mathbb{R}\to\mathbb{R}$ at s in the direction r given by

$$j^0(s;r) = \limsup_{y\to s,\, t\downarrow 0} \frac{j(y+tr) - j(y)}{t} \tag{1.2}$$

(cf. [1, Chapter 2]), and $K \subset V_0$ is some closed and convex subset. The operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $1 < p < \infty$, and F denotes the Nemytskij operator related to the function $f : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$F(u)(x) = f(x, u(x), u(x)). \tag{1.3}$$

In [2] the method of sub- and supersolutions was developed for variational-hemivariational inequalities of the form (1.1) with $F(u) \equiv f \in V_0^*$. The aim of this paper is the generalization for discontinuous Nemytskij operators $F : L^p(\Omega) \to L^q(\Omega)$. Let us consider some special cases of problem (1.1) as follows.

(i) For $f \in V_0^*$, (1.1) is also a variational-hemivariational inequality which is discussed in [2].

(ii) If $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying some growth condition and $j = 0$, then (1.1) is a classical variational inequality of the form

$$u \in K : \langle -\Delta_p u + F(u), v - u \rangle \geq 0, \quad \forall v \in K, \tag{1.4}$$

for which the method of sub- and supersolutions has been developed in [3, Chapter 5].

(iii) For $K = V_0$, $f \in V_0^*$, and $j : \mathbb{R} \to \mathbb{R}$ smooth, (1.1) becomes a variational equality of the form

$$u \in V_0 : \langle -\Delta_p u + f + j'(u), \varphi \rangle = 0, \quad \forall \varphi \in V_0, \tag{1.5}$$

for which the sub-supersolution method is well known.

2. Notations and hypotheses

For functions $u, v : \Omega \to \mathbb{R}$, we use the notation $u \wedge v = \min(u, v)$, $u \vee v = \max(u, v)$, $K \wedge K = \{u \wedge v : u, v \in K\}$, $K \vee K = \{u \vee v : u, v \in K\}$, and $u \wedge K = \{u\} \wedge K$, $u \vee K = \{u\} \vee K$ and introduce the following definitions.

Definition 2.1. A function $\underline{u} \in V$ is called a subsolution of (1.1) if the following holds:

(1) $\underline{u} \leq 0$ on $\partial\Omega$ and $F(\underline{u}) \in L^q(\Omega)$;

(2) $\langle -\Delta_p \underline{u} + F(\underline{u}), w - \underline{u} \rangle + \int_\Omega j^0(\underline{u}; w - \underline{u})\,dx \geq 0, \forall w \in \underline{u} \wedge K$.

Definition 2.2. A function $\overline{u} \in V$ is called a supersolution of (1.1) if the following holds:

(1) $\overline{u} \geq 0$ on $\partial\Omega$ and $F(\overline{u}) \in L^q(\Omega)$;

(2) $\langle -\Delta_p \overline{u} + F(\overline{u}), w - \overline{u} \rangle + \int_\Omega j^0(\overline{u}; w - \overline{u})\,dx \geq 0, \forall w \in \overline{u} \vee K$.

Definition 2.3. The multivalued operator $\partial j : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\varnothing\}$ is called Clarke's generalized gradient of j defined by

$$\partial j(s) := \{\xi \in \mathbb{R} : j^0(s; r) \geq \xi r, \forall r \in \mathbb{R}\}. \tag{2.1}$$

We impose the following hypotheses for j and the nonlinearity f in problem (1.1).

(A) There exists a constant $c_1 \geq 0$ such that

$$\xi_1 \leq \xi_2 + c_1 (s_2 - s_1)^{p-1} \tag{2.2}$$

for all $\xi_i \in \partial j(s_i)$, $i = 1, 2$, and for all s_1, s_2 with $s_1 < s_2$.

(B) There is a constant $c_2 \geq 0$ such that

$$\xi \in \partial j(s) : |\xi| \leq c_2\left(1+|s|^{p-1}\right), \quad \forall s \in \mathbb{R}. \tag{2.3}$$

(C) (i) $x \mapsto f(x,r,u(x))$ is measurable for all $r \in \mathbb{R}$ and for all measurable functions $u : \Omega \to \mathbb{R}$.

