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*Research Article*

# An Inexact Proximal-Type Method for the Generalized Variational Inequality in Banach Spaces

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We investigate an inexact proximal-type method, applied to the generalized variational inequality problem with maximal monotone operator in reflexive Banach spaces. Solodov and Svaiter (2000) first introduced a new proximal-type method for generating a strongly convergent sequence to the zero of maximal monotone operator in Hilbert spaces, and subsequently Kamimura and Takahashi (2003) extended Solodov and Svaiter algorithm and strong convergence result to the setting of uniformly convex and uniformly smooth Banach spaces. In this paper Kamimura and Takahashi's algorithm is extended to develop a generic inexact proximal point algorithm, and their convergence analysis is extended to develop a generic convergence analysis which unifies a wide class of proximal-type methods applied to finding the zeroes of maximal monotone operators in the setting of Hilbert spaces or Banach spaces.



## 1. Introduction

In this paper, we investigate a broad class of inexact proximal-type methods for solving the generalized variational inequality problem with maximal monotone operator in a reflexive Banach space. Let $E$ be a real reflexive Banach space with dual $E^*$. The notation $\langle x, f\rangle$ stands for the duality pairing $f(x)$ of $f \in E^*$ and $x \in E$. Given $T : E \to 2^{E^*}$, a maximal monotone operator, and $\Omega \subset E$, a nonempty closed and convex subset, the generalized variational inequality for $T$ and $\Omega$, GVI $(T,\Omega)$, is as follows. Find $x^*$ such that

$$x^* \in \Omega, \quad \exists u^* \in T(x^*) : \langle u^*, x - x^*\rangle \geq 0 \quad \forall x \in \Omega. \tag{1.1}$$

The set $\Omega$ will be called the feasible set for problem (1.1). In the particular case, in which $T$ is the subdifferential of a proper convex and lower semicontinuous function $\varphi : E \to (-\infty, \infty]$, (1.1) reduces to the convex optimization problem

$$\min_{x\in\Omega} \varphi(x). \tag{1.2}$$

It is well known that one of the most significant and important problems in the variational inequality theory is the development of an efficient iterative algorithm to compute approximate solutions. In 2005, Burachik et al. [1] studied the following generic outer approximation scheme for solving GVI $(T,\Omega)$.

*Algorithm 1.1* (BLS).

*Initialization.* Take $\Omega_1 \supset \Omega$.

*Iterations.* For $n = 1,2,\dots$, find $x_n \in \Omega_n$, a solution of the approximated problem $(P_n)$, defined as

$$\exists u_n \in T(x_n) \quad \text{with } \langle u_n, x - x_n\rangle \geq -\varepsilon_n \quad \forall x \in \Omega_n, \tag{1.3}$$

where there hold the following conditions:

(i) $\{\varepsilon_n\} \subset [0,\infty)$ satisfies $\lim_{n\to\infty}\varepsilon_n = 0$;

(ii) $\{\Omega_n\} \subset E$ is a sequence of closed convex subsets such that $\Omega \subset \Omega_n$ for all $n$.

The goal of their work in [1] is twofold. First, they developed a convergence analysis which can be applied to a more general and flexible Algorithm BLs for successive approximation of GVI $(T,\Omega)$, under the standard boundedness assumptions. They proved that Algorithm BLs generates a bounded sequence and that all weak accumulation points are solutions of GVI $(T,\Omega)$. Second, they obtained the same convergence results in the absence of boundedness assumptions. For doing this, they considered subproblems $(P_n)$, where the original operator is replaced by a suitable coercive regularization. Their work was built around the above generic outer approximation algorithm for solving GVI $(T,\Omega)$.

To present a convergence analysis of Algorithm BLs, they assumed that the solution set $S^*$ of GVI $(T,\Omega)$ is nonempty and that the sequence $\{x_n\}$ generated by Algorithm BLs is asymptotically feasible.

We recall that $\{x_n\}$ is called asymptotically feasible when all weak accumulation points of $\{x_n\}$ belong to $\Omega$.

In 2003, Kamimura and Takahashi [2] introduced and studied the following proximal-type algorithm in a smooth Banach space $E$.

*Algorithm 1.2* (KT).

$$\begin{gathered} x_0 \in E, \\ 0 = v_n + \frac{1}{r_n}(Jy_n - Jx_n), \quad v_n \in Ty_n, \\ H_n = \{z \in E : \langle v_n, z - y_n\rangle \leq 0\}, \end{gathered}$$

$$W_n = \{z \in E : \langle Jx_0 - Jx_n, z - x_n \rangle \le 0\},$$
$$x_{n+1} = Q_{H_n \cap W_n} x_0, \quad n = 0,1,2,\dots, \tag{1.4}$$

where $\{r_n\}$ is a sequence of positive real numbers and $x_{n+1} = Q_{H_n \cap W_n} x_0$ is the unique point such that

$$\phi(x_{n+1}, x_0) = \inf\{\phi(z, x_0) : z \in H_n \cap W_n\}. \tag{1.5}$$

They derived a strong convergence theorem which extends and improves Solodov and Svaiter results [3].

