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Research Article

Some Subordination Results of Multivalent Functions Defined by Integral Operator

Öznur Özkan

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The object of the present paper is to give some subordination properties of integral operator $\mathcal{P}^{\alpha}$ which was studied by Jung in 1993.

1. Introduction and definitions

Let $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, let $\mathcal{H}(\mathbb{U})$ be the set of all functions *analytic in* $\mathbb{U}$, and let

$$\mathcal{A}_p = \{f \in \mathcal{H}(\mathbb{U}) : f(z) = z^p + a_{p+1}z^{p+1} + \cdots\} \tag{1.1}$$

for all $z \in \mathbb{U}$ and $p \in \mathbb{N} = \{1,2,3,\dots\}$ with $\mathcal{A} := \mathcal{A}_1$.

For $p \in \mathbb{N}$, let

$$\mathcal{H}_p = \{f \in \mathcal{H}(\mathbb{U}) : f(z) = p + b_p z^p + \cdots\} \tag{1.2}$$

with $\mathcal{H} := \mathcal{H}_1$.

Given two functions f and g, which are analytic in $\mathbb{U}$, then we say that the function f is *subordinate* to g and write $f \prec g$ or (more precisely)

$$f(z) \prec g(z) \quad (z \in \mathbb{U}), \tag{1.3}$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with

$$w(0) = 0, \qquad |w(z)| < 1 \quad (z \in \mathbb{U}) \tag{1.4}$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}). \tag{1.5}$$

In particular, if g is univalent in $\mathbb{U}$, then we write the following equivalence:

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0), \qquad f(\mathbb{U}) \subset g(\mathbb{U}). \tag{1.6}$$

For analytic functions f_j $(j = 1,2)$ given by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k \quad (j = 1,2), \tag{1.7}$$

let $f_1 * f_2$ denote the *Hadamard* product *(or convolution)* of f_1 and f_2, defined by

$$(f_1 * f_2)(z) := z + \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k =: (f_2 * f_1)(z). \tag{1.8}$$

Next, for real parameters A and B such that $-1 \le B < A \le 1$, we define the function

$$\varphi_{A,B}(z) := \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}). \tag{1.9}$$

It is obvious that $\varphi_{A,B}(z)$ for $-1 \le B \le 1$ is the conformal map of the unit disk $\mathbb{U}$ onto the disk symmetrical with respect to the real axis having the center

$$\frac{1 - AB}{1 - B^2} \quad (B \neq \mp 1) \tag{1.10}$$

and the radius

$$\frac{A - B}{1 - B^2} \quad (B \neq \mp 1). \tag{1.11}$$

Furthermore, the boundary circle of this disk intersects the real axis at the points

$$\frac{1 - A}{1 - B}, \quad \frac{1 + A}{1 + B} \quad (B \neq \mp 1). \tag{1.12}$$

Let $(a)_\nu$ denote the *Pochhammer* symbol *(or the shifted factorial)*, since

$$(1)_n = n! \quad \text{for } n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \tag{1.13}$$

defined (for $a, \nu \in \mathbb{C}$ and in terms of the Gamma function) by

$$(a)_\nu := \frac{\Gamma(a+\nu)}{\Gamma(a)} = \begin{cases} 1 & (\nu = 0,\ a \in \mathbb{C} \backslash \{0\}), \\ a(a+1)\cdots(a+n-1) & (\nu = n \in \mathbb{N};\ a \in \mathbb{C}). \end{cases} \tag{1.14}$$

Recently, the following integral operator was studied by Jung et al. [1] for $p = 1$ and Shams et al. [2]:

$$\mathcal{P}^\alpha f = \mathcal{P}^\alpha f(z) := \frac{(p+1)^\alpha}{z\Gamma(\alpha)} \int_0^z \left(\log \frac{z}{t}\right)^{\alpha-1} f(t)dt = z^p + \sum_{n=p+1}^{\infty} \left(\frac{p+1}{n+1}\right)^\alpha a_n z^n \tag{1.15}$$

for $f \in \mathcal{A}_p$ and $\alpha > 0$. Moreover, Jung et al. [1] have shown that

$$\mathcal{P}^{\alpha} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^{\alpha} a_n z^n \tag{1.16}$$

for $f \in \mathcal{A}$ and $\alpha > 0$.

Therefore, Shams et al. [2] showed the following equality:

$$z\left(\mathcal{P}^{\alpha} f(z)\right)' = (p+1)\mathcal{P}^{\alpha-1} f(z) - \mathcal{P}^{\alpha} f(z) \tag{1.17}$$

using (1.15).