(ii) $r \mapsto f(x,r,s)$ is continuous for all $s \in \mathbb{R}$ and for almost all $x \in \Omega$.

(iii) $s \mapsto f(x,r,s)$ is decreasing for all $r \in \mathbb{R}$ and for almost all $x \in \Omega$.

(iv) For a given ordered pair of sub- and supersolutions $\underline{u},\overline{u}$ of problem (1.1), there exists a function $k_1 \in L_+^q(\Omega)$ such that $|f(x,r,s)| \leq k_1(x)$ for all $r,s \in [\underline{u}(x),\overline{u}(x)]$ and for almost all $x \in \Omega$.

By [4] the mapping $x \mapsto f(x,u(x),u(x))$ is measurable for $x \mapsto u(x)$ measurable, but the associated Nemytskij operator $F : L^p(\Omega) \to L^q(\Omega)$ needs not necessarily be continuous. In this paper we assume K has lattice structure, that is, K fulfills

$$K \vee K \subset K, \qquad K \wedge K \subset K. \tag{2.4}$$

We recall that the normed space $L^p(\Omega)$ is equipped with the natural partial ordering of functions defined by $u \leq v$ if and only if $v - u \in L_+^p(\Omega)$, where $L_+^p(\Omega)$ is the set of all nonnegative functions of $L^p(\Omega)$.

3. Preliminaries

Here we consider (1.1) for a Carathéodory function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ (i.e., $x \mapsto h(x,s)$ is measurable in Ω for all $s \in \mathbb{R}$ and $s \mapsto h(x,s)$ is continuous on $\mathbb{R}$ for almost all $x \in \Omega$), which fulfills the following growth condition:

$$|h(x,s)| \leq k_2(x), \quad \forall s \in [\underline{u}(x),\overline{u}(x)] \text{ and for a.e. } x \in \Omega, \tag{3.1}$$

where $k_2 \in L_+^q(\Omega)$ and $[\underline{u},\overline{u}]$ is some ordered pair in $L^p(\Omega)$, specified later. Note that the associated Nemytskij operator H defined by $H(u)(x) = h(x,u(x))$ is continuous and bounded from $[\underline{u},\overline{u}] \subset L^p(\Omega)$ to $L^q(\Omega)$ (cf. [5]). Next we introduce the indicator function $I_K : V_0 \to \mathbb{R} \cup \{+\infty\}$ related to the closed convex set $K \neq \emptyset$ given by

$$I_K(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{if } u \notin K, \end{cases} \tag{3.2}$$

which is known to be proper, convex, and lower semicontinuous. The variational-hemivariational inequality (1.1) can be rewritten as follows: find $u \in V_0$ such that

$$\langle -\Delta_p u + H(u), v - u \rangle + I_K(v) - I_K(u) + \int_\Omega j^0(u; v-u)\,dx \geq 0, \quad \forall v \in V_0. \tag{3.3}$$

If $H(u) \equiv h \in V_0^*$, problem (3.3) is a special case of the elliptic variational-hemivariational inequality in [3, Corollary 7.15] for which the method of sub- and supersolutions was developed. In the next result, we show the existence of extremal solutions of (3.3) for a Carathéodory function $h = h(x,s)$.

Lemma 3.1. *Let hypotheses (A),(B), and (2.4) be satisfied and assume the existence of sub- and supersolutions $\underline{u}$ and $\overline{u}$ satisfying $\underline{u} \le \overline{u}$, $\underline{u} \vee K \subset K$, and $\overline{u} \wedge K \subset K$. Furthermore we suppose that the Carathéodory function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies (3.1). Then, (3.3) has a greatest solution u^* and a smallest solution u_* such that*

$$\underline{u} \le u_* \le u^* \le \overline{u}, \tag{3.4}$$

that is, u_* and u^* are solutions of (3.3) that satisfy (3.4), and if u is any solution of (3.3) such that $\underline{u} \le u \le \overline{u}$, then $u_* \le u \le u^*$.