In this paper, Kamimura and Takahashis convergence analysis [2] is extended to develop a generic convergence analysis which unifies a wide class of proximal-type methods applied to finding the zeroes of maximal monotone operators in the setting of Hilbert spaces or Banach spaces. Our work is built around a generic inexact proximal point algorithm (see Section 3, Algorithm (I)). First, by virtue of Burachik et al. technique [1, Lemma 3.2], we prove that all weak accumulation points of the iterative sequence $\{x_n\}$ are solutions of GVI $(T,\Omega)$. Second, utilizing Kamimura and Takahashi technique [2, Theorem 3.1], we prove that the whole sequence $\{x_n\}$ converges strongly to a solution of GVI $(T,\Omega)$.

We recall the main basic notions that will be used in the sequel. Let $T : E \to 2^{E^*}$ be a multivalued operator.

$D(T) := \{x \in E \mid Tx \neq \varnothing\}$ is the domain of $T$; $G(T) := \{(x,u) \in E \times E^* \mid u \in Tx\}$ and $R(T) := \{u \in E^* \mid u \in Tx$ for some $x \in E\}$ are the graph and the range of $T$, respectively;

$T$ is monotone if for all $x, y \in E$, $u \in Tx$, and $v \in Ty$,

$$\langle u - v, x - y \rangle \ge 0; \tag{1.6}$$

if this inequality holds strictly whenever $x, y \in E$, $u \in Tx$, $v \in Ty$, and $x \neq y$, then $T$ is strictly monotone;

$T$ is maximal monotone if it is monotone and for any monotone $\widetilde{T} : E \to 2^{E^*}$, $G(T) \subset G(\widetilde{T}) \Rightarrow T = \widetilde{T}$.

## 2. Preliminaries

To proceed, we establish some preliminaries. Let $E$ be a real Banach space, and $E^*$ the dual space of $E$. The notion of paramonotonicity was introduced in [4, 5] and further studied in [6]. It is defined as follows.

*Definition 2.1* [1, page 2075]. The operator $T$ is paramonotone in $\Omega$ if it is monotone and $\langle v - u, y - z \rangle = 0$ with $y, z \in \Omega, v \in T(y), u \in T(z)$ implies that $u \in T(y)$, $v \in T(z)$. The operator $T$ is paramonotone if this property holds in the whole space.

Proposition 2.2 (see [6, Proposition 4]). *Assume that $T$ is paramonotone on $\Omega$ and $\overline{x}$ is a solution of GVI $(T,\Omega)$. Let $x^* \in \Omega$ be such that there exists an element $u^* \in T(x^*)$ with $\langle u^*, x^* - \overline{x} \rangle \le 0$. Then $x^*$ also solves GVI $(T,\Omega)$.*

Paramonotonicity can be seen in a condition which is weaker than strict monotonicity. The remark below contains some examples of operators which are paramonotone.

*Remark 2.3.* If $T$ is the subdifferential of a convex function $\varphi : E \to (-\infty,\infty]$, then $T$ is paramonotone. When $E = R^n$, a condition which guarantees paramonotonicity of $T : E \to 2^E$, is when $T$ is differentiable and the symmetrization of its Jacobian matrix has the same rank as the Jacobian matrix itself. However, relevant operators fail to satisfy this condition.

Recall the definition of pseudomonotonicity, which was taken from [7] and should not be confused with other uses of the same word (see, e.g., [8]).

*Definition 2.4* [1, page 2075]. Let $E$ be a reflexive Banach space and the operator $T$ such that $D(T)$ is closed and convex. $T$ is said to be pseudomonotone if it satisfies the following condition. If the sequence $\{(x_n,u_n)\} \subset G(T)$ satisfies that

(a) $\{x_n\}$ converges weakly to $x^* \in D(T)$,

(b) $\limsup_n \langle u_n, x_n - x^*\rangle \leq 0$,

then for every $w \in D(T)$ there exists an element $u^* \in T(x^*)$ such that

$$\langle u^*, x^* - w\rangle \leq \liminf_n \langle u_n, x_n - w\rangle. \tag{2.1}$$

*Remark 2.5.* If $T$ is the gradient of a Gâteaux differentiable convex function $\varphi : R^n \to (-\infty,\infty]$, then $T$ is pseudomonotone. Indeed, $T \equiv \nabla\varphi$ is hemicontinuous according to [9, page 94]. Thus $T \equiv \nabla\varphi$ is pseudomonotone according to [9, page 107]. Combining the latter statement with Remark 2.3, we conclude that every $T$ of this kind is both para- and pseudomonotone. An example of a nonstrictly monotone operator, which is both para- and pseudomonotone, is the subdifferential of the function $\varphi : (-\infty,\infty) \to (-\infty,\infty)$ defined by $\varphi(t) = |t|$ for all $t$.