Let $\mathcal{P}(\gamma)$ denote the subclass of $\mathcal{H}$ consisting of functions f with the following condition:

$$\mathfrak{Re}\{f'(z)\} > \gamma \tag{1.18}$$

for $0 \leq \gamma < 1$ and for all $z \in \mathbb{U}$.

2. Main results

In proving our main results, we need the following lemmas.

Lemma 2.1 [3, page 71]. *Let h be analytic, univalent, convex in $\mathbb{U}$ with $h(0) = 1$. Also let p be analytic in $\mathbb{U}$ with $p(0) = h(0)$. If*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z) \quad (z \in \mathbb{U};\ \gamma \neq 0), \tag{2.1}$$

then

$$p(z) \prec q(z) \prec h(z), \tag{2.2}$$

where

$$q(z) = \frac{\gamma}{z^{\gamma}} \int_0^z t^{\gamma-1} h(t)\,dt \quad (z \in \mathbb{U};\ \mathfrak{Re}(\gamma) \geq 0;\ \gamma \neq 0). \tag{2.3}$$

Lemma 2.2 [4]. *If $f \in \mathcal{P}(\gamma)$ for $0 \leq \gamma < 1$, then*

$$\mathfrak{Re}\{f(z)\} \geq 2\gamma - 1 + \frac{2(1-\gamma)}{1+|z|} \quad (z \in \mathbb{U}). \tag{2.4}$$

Lemma 2.3 [3, page 132]. *Let q be univalent in $\mathbb{U}$ and let θ and ϕ be analytic in a domain $\mathcal{D}$ containing $q(\mathbb{U})$ with $\phi(w) \neq 0$, when $w \in q(\mathbb{U})$. Set*

$$Q(z) = zq'(z) \cdot \phi[q(z)], \qquad h(z) = \theta[q(z)] + Q(z), \tag{2.5}$$

and suppose that either

(i) *Q is starlike, or*

(ii) *h is convex.*

In addition, assume that

(iii)

$$\mathfrak{Re}\frac{zh'(z)}{Q(z)}=\mathfrak{Re}\left[\frac{\theta'[q(z)]}{\phi[q(z)]}+\frac{zQ'(z)}{Q(z)}\right]>0. \tag{2.6}$$

If P is analytic in $\mathbb{U}$, with $P(0)=q(0)$, $P(\mathbb{U})\subset\mathfrak{D}$, and

$$\theta[P(z)]+zP'(z)\cdot\phi[P[z]]\prec\theta[q(z)]+zq'(z)\cdot\phi[q[z]]=h(z), \tag{2.7}$$

then $P\prec q$, and q is the best dominant.

Theorem 2.4. *Let $\alpha>0$, $\lambda>0$, and $-1\le B_j<A_j\le 1$ $(j=1,2)$. If each of the functions $f_j(z)\in\mathcal{A}_p$ $(j=1,2)$ satisfies the following subordination condition:*

$$\lambda\frac{\mathcal{P}^{\alpha-1}f_j(z)}{z^p}+(1-\lambda)\frac{\mathcal{P}^{\alpha}f_j(z)}{z^p}\prec\varphi_{A_j,B_j}(z), \tag{2.8}$$

then

$$\lambda\frac{\mathcal{P}^{\alpha-1}\Omega(z)}{z^p}+(1-\lambda)\frac{\mathcal{P}^{\alpha}\Omega(z)}{z^p}\prec\varphi_{1-2\beta,-1}(z), \tag{2.9}$$

where

$$\Omega(z):=\mathcal{P}^{\alpha}(f_1*f_2)(z), \tag{2.10}$$

$$\beta:=1-\frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left(1-\frac{p+1}{\lambda}\int_0^1\frac{u^{(p+1)/\lambda-1}}{1+u}du\right). \tag{2.11}$$

The result is sharp for $B_1=B_2=-1$.

Proof. We assume that each of the functions $f_j(z)\in\mathcal{A}_p$ $(j=1,2)$ satisfies the subordination condition (2.8). If we take

$$\varphi_j(z):=\lambda\frac{\mathcal{P}^{\alpha-1}f_j(z)}{z^p}+(1-\lambda)\frac{\mathcal{P}^{\alpha}f_j(z)}{z^p}, \tag{2.12}$$

then we write

$$\varphi_j(z)\in\mathcal{P}(\gamma_j) \tag{2.13}$$

for $\gamma_j=(1-A_j)/(1-B_j)$ and $j=1,2$. Using (1.17) and (2.12), we obtain the equality

$$\mathcal{P}^{\alpha}f_j(z)=\frac{p+1}{\lambda}z^{p-(p+1)/\lambda}\int_0^z t^{(p+1)/\lambda-1}\varphi_j(t)dt \tag{2.14}$$