Proof. The proof follows the same ideas as in the proof for $H(u) \equiv h \in V_0^*$ with an additional modification. We only introduce a truncation operator related to the functions $\underline{u}$ and $\overline{u}$ defined by

$$Tu(x) = \begin{cases} \overline{u}(x) & \text{if } u(x) > \overline{u}(x), \\ u(x) & \text{if } \underline{u}(x) \le u(x) \le \overline{u}(x), \\ \underline{u}(x) & \text{if } u(x) < \underline{u}(x). \end{cases} \tag{3.5}$$

The mapping T is continuous and bounded from V into V which follows from the fact that the functions $\min(\cdot,\cdot)$ and $\max(\cdot,\cdot)$ are continuous from V to itself and that T can be represented as $Tu = \max(u,\underline{u}) + \min(u,\overline{u}) - u$ (cf. [6]). In the auxiliary problems of the proof of [3, Corollary 7.15], we replace $h \in V_0^*$ by $(H \circ T)(u)$ and argue in an analogous way. □

An important tool in extending the previous result to discontinuous Nemytskij operators is the next fixed point result. The proof of this Lemma can be found in [7, Theorem 1.1.1].

Lemma 3.2. *Let P be a subset of an ordered normed space, $G : P \to P$ an increasing mapping, and $G[P] = \{Gx \mid x \in P\}$.*

(1) *If $G[P]$ has a lower bound in P and the increasing sequences of $G[P]$ converge weakly in P, then G has the least fixed point x_*, and $x_* = \min\{x \mid Gx \le x\}$.*
(2) *If $G[P]$ has an upper bound in P and the decreasing sequences of $G[P]$ converge weakly in P, then G has the greatest fixed point x^*, and $x^* = \max\{x \mid x \le Gx\}$.*

4. Main results

One of our main results is the following theorem.

Theorem 4.1. *Let hypotheses (A)–(C), (2.4) be satisfied and assume the existence of sub- and supersolutions $\underline{u}$ and $\overline{u}$ satisfying $\underline{u} \le \overline{u}$, $\underline{u} \vee K \subset K$, and $\overline{u} \wedge K \subset K$. If f is right-continuous (resp., left-continuous) in the third argument, then there exists a greatest solution u^* (resp., a smallest solution u_*) of (1.1) in the order interval $[\underline{u},\overline{u}]$.*

Proof. We choose a fixed element $z \in [\underline{u},\overline{u}]$ which is a supersolution of (1.1) satisfying $z \wedge K \subset K$ and consider the following auxiliary problem:

$$u \in K : \langle -\Delta_p u + F_z(u), v - u \rangle + \int_\Omega j^0(u; v-u)\,dx \ge 0, \quad \forall v \in K, \tag{4.1}$$

where $F_z(u)(x) = f(x,u(x),z(x))$. It is readily seen that the mapping $(x,u) \mapsto f(x,u,z(x))$ is a Carathéodory function satisfying some growth condition as in (3.1). Since $F_z(z) = F(z)$, z is also a supersolution of (4.1). By Definition 2.1, we have for a given subsolution $\underline{u}$ of (1.1)

$$\langle -\Delta_p \underline{u} + F(\underline{u}), w - \underline{u}\rangle + \int_\Omega j^0(\underline{u}; w - \underline{u})\,dx \geq 0, \quad \forall w \in \underline{u} \wedge K. \tag{4.2}$$

Setting $w = \underline{u} - (\underline{u} - v)^+$ for all $v \in K$ and using the monotonicity of f with respect to s, we get

$$\begin{aligned} 0 &\geq \langle -\Delta_p \underline{u} + F(\underline{u}), (\underline{u} - v)^+\rangle - \int_\Omega j^0(\underline{u}; -(\underline{u} - v)^+)\,dx \\ &\geq \langle -\Delta_p \underline{u} + F_z(\underline{u}), (\underline{u} - v)^+\rangle - \int_\Omega j^0(\underline{u}; -(\underline{u} - v)^+)\,dx, \quad \forall v \in K, \end{aligned} \tag{4.3}$$

which shows that $\underline{u}$ is also a subsolution of (4.1). Lemma 3.1 implies the existence of a greatest solution $u^* \in [\underline{u}, z]$ of (4.1). Now we introduce the set A given by $A := \{z \in V : z \in [\underline{u}, \overline{u}]$ and z is a supersolution of (1.1) satisfying $z \wedge K \subset K\}$ and define the operator $L : A \to K$ by $z \mapsto u^* =: Lz$. This means that the operator L assigns to each $z \in A$ the greatest solution u^* of (4.1) in $[\underline{u}, z]$. In the next step we construct a decreasing sequence as follows:

$$\begin{aligned} &u_0 := \overline{u} \\ &u_1 := Lu_0 \quad \text{with } u_1 \in [\underline{u}, u_0] \\ &u_2 := Lu_1 \quad \text{with } u_2 \in [\underline{u}, u_1] \\ &\vdots \\ &u_n := Lu_{n-1} \quad \text{with } u_n \in [\underline{u}, u_{n-1}]. \end{aligned} \tag{4.4}$$