On the other hand, recall that $E$ is said to be smooth if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.2}$$

exists for each $x, y \in S_E$, where $S_E := \{x \in E : \|x\| = 1\}$ is the unit sphere of $E$. If $E$ is smooth, then the normalized duality mapping $J : E \to 2^{E^*}$ is single valued, and continuous from the norm topology of $E$ to the weak$^*$ topology of $E^*$, that is, norm-to-weak$^*$ continuous. In general, the normalized duality mapping $J$ has the following well-known property:

$$\|x\|^2 - \|y\|^2 \geq 2\langle jy, x - y\rangle \tag{2.3}$$

for all $x, y \in E$ and $jy \in Jy$. Recall also that $E$ is said to be uniformly smooth if $E$ is smooth and the limit (2.2) is attained uniformly for $x, y \in S_E$.

A Banach space $E$ is said to be strictly convex if $\|(x+y)/2\| < 1$ for all $x, y \in S_E$ with $x \neq y$. It is also said to be uniformly convex if for any given $\varepsilon > 0$, there exists some $\delta > 0$

such that for each $x, y \in S_E$,

$$\|x+y\| > 2-\delta \Longrightarrow \|x-y\| < \varepsilon. \tag{2.4}$$

It is known that a uniformly convex Banach space is reflexive and strictly convex.

Next, we recall some propositions involving the function $\phi : E \times E \to (-\infty, \infty)$ defined by

$$\phi(x,y) = \|x\|^2 - 2\langle Jy, x\rangle + \|y\|^2 \quad \forall x, y \in E, \tag{2.5}$$

where $E$ is a smooth Banach space. When $\{x_n\}$ is a sequence in $E$, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$, and the weak convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ weakly.

Proposition 2.6 [2]. *Let $E$ be a uniformly convex and smooth Banach space, and let $\{y_n\}$ and $\{z_n\}$ be two sequences of $E$. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Proposition 2.7 [10]. *Let $E$ be a reflexive, strictly convex, and smooth Banach space. Let $C$ be a nonempty closed convex subset of $E$ and $x \in E$. Then there exists a unique element $x_0 \in C$ such that*

$$\phi(x_0, x) = \inf\{\phi(z,x) : z \in C\}. \tag{2.6}$$

For each nonempty closed convex subset $C$ of a reflexive, strictly convex, and smooth Banach space $E$ and $x \in E$, we defined the mapping $Q_C$ of $E$ onto $C$ by $Q_C x = x_0$ where $x_0$ is defined by (2.6). It is easy to see that, in a Hilbert space, the mapping $Q_C$ is coincident with the metric projection. In our discussion, instead of the metric projection, we make use of the mapping $Q_C$. Finally, we recall two results concerning Proposition 2.7 and the mapping $Q_C$.

Proposition 2.8 [2]. *Let $E$ be a smooth Banach space and $C$ a convex subset of $E$. Let $x \in E$ and $\overline{x} \in C$. Then*

$$\phi(\overline{x}, x) = \inf\{\phi(z,x) : z \in C\} \tag{2.7}$$

*if and only if*

$$\langle J\overline{x} - Jx, z - \overline{x}\rangle \geq 0 \quad \forall z \in C. \tag{2.8}$$

Proposition 2.9 [2]. *Let $E$ be a reflexive, strictly convex, and smooth Banach space. Let $C$ be a nonempty closed convex subset of $E$ and $x \in E$. Then*

$$\phi(y, Q_c x) + \phi(Q_c x, x) \leq \phi(y, x) \quad \forall y \in C. \tag{2.9}$$

## 3. Inexact proximal-type method and its convergence

In the remainder of this paper, we always assume that $E$ is a real smooth Banach space, $\Omega \subset E$ a nonempty closed and convex set, and $T : E \to 2^{E^*}$ a maximal monotone operator.

To present a convergence analysis for the GVI $(T,\Omega)$ (1.1) which can be applied to a wide family of proximal point schemes, we fix a sequence $\{\Omega_n\}$ of closed convex subsets of $E$, and a sequence $\{\varepsilon_n\}\subset[0,\infty)$ verifying

(i) $\Omega\subseteq\Omega_n$ for all $n\geq 0$, where $\Omega_0=\Omega$,

(ii) $\lim_n \varepsilon_n=0$.

Let $A:E\to E^*$ be monotone such that

$$R(J+A+rT)=E^* \quad \forall r>0. \tag{3.1}$$

We will make the following assumptions:

$(\mathrm{H}_1)$ $D(T)\cap\operatorname{int}(\Omega)\neq\varnothing$ or $\operatorname{int}(D(T))\cap\Omega\neq\varnothing$;

$(\mathrm{H}_2)$ $T$ paramonotone and pseudomonotone with closed domain;

$(\mathrm{H}_3)$ the solution set $S^*$ of GVI $(T,\Omega)$ is nonempty.

Now, we introduce the following inexact proximal point algorithm.