for $j=1,2$. Then, from definition (2.10) we write

$$\mathcal{P}^{\alpha}\Omega(z)=\frac{p+1}{\lambda}z^{p-(p+1)/\lambda}\int_0^z t^{(p+1)/\lambda-1}\varphi_0(t)dt, \tag{2.15}$$

where, for convenience,

$$\varphi_0(z) = \lambda \frac{\mathcal{P}^{\alpha-1}\Omega(z)}{z^p} + (1-\lambda)\frac{\mathcal{P}^{\alpha}\Omega(z)}{z^p} = \frac{p+1}{\lambda} z^{-(p+1)/\lambda} \int_0^z t^{(p+1)/\lambda-1}(\varphi_1 * \varphi_2)(t)dt. \tag{2.16}$$

Since $\varphi_1(z) \in \mathcal{P}(\gamma_1)$ and

$$\varphi_3(z) := \frac{\varphi_2(z) - \gamma_2}{2(1-\gamma_2)} + \frac{1}{2} \in \mathcal{P}\left(\frac{1}{2}\right), \tag{2.17}$$

we see that $(\varphi_1 * \varphi_3)(z) \in \mathcal{P}(\gamma_1)$ by applying the well-known Herglotz formula. Thus,

$$(h_1 * h_2)(z) \in \mathcal{P}(\gamma_3) \quad \text{for } \gamma_3 := 1 - 2(1-\gamma_1)(1-\gamma_2). \tag{2.18}$$

If we change variable $t = uz$ and take real part in (2.16), then we obtain the following inequality:

$$\mathfrak{Re}\{h_0(z)\} = \frac{p+1}{\lambda} \int_0^z u^{(p+1)/\lambda-1} \mathfrak{Re}\{(\varphi_1 * \varphi_2)(uz)\} du. \tag{2.19}$$

Using Lemma 2.2 in last equality, we obtain

$$\begin{aligned} \mathfrak{Re}\{\varphi_0(z)\} &\geq \frac{p+1}{\lambda} \int_0^z u^{(p+1)/\lambda-1}\left[2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u|z|}\right] du \\ &> \frac{p+1}{\lambda} \int_0^z u^{(p+1)/\lambda-1}\left[2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u}\right] du \\ &= 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left(1 - \frac{p+1}{\lambda}\int_0^1 \frac{u^{(p+1)/\lambda-1}}{1+u} du\right) = \beta. \end{aligned} \tag{2.20}$$

Thus, we have obtained (2.11).

Now, we must prove that this result is sharp when $B_1 = B_2 = -1$. Let $B_1 = B_2 = -1$. From (2.14), we can write

$$\mathcal{P}^{\alpha} f_j(z) = \frac{p+1}{\lambda} z^{p-(p+1)/\lambda} \int_0^z t^{(p+1)/\lambda-1} \frac{1+A_j t}{1-t} dt \tag{2.21}$$

for $j = 1, 2$. Since

$$\left(\frac{1+A_1 z}{1-z}\right) * \left(\frac{1+A_2 z}{1-z}\right) = 1 - (1+A_1)(1+A_2) + \frac{(1+A_1)(1+A_2)}{1-z} \tag{2.22}$$

and with change of variable $t = uz$ in (2.16),

$$\varphi_0(z) = \frac{p+1}{\lambda} z^{-(p+1)/\lambda} \int_0^z t^{(p+1)/\lambda-1}\left[1 - (1+A_1)(1+A_2) + \frac{(1+A_1)(1+A_2)}{1-uz}\right] du. \tag{2.23}$$

If we choose z on the real axis and let $z \to -1$, we obtain

$$h_0(z) \longrightarrow 1-(1+A_1)(1+A_2)\left(1-\frac{p+1}{\lambda}\int_0^1 \frac{u^{(p+1)/\lambda-1}}{1+u}du\right). \tag{2.24}$$

Thus, we complete the proof of Theorem 2.4. □

Corollary 2.5. *Let $\alpha > 0$ and the function $f_j(z) \in \mathcal{A}_p$ $(j=1,2)$. If the condition*

$$\frac{\mathcal{P}^{\alpha-1} f_j(z)}{z} \prec \frac{1+z}{1-z} \tag{2.25}$$

is satisfied, then

$$\frac{\mathcal{P}^{\alpha-1}\Omega(z)}{z} \prec \frac{1+(16\ln 2-9)z}{1-z}, \tag{2.26}$$

where

$$\Omega := \mathcal{P}^{\alpha}(f_1 * f_2)(z), \qquad \beta = 5-8\ln 2. \tag{2.27}$$

Proof. By putting $p=1$, $\lambda=1$, $A=1$, $B=-1$ in Theorem 2.4, we obtain Corollary 2.5. □

Theorem 2.6. *Let $0 \le \rho < 1$. For $f \in \mathcal{A}_p$, if*

$$\Re\mathfrak{e}\left(\frac{\mathcal{P}^{\alpha-1} f(z)}{\mathcal{P}^{\alpha} f(z)}\right) > \rho, \tag{2.28}$$

then

$$\left(\frac{\mathcal{P}^{\alpha} f(z)}{z^p}\right)^{\gamma} \prec \frac{1}{(1-z)^{2\gamma(1-\rho^2)}} = q(z), \tag{2.29}$$

where $q(z)$ is the best dominant.