As $u_n \in [\underline{u}, u_{n-1}]$, we get $u_n(x) \searrow u(x)$ a.e. $x \in \Omega$. Furthermore, the sequence u_n is bounded in V_0, that is, $\|u_n\|_{V_0} \leq C$ for all n and due to the monotony of u_n and the compact embedding $V_0 \hookrightarrow L^p(\Omega)$, we obtain

$$u_n \rightharpoonup u \quad \text{in } V_0, \qquad u_n \longrightarrow u \quad \text{in } L^p(\Omega) \text{ and a.e. pointwise in } \Omega. \tag{4.5}$$

The fact that u_n is a solution of (4.1) with $z = u_{n-1}$ and $v = u \in K$ results in

$$\langle -\Delta_p u_n, u_n - u\rangle \leq \langle F_{u_{n-1}}(u_n), u - u_n\rangle + \int_\Omega j^0(\underline{u}; u - u_n)\,dx. \tag{4.6}$$

Applying Fatou's Lemma, (4.5), and the upper semicontinuity of $j^0(\cdot,\cdot)$ yields

$$\limsup_{n\to\infty}\langle -\Delta_p u_n, u_n - u\rangle \leq \underbrace{\limsup_{n\to\infty}\|k\|_{L^q(\Omega)}\|u - u_n\|_{L^p(\Omega)}}_{\to 0} + \int_\Omega \underbrace{\limsup_{n\to\infty} j^0(\underline{u}; u - u_n)}_{\leq j^0(u;0)=0}\,dx \leq 0, \tag{4.7}$$

which by the S_+-property of $-\Delta_p$ on V_0 along with (4.5) implies

$$u_n \longrightarrow u \quad \text{in}\, V_0. \tag{4.8}$$

The right-continuity of f and the strong convergence of the decreasing sequence (u_n) along with the upper semicontinuity of $j^0(\cdot;\cdot)$ allow us to pass to the lim sup in (4.1), where u (resp., z) is replaced by u_n (resp., u_{n-1}). We have

$$\begin{aligned} 0 &\le \limsup_{n\to\infty}\langle -\Delta_p u_n + F_{u_{n-1}}(u_n), v-u_n\rangle + \limsup_{n\to\infty}\int_\Omega j^0(u_n; v-u_n)\,dx \\ &\le \lim_{n\to\infty}\langle -\Delta_p u_n + F_{u_{n-1}}(u_n), v-u_n\rangle + \int_\Omega \limsup_{n\to\infty} j^0(u_n; v-u_n)\,dx \\ &\le \langle -\Delta_p u + F_u(u), v-u\rangle + \int_\Omega j^0(u; v-u)\,dx, \quad \forall v\in K. \end{aligned} \tag{4.9}$$

This shows that u is a solution of (1.1) in the order interval $[\underline{u},\overline{u}]$. Now, we still have to prove that u is the greatest solution of (1.1) in $[\underline{u},\overline{u}]$. Let $\tilde{u}$ be any solution of (1.1) in $[\underline{u},\overline{u}]$. Because of the fact that K has lattice structure, $\tilde{u}$ is also a subsolution of (1.1), respectively, a subsolution of (4.1). By the same construction as in (4.4), we obtain

$$\begin{aligned} &\widetilde{u_0} := \overline{u} \\ &\widetilde{u_1} := Lu_0 \quad \text{with}\, \widetilde{u_1} \in [\tilde{u}, u_0] \\ &\widetilde{u_2} := Lu_1 \quad \text{with}\, \widetilde{u_2} \in [\tilde{u}, u_1] \\ &\vdots \\ &\widetilde{u_n} := Lu_{n-1} \quad \text{with}\, \widetilde{u_n} \in [\tilde{u}, u_{n-1}]. \end{aligned} \tag{4.10}$$

Obviously, the sequences in (4.4) and (4.10) create the same extremal solutions u_n and $\widetilde{u_n}$, which implies that $\tilde{u} \le \widetilde{u_n} = u_n$ for all n. Passing to the limit delivers the assertion. The existence of a smallest solution can be shown in a similar way. □

In the next theorem we will prove that only the monotony of f in the third argument is sufficient for the existence of extremal solutions. The function f needs neither be right-continuous nor left-continuous.