*Algorithm 3.1* (I).

$$\begin{gathered}
x_0\in E,\\
\exists v_n\in Ty_n \quad \text{with } \left\langle v_n+\frac{1}{r_n}(Jy_n-Jx_n)+\frac{1}{r_n}(Ay_n-Ax_n)+\frac{1}{r_n}\cdot e_n, y-y_n\right\rangle \geq -\varepsilon_n \quad \forall y\in\Omega_n,\\
H_n=\{z\in\Omega_n:\langle v_n,z-y_n\rangle\leq 0\},\\
W_n=\{z\in\Omega_n:\langle Jx_0-Jx_n,z-x_n\rangle\leq 0\},\\
x_{n+1}=Q_{H_n\cap W_n}x_0,\quad n=0,1,2,\ldots,
\end{gathered} \tag{3.2}$$

where $\{r_n\}$ is a sequence of positive real numbers and $\{e_n\}$ is regarded as an error sequence in $E^*$.

First, we investigate the conditions under which Algorithm (I) is well defined.

Proposition 3.2. *Let $E$ be a reflexive, strictly convex, and smooth Banach space. If $D(T)\subset\Omega$ such that $(\mathrm{H}_3)$ holds, then the sequences $\{x_n\}$ and $\{y_n\}$ generated by Algorithm (I) are both well defined.*

*Proof.* It is obvious that both $H_n$ and $W_n$ are closed convex sets. Let $\hat{x}\in S^*$. Then there exists $\hat{u}\in T\hat{x}$ such that

$$\langle\hat{u},x-\hat{x}\rangle\geq 0 \quad \forall x\in\Omega. \tag{3.3}$$

From the assumption $(*)$, there exists $(y_n,v_n)\in E\times E^*$ such that $v_n\in Ty_n$ and

$$\left\langle v_n+\frac{1}{r_n}(Jy_n-Jx_n)+\frac{1}{r_n}(Ay_n-Ax_n)+\frac{1}{r_n}\cdot e_n, y-y_n\right\rangle \geq -\varepsilon_n \quad \forall y\in\Omega_n. \tag{3.4}$$

Since $\hat{x}\in\Omega\subset\Omega_n$ for all $n\geq 1$ and $y_n\in D(T)\subset\Omega$, the monotonicity of $T$ implies that

$$\langle v_n,y_n-\hat{x}\rangle\geq\langle\hat{u},y_n-\hat{x}\rangle\geq 0. \tag{3.5}$$

Hence $\hat{x} \in H_n$ for each $n \geq 0$. It is clear that $\hat{x} \in H_0 \cap W_0$. Thus it follows from Proposition 2.8 that

$$\langle Jx_0 - Jx_1, \hat{x} - x_1 \rangle = \langle Jx_0 - JQ_{H_0 \cap W_0} x_0, \hat{x} - Q_{H_0 \cap W_0} x_0 \rangle \leq 0. \tag{3.6}$$

Therefore, $\hat{x} \in H_1 \cap W_1$. By induction, we obtain

$$\langle Jx_0 - Jx_n, \hat{x} - x_n \rangle = \langle Jx_0 - JQ_{H_{n-1} \cap W_{n-1}} x_0, \hat{x} - Q_{H_{n-1} \cap W_{n-1}} x_0 \rangle \leq 0 \tag{3.7}$$

which implies $\hat{x} \in H_n \cap W_n$ and hence $x_{n+1} = Q_{H_n \cap W_n} x_0$ is well defined. Thus by induction again, the sequence $\{x_n\}$ generated by Algorithm (I) is well defined for each $n \geq 0$. Furthermore, it is clear that the sequence $\{y_n\}$ is also well defined. □

*Remark 3.3.* From the above proof, it follows that $S^* \subset H_n \cap W_n$ for all $n \geq 0$ under the assumption of Proposition 3.2.

*Definition 3.4.* Fix $\{\Omega_n\}$ and $\{\varepsilon_n\}$ as in (i) and (ii).

(a) A sequence $\{x_n\}$ generated by Algorithm (I) will be called an orbit for GVI $(T, \Omega)$.

(b) An orbit $\{x_n\}$ will be called asymptotically feasible (AF) for GVI $(T, \Omega)$ when all weak accumulation points of $\{x_n\}$ belong to $\Omega$.

A relevant question regarding AF orbits for GVI $(T, \Omega)$ is which extra conditions guarantee optimality of all weak accumulation points. In our analysis, we use the assumption of para- and pseudomonotonicity.

We are now in a position to prove the main theorem in this paper.