Proof. Let $f \in \mathcal{A}_p$ and

$$p(z) := \left(\frac{\mathcal{P}^{\alpha} f(z)}{z^p}\right)^{\gamma}. \tag{2.30}$$

Taking logarithmic derivative and multiplying by z, we find that

$$\frac{zp'(z)}{p(z)} = (p+1)\gamma\left(\frac{\mathcal{P}^{\alpha-1} f(z)}{\mathcal{P}^{\alpha} f(z)}-1\right), \tag{2.31}$$

$$1+\frac{1}{(p+1)\gamma}\frac{zp'(z)}{p(z)} = \frac{\mathcal{P}^{\alpha-1} f(z)}{\mathcal{P}^{\alpha} f(z)}. \tag{2.32}$$

Therefore, from (2.28) we write

$$\frac{\mathcal{P}^{\alpha-1} f(z)}{\mathcal{P}^{\alpha} f(z)} \prec \frac{1+(1-2\rho)z}{1-z}. \tag{2.33}$$

Thus, using (2.33) in (2.32), we obtain

$$1+\frac{1}{(p+1)\gamma}\frac{zp'(z)}{p(z)} \prec \frac{1+(1-2\rho)z}{1-z}. \tag{2.34}$$

Now, define the functions θ and ϕ by

$$\theta(w) := 1, \qquad \phi(w) := \frac{1}{(p+1)\gamma w}, \qquad \mathcal{D} = \{w : w \neq 0\} \tag{2.35}$$

in Lemma 2.3. If we take

$$q(z) := \frac{1}{(1-z)^{2\gamma(1-\rho^2)}}, \tag{2.36}$$

then q satisfies the conditions of Lemma 2.3. Thus, the following functions:

$$\begin{aligned} Q(z) &= zq'(z)\cdot\phi[q(z)] = \frac{1}{(p+1)\gamma}\frac{zq'(z)}{q(z)} = \frac{2(1-\rho)z}{1-z}, \\ h(z) &= \theta[q(z)] + Q(z) = 1+\frac{1}{(p+1)\gamma}\frac{zq'(z)}{q(z)} = \frac{1+(1-2\rho)z}{1-z}. \end{aligned} \tag{2.37}$$

Since h is convex, preconditions of Lemma 2.3 are satisfied. Consequently, from Lemma 2.3 we write $p \prec q$, and q is the best dominant. □

Corollary 2.7. *Let $0 \leq \rho < 1$. If $f \in \mathcal{A}_p$ satisfies (2.28), then*

$$\mathfrak{Re}\left\{\frac{\mathcal{P}^{\alpha-1} f_j(z)}{z^p}\right\}^{\gamma/2\left(1-\rho^2\right)} > 2^{-1/\gamma} \tag{2.38}$$

and $2^{-1/\gamma}$ is the best possible.

Proof. From (2.29), there exists a Schwarz function $w(z)$ analytic in $\mathbb{U}$ with

$$w(0) = 0, \qquad |w(z)| < 1 \quad (z \in \mathbb{U}) \tag{2.39}$$

such that

$$\left(\frac{\mathcal{P}^{\alpha} f(z)}{z^p}\right)^{\gamma} = \frac{1}{(1-w(z))^{2\gamma\left(1-\rho^2\right)}}, \tag{2.40}$$

that is,

$$\left(\frac{\mathcal{P}^{\alpha} f(z)}{z^p}\right)^{\gamma/2(1-\rho^2)} = (1-w(z))^{-\gamma}. \tag{2.41}$$

In the last equality, if we take a real part and use the following inequality:

$$\mathfrak{Re}\left(w^{1/m}\right) \geq \left[\mathfrak{Re}(w)\right]^{1/m} \quad (\mathfrak{Re}(w) > 0), \tag{2.42}$$

then (2.38) is obtained. □

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Öznur Özkan: Department of Statistics and Computer Sciences, Başkent University, Baglıca, 06530 Ankara, Turkey
Email address: oznur@baskent.edu.tr