Theorem 4.2. *Assume that hypotheses (A)–(C), (2.4) are valid and let $\underline{u}$ and $\overline{u}$ be sub- and supersolutions of (1.1) satisfying $\underline{u} \le \overline{u}$, $\underline{u} \vee K \subset K$, and $\overline{u} \wedge K \subset K$. Then there exist extremal solutions u^* and u_* of (1.1) with $\underline{u} \le u_* \le u^* \le \overline{u}$.*

Proof. As in the proof of Theorem 4.1, we consider the following auxiliary problem:

$$u\in K : \langle -\Delta_p u + F_z(u), v-u\rangle + \int_\Omega j^0(u; v-u)\,dx \ge 0, \quad \forall v\in K, \tag{4.11}$$

where $F_z(u)(x) = f(x,u(x),z(x))$. We define again the set $A := \{z \in V : z \in [\underline{u},\overline{u}]$ and z is a supersolution of (1.1) satisfying $z \wedge K \subset K\}$ and introduce the fixed point operator $L : A \to K$ by $z \mapsto u^* =: Lz$. For a given supersolution $z \in A$, the element Lz is the greatest solution of (4.11) in $[\underline{u}, z]$, and thus it holds that $\underline{u} \le Lz \le z$ for all $z \in A$ which implies $L : A \to [\underline{u}, \overline{u}]$. Because of (2.4), Lz is also a supersolution of (4.11) satisfying

$$\langle -\Delta_p Lz + F_z(Lz), w - Lz \rangle + \int_\Omega j^0(Lz; w - Lz)\,dx \ge 0, \quad \forall w \in Lz \vee K. \tag{4.12}$$

By the monotonicity of f with respect to $Lz \le z$ and using the representation $w = Lz + (v - Lz)^+$ for any $v \in K$, we obtain

$$\begin{aligned} 0 &\le \langle -\Delta_p Lz + F_z(Lz), (v - Lz)^+ \rangle + \int_\Omega j^0(Lz; (v - Lz)^+)\,dx \\ &\le \langle -\Delta_p Lz + F_{Lz}(Lz), (v - Lz)^+ \rangle + \int_\Omega j^0(Lz; (v - Lz)^+)\,dx, \quad \forall v \in K. \end{aligned} \tag{4.13}$$

Consequently, Lz is a supersolution of (1.1). This shows $L : A \to A$.

Let $v_1, v_2 \in A$ and assume that $v_1 \le v_2$. Then we have

$$\begin{aligned} &Lv_1 \in [\underline{u}, v_1] \text{ is the greatest solution of} \\ &\qquad \langle -\Delta_p u + F_{v_1}(u), v - u \rangle + \int_\Omega j^0(u; v - u)\,dx \ge 0, \quad \forall v \in K, \end{aligned} \tag{4.14}$$

$$\begin{aligned} &Lv_2 \in [\underline{u}, v_2] \text{ is the greatest solution of} \\ &\qquad \langle -\Delta_p u + F_{v_2}(u), v - u \rangle + \int_\Omega j^0(u; v - u)\,dx \ge 0, \quad \forall v \in K. \end{aligned} \tag{4.15}$$

Since $v_1 \le v_2$, it follows that $Lv_1 \le v_2$ and due to (2.4), Lv_1 is also a subsolution of (4.14), that is, (4.14) holds, in particular, for $v \in Lv_1 \wedge K$, that is,

$$\langle -\Delta_p Lv_1 + F_{v_1}(Lv_1), (Lv_1 - v)^+ \rangle - \int_\Omega j^0(Lv_1; -(Lv_1 - v)^+)\,dx \le 0, \quad \forall v \in K. \tag{4.16}$$