THEOREM 3.5. *Let $E$ be a uniformly convex and uniformly smooth Banach space. Let $T : E \to 2^{E^*}$ be a maximal monotone operator with $D(T) \subset \Omega$ and $A : E \to E^*$ monotone and uniformly norm-norm continuous on any bounded subset of $E$ such that $R(J + A + rT) = E^*$ for all $r > 0$. Suppose that ($H_2$) and ($H_3$) hold, and that for an arbitrary $x_0 \in E$, $\{x_n\}$ and $\{y_n\}$ are the sequences generated by Algorithm (I), where $\{r_n\}$ is a positive bounded sequence with $\liminf_{n\to\infty} r_n > 0$, and $\{e_n\}$ is an error sequence of $E^*$ with $\lim_{n\to\infty} \|e_n\| = 0$. If $\{x_n\}$ is an AF orbit for GVI $(T, \Omega)$ such that $\{x_n - y_n\}$ is bounded, then there holds one of the following statements:*

(a) $\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = +\infty$;

(b) *both $\{x_n\}$ and $\{y_n\}$ converge strongly to $Q_{S^*} x_0$.*

*Proof.* It follows from the definition of $W_n$ and Proposition 2.8 that $Q_{W_n} x_0 = x_n$. Further, from $x_{n+1} \in W_n$ and Proposition 2.9, we deduce that

$$\phi(x_{n+1}, Q_{W_n} x_0) + \phi(Q_{W_n} x_0, x_0) \leq \phi(x_{n+1}, x_0), \tag{3.8}$$

and hence

$$\phi(x_{n+1}, x_n) + \phi(x_n, x_0) \leq \phi(x_{n+1}, x_0). \tag{3.9}$$

Since $\phi(x_{n+1}, x_n) \geq 0$ for all $n \geq 0$, from (3.9), we know that $\{\phi(x_n, x_0)\}$ is nondecreasing. Consequently, we have $\lim_{n\to\infty} \phi(x_n, x_0) = +\infty$ or $\lim_{n\to\infty} \phi(x_n, x_0) < +\infty$.

Next, we discuss the two possible cases.

*Case 1.* $\lim_{n\to\infty}\phi(x_n,x_0)=+\infty$. Observe that

$$\phi(x_n,x_0)=\|x_n\|^2-2\langle Jx_0,x_n\rangle+\|x_0\|^2\le(\|x_n\|+\|x_0\|)^2. \tag{3.10}$$

Hence we have $\sqrt{\phi(x_n,x_0)}-\|x_0\|\le\|x_n\|$. This implies that $\lim_{n\to\infty}\|x_n\|=+\infty$. Since $\|x_n\|\le\|x_n-y_n\|+\|y_n\|$, it follows that $\lim_{n\to\infty}\|y_n\|=+\infty$.

*Case 2.* $\lim_{n\to\infty}\phi(x_n,x_0)<+\infty$. In this case, it is clear that $\{\phi(x_n,x_0)\}$ is bounded. Also, it follows from (3.9) that as $n\to\infty$,

$$0\le\phi(x_{n+1},x_n)\le\phi(x_{n+1},x_0)-\phi(x_n,x_0)\longrightarrow 0, \tag{3.11}$$

that is, $\lim_{n\to\infty}\phi(x_{n+1},x_n)=0$. Now, observe that

$$(\|x_n\|-\|x_0\|)^2\le\|x_n\|^2-2\langle Jx_0,x_n\rangle+\|x_0\|^2=\phi(x_n,x_0). \tag{3.12}$$

This shows that $\|x_n\|\le\|x_0\|+\sqrt{\phi(x_n,x_0)}$, and so $\{x_n\}$ is bounded. Thus from Proposition 2.6, we derive $x_{n+1}-x_n\to 0$ as $n\to\infty$.

On the other hand, observe that

$$\begin{aligned}\phi(Q_{H_n}x_n,x_n)-\phi(y_n,x_n)&=\|Q_{H_n}x_n\|^2-\|y_n\|^2+2\langle Jx_n,y_n-Q_{H_n}x_n\rangle\\&\ge 2\langle Jy_n,Q_{H_n}x_n-y_n\rangle+2\langle Jx_n,y_n-Q_{H_n}x_n\rangle\\&=2\langle Jx_n-Jy_n,y_n-Q_{H_n}x_n\rangle.\end{aligned} \tag{3.13}$$

Moreover, utilizing Algorithm (I), we have

$$\langle Jx_n-Jy_n,y_n-Q_{H_n}x_n\rangle\ge\langle r_nv_n+Ay_n-Ax_n+e_n,y_n-Q_{H_n}x_n\rangle-r_n\varepsilon_n. \tag{3.14}$$

Note that from $x_{n+1}\in H_n$, we have $\phi(x_{n+1},x_n)\ge\phi(Q_{H_n}x_n,x_n)$. Thus we deduce that

$$\phi(Q_{H_n}x_n,x_n)\le\phi(x_{n+1},x_n)\longrightarrow 0\quad\text{as } n\longrightarrow\infty. \tag{3.15}$$

Further, it follows from the boundedness of $\{x_n\}$ and Proposition 2.6 that $Q_{H_n}x_n-x_n\to 0$. This immediately implies that $\{Q_{H_n}x_n\}$ is bounded. Since $Q_{H_n}x_n\in H_n$ and $A$ is monotone, it follows from (3.13), (3.14), and the definition of $H_n$ that

$$\begin{aligned}&\phi(x_{n+1},x_n)-\phi(y_n,x_n)\\&\quad\ge\phi(Q_{H_n}x_n,x_n)-\phi(y_n,x_n)\\&\quad\ge 2\langle Jx_n-Jy_n,y_n-Q_{H_n}x_n\rangle\\&\quad\ge 2r_n\langle v_n,y_n-Q_{H_n}x_n\rangle+2\langle Ay_n-Ax_n,y_n-Q_{H_n}x_n\rangle\\&\qquad+2\langle e_n,y_n-Q_{H_n}x_n\rangle-2r_n\varepsilon_n\\&\quad=2r_n\langle v_n,y_n-Q_{H_n}x_n\rangle+2\langle Ay_n-AQ_{H_n}x_n,y_n-Q_{H_n}x_n\rangle\\&\qquad+2\langle AQ_{H_n}x_n-Ax_n,y_n-Q_{H_n}x_n\rangle+2\langle e_n,y_n-Q_{H_n}x_n\rangle-2r_n\varepsilon_n\\&\quad\ge 2\langle AQ_{H_n}x_n-Ax_n,y_n-Q_{H_n}x_n\rangle+2\langle e_n,y_n-Q_{H_n}x_n\rangle-2r_n\varepsilon_n\end{aligned} \tag{3.16}$$

which yields

$$\begin{aligned}\phi(y_n,x_n) \le \phi(x_{n+1},x_n) &- 2\langle e_n, y_n - Q_{H_n}x_n\rangle \\ &- 2\langle AQ_{H_n}x_n - Ax_n, y_n - Q_{H_n}x_n\rangle + 2r_n\varepsilon_n.\end{aligned} \tag{3.17}$$

Note that $\{x_n - y_n\}$ is bounded. Hence $\{y_n\}$ is bounded. Since $\|e_n\| \to 0$, $\{r_n\}$ is bounded and $A$ is uniformly norm-to-norm continuous on any bounded subset of $E$, from (3.17), we derive

$$\begin{aligned}&\limsup_{n\to\infty}\phi(y_n,x_n) \\ &\quad \le \limsup_{n\to\infty}\{\phi(x_{n+1},x_n) - 2\langle e_n, y_n - Q_{H_n}x_n\rangle - 2\langle AQ_{H_n}x_n - Ax_n, y_n - Q_{H_n}x_n\rangle + 2r_n\varepsilon_n\} \\ &\quad \le \limsup_{n\to\infty}\phi(x_{n+1},x_n) + 2\limsup_{n\to\infty}\|e_n\|\,\|y_n - Q_{H_n}x_n\| \\ &\qquad + 2\limsup_{n\to\infty}\|AQ_{H_n}x_n - Ax_n\|\,\|y_n - Q_{H_n}x_n\| + 2\limsup_{n\to\infty} r_n\varepsilon_n = 0,\end{aligned} \tag{3.18}$$

and hence $\lim_{n\to\infty}\phi(y_n,x_n) = 0$. From Proposition 2.6, we obtain $y_n - x_n \to 0$.

To prove the strong convergence of $\{x_n\}$ to $Q_{S^*}x_0$, we will proceed in the following two steps.

Firstly, we claim that $\omega_w(x_n) \subset S^*$ where $\omega_w(x_n)$ denote the weak $\omega$-limit set of $\{x_n\}$, that is,

$$\omega_w(x_n) = \{y \in E : y = \text{weak-}\lim_{i\to\infty} x_{n_i} \text{ for some } n_i \uparrow \infty\}. \tag{3.19}$$

Indeed, since $E$ is reflexive, it follows from the boundedness of $\{x_n\}$ that $\omega_w(x_n) \ne \varnothing$. Let $x^*$ be an arbitrary element of $\omega_w(x_n)$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x^*$ weakly as $i \to \infty$. Note that $y_n - x_n \to 0$ as $n \to \infty$. Hence it is clear that $y_{n_i} \to x^*$ weakly as $i \to \infty$. Since $A, J : E \to E^*$ are uniformly norm-to-norm continuous on any bounded subset of $E$, we conclude that $\|Jy_n - Jx_n\| \to 0$ and $\|Ay_n - Ax_n\| \to 0$ as $n \to \infty$. Note that for each $i$, there exists $v_{n_i} \in Ty_{n_i}$ such that

$$\left\langle v_{n_i} + \frac{1}{r_{n_i}}(Jy_{n_i} - Jx_{n_i}) + \frac{1}{r_{n_i}}(Ay_{n_i} - Ax_{n_i}) + \frac{1}{r_{n_i}}e_{n_i}, y - y_{n_i}\right\rangle \ge -\varepsilon_{n_i} \quad \forall y \in \Omega_{n_i},\ \forall n_{n_i}. \tag{3.20}$$