Using the monotonicity of f with respect to s yields

$$\begin{aligned} 0 &\ge \langle -\Delta_p Lv_1 + F_{v_1}(Lv_1), (Lv_1 - v)^+ \rangle - \int_\Omega j^0(Lv_1; -(Lv_1 - v)^+)\,dx \\ &\ge \langle -\Delta_p Lv_1 + F_{v_2}(Lv_1), (Lv_1 - v)^+ \rangle - \int_\Omega j^0(Lv_1; -(Lv_1 - v)^+)\,dx, \quad \forall v \in K, \end{aligned} \tag{4.17}$$

and hence Lv_1 is a subsolution of (4.15). By Lemma 3.1, we know there exists a greatest solution of (4.15) in $[Lv_1, v_2]$. But Lv_2 is the greatest solution of (4.15) in $[\underline{u}, v_2] \supseteq [Lv_1, v_2]$ and therefore, $Lv_1 \le Lv_2$. This shows that L is increasing.

In the last step we have to prove that any decreasing sequence of $L(A)$ converges weakly in A. Let $(u_n) = (Lz_n) \subset L(A) \subset A$ be a decreasing sequence. The same argument as in the proof of Theorem 4.1 delivers $u_n(x) \searrow u(x)$ a.e. $x \in \Omega$. The boundedness of u_n in V_0, and the compact imbedding $V_0 \hookrightarrow L^p(\Omega)$ along with the monotony of u_n implies

$$u_n \rightharpoonup u \quad \text{in}\, V_0, \qquad u_n \longrightarrow u \quad \text{in}\, L^p(\Omega) \text{ and a.e. } x \in \Omega. \tag{4.18}$$

Since $u_n \in K$ solves (4.11), it follows $u \in K$. From (4.11) with u replaced by u_n and v by u and with the fact that $(s,r) \mapsto j^0(s;r)$ is upper semicontinuous, we obtain by applying Fatou's Lemma

$$\begin{aligned}\limsup_{n\to\infty}\langle -\Delta_p u_n, u_n - u\rangle &\le \limsup_{n\to\infty}\langle F_{z_n}(u_n), u-u_n\rangle + \limsup_{n\to\infty}\int_\Omega j^0(u_n;u-u_n)\,dx \\ &\le \underbrace{\limsup_{n\to\infty}\langle F_{z_n}(u_n), u-u_n\rangle}_{\to 0} + \int_\Omega \underbrace{\limsup_{n\to\infty} j^0(u_n;u-u_n)}_{\le j^0(u;0)=0}\,dx \le 0.\end{aligned} \tag{4.19}$$

The S_+-property of $-\Delta_p$ provides the strong convergence of (u_n) in V_0. As $Lz_n = u_n$ is also a supersolution of (4.11), Definition 2.2 yields

$$\langle -\Delta_p u_n + F_{z_n}(u_n), (v-u_n)^+\rangle + \int_\Omega j^0(u_n;(v-u_n)^+)\,dx \ge 0, \quad \forall v \in K. \tag{4.20}$$

Due to $z_n \ge u_n \ge u$ and the monotonicity of f, we get

$$\begin{aligned}0 &\le \langle -\Delta_p u_n + F_{z_n}(u_n), (v-u_n)^+\rangle + \int_\Omega j^0(u_n;(v-u_n)^+)\,dx \\ &\le \langle -\Delta_p u_n + F_{u}(u_n), (v-u_n)^+\rangle + \int_\Omega j^0(u_n;(v-u_n)^+)\,dx, \quad \forall v \in K,\end{aligned} \tag{4.21}$$

and, since the mapping $u \mapsto u^+ = \max(u,0)$ is continuous from V_0 to itself (cf. [6]), we can pass to the upper limit on the right-hand side for $n\to\infty$. This yields

$$\langle -\Delta_p u + F_u(u), (v-u)^+\rangle + \int_\Omega j^0(u;(v-u)^+)\,dx \ge 0, \quad \forall v \in K, \tag{4.22}$$

which shows that u is a supersolution of (1.1), that is, $u \in A$. As $\overline{u}$ is an upper bound of $L(A)$, we can apply Lemma 3.2, which yields the existence of a greatest fixed point u^* of L in A. This implies that u^* must be the greatest solution of (1.1) in $[\underline{u},\overline{u}]$. By analogous reasoning, one shows the existence of a smallest solution u_* of (1.1). This completes the proof of the theorem. □