Then by (i), we have

$$\langle v_{n_i}, y_{n_i} - y\rangle \le \varepsilon_{n_i} + \frac{1}{r_{n_i}}\langle (Jy_{n_i} - Jx_{n_i}) + (Ay_{n_i} - Ax_{n_i}) + e_{n_i}, y - y_{n_i}\rangle \quad \forall y \in \Omega,\ \forall n_i. \tag{3.21}$$

Since $\{x_n\}$ is AF, $x^* \in \Omega$ and hence

$$\langle v_{n_i}, y_{n_i} - x^*\rangle \le \varepsilon_{n_i} + \frac{1}{r_{n_i}}\langle (Jy_{n_i} - Jx_{n_i}) + (Ay_{n_i} - Ax_{n_i}) + e_{n_i}, x^* - y_{n_i}\rangle \quad \forall n_i. \tag{3.22}$$

Using also (ii) and the conditions that $\lim_{n\to\infty}\|e_n\| = 0$ and $\liminf_{n\to\infty} r_n > 0$, we have

$$\begin{aligned}
&\limsup_i \langle v_{n_i}, y_{n_i} - x^*\rangle \\
&\quad\le \limsup_i \varepsilon_{n_i} + \limsup_i \left\{\frac{1}{r_{n_i}}\langle (Jy_{n_i} - Jx_{n_i}) + (Ay_{n_i} - Ax_{n_i}) + e_{n_i}, x^* - y_{n_i}\rangle\right\} \\
&\quad\le \limsup_i \frac{1}{r_{n_i}} \cdot \limsup_i \|(Jy_{n_i} - Jx_{n_i}) + (Ay_{n_i} - Ax_{n_i}) + e_{n_i}\| \, \|x^* - y_{n_i}\| = 0.
\end{aligned} \tag{3.23}$$

Take $\overline{x} \in S^*$. By pseudomonotonicity of $T$, we conclude that there exists $u^* \in T(x^*)$ such that

$$\liminf_i \langle v_{n_i}, y_{n_i} - \overline{x}\rangle \ge \langle u^*, x^* - \overline{x}\rangle. \tag{3.24}$$

Since $\overline{x} \in \Omega$, (3.21) implies that

$$\begin{aligned}
&\liminf_i \langle v_{n_i}, y_{n_i} - \overline{x}\rangle \\
&\quad\le \liminf_i \varepsilon_{n_i} + \liminf_i \frac{1}{r_{n_i}}\langle (Jy_{n_i} - Jx_{n_i}) + (Ay_{n_i} - Ax_{n_i}) + e_{n_i}, \overline{x} - y_{n_i}\rangle \\
&\quad\le \limsup_i \frac{1}{r_{n_i}}\|(Jy_{n_i} - Jx_{n_i}) + (Ay_{n_i} - Ax_{n_i}) + e_{n_i}\| \, \|\overline{x} - y_{n_i}\| = 0.
\end{aligned} \tag{3.25}$$

Combining the last two inequalities, we have that

$$\langle u^*, x^* - \overline{x}\rangle \le 0. \tag{3.26}$$

Finally, by paramonotonicity of $T$ and Proposition 2.2, we conclude that $x^*$ is a solution of the GVI $(T, \Omega)$, that is, $x^* \in S^*$. This shows that $\omega_w(x_n) \subset S^*$.

Secondly, we claim that $x_n \to Q_{S^*}x_0$ as $n \to \infty$. Indeed, set $w^* =: Q_{S^*}x_0$. Let $\{x_{n_i}\}$ be any weakly convergent subsequence of $\{x_n\}$ such that $x_{n_i} \to w$ weakly as $i \to \infty$ for some $w \in \omega_w(x_n)$. According to the above argument, we conclude that $w \in S^*$. Now, from $x_{n+1} = Q_{H_n \cap W_n}x_0$ and $w^* \in S^* \subset H_n \cap W_n$, we have

$$\phi(x_{n+1}, x_0) \le \phi(w^*, x_0). \tag{3.27}$$

Then it is readily seen that

$$\begin{aligned}
\phi(x_n, w^*) &= \phi(x_n, x_0) + \phi(x_0, w^*) - 2\langle Jw^* - Jx_0, x_n - x_0\rangle \\
&\le \phi(w^*, x_0) + \phi(x_0, w^*) - 2\langle Jw^* - Jx_0, x_n - x_0\rangle,
\end{aligned} \tag{3.28}$$

which yields

$$\limsup_{i\to\infty} \phi(x_{n_i}, w^*) \le \phi(w^*, x_0) + \phi(x_0, w^*) - 2\langle Jw^* - Jx_0, w - x_0\rangle. \tag{3.29}$$

From Proposition 2.8, it follows that

$$\begin{aligned}\phi(w^*,x_0)+\phi(x_0,w^*)-2\langle Jw^*-Jx_0,w-x_0\rangle \\ =2(\|w^*\|^2-\langle Jx_0,w^*\rangle-\langle Jw^*,w\rangle+\langle Jx_0,w\rangle) \\ =2\langle Jx_0-Jw^*,w-w^*\rangle\leq 0.\end{aligned} \tag{3.30}$$