Application. In the last part, we give an example of the construction of sub- and supersolutions of problem (1.1). We denote by $\lambda_1 > 0$ the first eigenvalue of $(-\Delta_p, V_0)$ and by φ_1 the eigenfunction of $(-\Delta_p, V_0)$ corresponding to λ_1 satisfying $\varphi_1 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ and $\|\varphi\|_p = 1$ (cf. [8]). Here, $\operatorname{int}(C_0^1(\overline{\Omega})_+)$ describes the interior of the positive cone $C_0^1(\overline{\Omega})_+$ given by

$$\operatorname{int}(C_0^1(\overline{\Omega})_+) = \left\{u \in C_0^1(\overline{\Omega}) : u(x) > 0, \forall x \in \Omega, \text{ and } \frac{\partial u}{\partial n}(x) < 0, \forall x \in \partial\Omega\right\}. \tag{4.23}$$

We suppose the following conditions for f and Clarke's generalized gradient of j, where $\lambda > \lambda_1$ is any fixed constant:

(D) (i)

$$\lim_{|s|\to\infty}\left(\frac{f(x,s,s)}{|s|^{p-2}s}\right) = +\infty, \tag{4.24}$$

uniformly with respect to a.a. $x \in \Omega$,

(ii)

$$\lim_{s\to 0}\left(\frac{f(x,s,s)}{|s|^{p-2}s}\right) = -\lambda, \tag{4.25}$$

uniformly with respect to a.a. $x \in \Omega$,

(iii)

$$\lim_{s\to 0}\left(\frac{\xi}{|s|^{p-2}s}\right) = 0, \tag{4.26}$$

uniformly with respect to a.a. $x \in \Omega$, for all $\xi \in \partial j(s)$,

(iv) f is bounded on bounded sets.

Proposition 4.3. *Assume hypotheses (A), (B), (C)(i)–(iv), and (D). Then there exists a constant a_λ such that $a_\lambda e$ and $-a_\lambda e$ are supersolution and subsolution of problem (1.1), where $e \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ is the unique solution of $-\Delta_p u = 1$ in V_0. Moreover, $-\varepsilon\varphi_1$ is a supersolution and $\varepsilon\varphi_1$ is a subsolution of (1.1) provided that $\varepsilon > 0$ is sufficiently small.*

Proof. A sufficient condition for a subsolution $\underline{u} \in V$ of problem (1.1) is $\underline{u} \le 0$ on $\partial\Omega$, $F(\underline{u}) \in L^q(\Omega)$, and

$$-\Delta_p \underline{u} + F(\underline{u}) + \xi \le 0 \quad \text{in } V_0^*, \forall \xi \in \partial j(\underline{u}). \tag{4.27}$$

Multiplying (4.27) with $(\underline{u} - v)^+ \in V_0 \cap L_+^p(\Omega)$ and using the fact $j^0(\underline{u}; -1) \ge -\xi$, for all $\xi \in \partial j(\underline{u})$, yield

$$\begin{aligned} 0 &\ge \langle -\Delta_p \underline{u} + F(\underline{u}) + \xi, (\underline{u} - v)^+ \rangle = \langle -\Delta_p \underline{u} + F(\underline{u}), (\underline{u} - v)^+ \rangle + \int_\Omega \xi(\underline{u} - v)^+ dx \\ &\ge \langle -\Delta_p \underline{u} + F(\underline{u}), (\underline{u} - v)^+ \rangle - \int_\Omega j^0(\underline{u}; -1)(\underline{u} - v)^+ dx \\ &= \langle -\Delta_p \underline{u} + F(\underline{u}), (\underline{u} - v)^+ \rangle - \int_\Omega j^0(\underline{u}; -(\underline{u} - v)^+) dx, \quad \forall v \in K, \end{aligned} \tag{4.28}$$

and thus, $\underline{u}$ is a subsolution of (1.1). Analogously, $\overline{u} \in V$ is a supersolution of problem (1.1) if $\overline{u} \ge 0$ on $\partial\Omega$, $F(\overline{u}) \in L^q(\Omega)$, and if the following inequality is satisfied,

$$-\Delta_p \overline{u} + F(\overline{u}) + \xi \ge 0 \quad \text{in } V_0^*, \ \forall \xi \in \partial j(\overline{u}). \tag{4.29}$$

The main idea of this proof is to show the applicability of [9, Lemmas 2.1–2.3]. We put $g(x,s) = f(x,s,s) + \xi + \lambda|s|^{p-2}s$ for $\xi \in \partial j(s)$ and notice that in our considerations the nonlinearity g needs not be a continuous function. In view of assumption (B), we see at once that

$$\frac{|\xi|}{|s|^{p-1}} \le c, \quad \text{for} |s| \ge k > 0,\ \forall \xi \in \partial j(s), \tag{4.30}$$

where c is a positive constant. This fact and the condition (D) yield the following limit values:

$$\lim_{|s|\to\infty} \frac{g(x,s)}{|s|^{p-2}s} = +\infty, \qquad \lim_{s\to 0} \frac{g(x,s)}{|s|^{p-2}s} = 0. \tag{4.31}$$

By [9, Lemmas 2.1–2.3], we obtain a pair of positive sub- and supersolutions given by $\underline{u} = \varepsilon\varphi_1$ and $\overline{u} = a_\lambda e$, respectively, a pair of negative sub- and supersolutions given by $\underline{u} = -a_\lambda e$ and $\overline{u} = -\varepsilon\varphi_1$. □

In order to apply Theorem 4.2, we need to satisfy the assumptions

$$\underline{u} \vee K \subset K, \qquad \overline{u} \wedge K \subset K, \qquad K \vee K \subset K, \qquad K \wedge K \subset K, \tag{4.32}$$

which depend on the specific K. For example, we consider an obstacle problem given by

$$K = \{v \in V_0 : v(x) \le \psi(x) \text{ for a.e. } x \in \Omega\}, \quad \psi \in L^\infty(\Omega),\ \psi \ge C > 0, \tag{4.33}$$

where C is a positive constant. One can show that for the positive pair of sub- and supersolutions in Proposition 4.3, all these conditions in (4.32) with respect to the closed convex set K defined in (4.33) can be satisfied.

Example 4.4. The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(r,s) = \begin{cases} -(\lambda+1)|s|^{p-2}s + |r|^{p-1}r & \text{for } s < -1, \\ -\lambda|s|^{p-2}s + |r|^{p-1}r & \text{for } -1 \le s \le 1, \\ -(\lambda+1)|s|^{p-2}s + |r|^{p-1}r & \text{for } s > 1 \end{cases} \tag{4.34}$$

fulfills the assumption (C)(i)–(iv) with respect to $\underline{u}$, $\overline{u}$ defined in Proposition 4.3. Moreover f satisfies the conditions (D)(i)-(ii), (D)(iv), where $\lambda > \lambda_1$ is fixed.

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References

[1] F. H. Clarke, *Optimization and Nonsmooth Analysis*, vol. 5 of *Classics in Applied Mathematics*, SIAM, Philadelphia, Pa, USA, 2nd edition, 1990.

[2] S. Carl, "Existence and comparison results for variational-hemivariational inequalities," *Journal of Inequalities and Applications*, no. 1, pp. 33–40, 2005.

[3] S. Carl, V. K. Le, and D. Motreanu, *Nonsmooth Variational Problems and Their Inequalities*, Springer Monographs in Mathematics, Springer, New York, NY, USA, 2007.

[4] J. Appell and P. P. Zabrejko, *Nonlinear Superposition Operators*, vol. 95 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, UK, 1990.

[5] E. Zeidler, *Nonlinear Functional Analysis and Its Applications Volume II/B*, Springer, Berlin, Germany, 1990.

[6] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, NY, USA, 1993.

[7] S. Carl and S. Heikkilä, *Nonlinear Differential Equations in Ordered Spaces*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2000.

[8] A. Anane, "Simplicité et isolation de la première valeur propre du p-laplacien avec poids," *Comptes Rendus des Séances de l'Académie des Sciences. Série I. Mathématique*, vol. 305, no. 16, pp. 725–728, 1987.

[9] S. Carl and D. Motreanu, "Sign-changing and extremal constant-sign solutions of nonlinear elliptic problems with supercritical nonlinearities," *Communications on Applied Nonlinear Analysis*, vol. 14, no. 4, pp. 85–100, 2007.

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