Therefore, we obtain $\limsup_{i\to\infty}\phi(x_{n_i},w^*)\leq 0$ and hence $\phi(x_{n_i},w^*)\to 0$ as $i\to\infty$. It follows from Proposition 2.6 that $x_{n_i}\to w^*$ as $i\to\infty$. In view of the arbitrariness of the subsequence $\{x_{n_i}\}$, we know that any weakly convergent subsequence of $\{x_n\}$ converges strongly to $w^*$, and also the whole sequence $\{x_n\}$ converges weakly to $w^*$. Therefore, $\{x_n\}$ converges strongly to $w^*=:Q_{S^*}x_0$. Finally, it follows from $y_n-x_n\to 0$ that $\{y_n\}$ also converges strongly to $Q_{S^*}x_0$. □

Theorem 3.6. *Let $E$ be a uniformly convex and uniformly smooth Banach space, and $T:E\to 2^{E^*}$ a maximal monotone operator with $D(T)\subset\Omega$. Suppose that $(H_2)$ and $(H_3)$ hold and that for an arbitrary $x_0\in E$, $\{x_n\}$ and $\{y_n\}$ are the sequences generated by Algorithm (I) where we have put $A\equiv 0$, $\{r_n\}$ a positive bounded sequence with $\liminf_{n\to\infty}r_n>0$, and $\{e_n\}$ an error sequence of $E^*$ with $\lim_{n\to\infty}\|e_n\|=0$. If $\{x_n\}$ is an AF bounded orbit for $GVI(T,\Omega)$ and $\liminf_{n\to\infty}\langle e_n,y_n\rangle\geq 0$, then both $\{x_n\}$ and $\{y_n\}$ converge strongly to $Q_{S^*}x_0$.*

*Proof.* As in the proof of Theorem 3.5, we can conclude that $\lim_{n\to\infty}\phi(x_n,x_0)=+\infty$ or $\lim_{n\to\infty}\phi(x_n,x_0)<+\infty$. Observe that for each $n\geq 0$,

$$0\leq\phi(x_n,x_0)=\|x_n\|^2-2\langle x_n,Jx_0\rangle+\|x_0\|^2\leq 4\sup_{n\geq 0}\|x_n\|^2<+\infty, \tag{3.31}$$

since $\{x_n\}$ is bounded. This shows that $\{\phi(x_n,x_0)\}$ is bounded. Therefore, $\lim_{n\to\infty}\phi(x_n,x_0)<+\infty$ and hence it follows from (3.9) that $\lim_{n\to\infty}\phi(x_{n+1},x_n)=0$. Also, observe that

$$\begin{aligned}\phi(x_{n+1},x_n)-\phi(y_n,x_n)&=\|x_{n+1}\|^2-\|y_n\|^2+2\langle Jx_n,y_n-x_{n+1}\rangle \\ &\geq 2\langle Jy_n,x_{n+1}-y_n\rangle+2\langle Jx_n,y_n-x_{n+1}\rangle \\ &=2\langle Jx_n-Jy_n,y_n-x_{n+1}\rangle.\end{aligned} \tag{3.32}$$

Moreover, utilizing Algorithm (I), we have

$$\langle Jx_n-Jy_n,y_n-x_{n+1}\rangle\geq\langle r_nv_n+e_n,y_n-x_{n+1}\rangle-r_n\varepsilon_n. \tag{3.33}$$

Since $x_{n+1}\in H_n$, it follows from (3.32), (3.33), and the definition of $H_n$ that

$$\begin{aligned}\phi(x_{n+1},x_n)-\phi(y_n,x_n)&\geq 2\langle Jx_n-Jy_n,y_n-x_{n+1}\rangle \\ &\geq 2r_n\langle v_n,y_n-x_{n+1}\rangle+2\langle e_n,y_n-x_{n+1}\rangle \\ &\geq 2\langle e_n,y_n-x_{n+1}\rangle\end{aligned} \tag{3.34}$$

and hence

$$\begin{aligned}\phi(y_n,x_n) &\le \phi(x_{n+1},x_n) - 2\langle e_n, y_n - x_{n+1}\rangle \\ &\le \phi(x_{n+1},x_n) - 2\langle e_n, y_n\rangle + 2\langle e_n, x_{n+1},\rangle \\ &\le \phi(x_{n+1},x_n) - 2\langle e_n, y_n\rangle + 2\|x_{n+1}\|\,\|e_n\|.\end{aligned} \tag{3.35}$$

Since $\liminf_{n\to\infty}\langle e_n, y_n\rangle \ge 0$ and $\lim_{n\to\infty}\|e_n\| = 0$, from (3.35), we obtain

$$\begin{aligned}\limsup_{n\to\infty}\phi(y_n,x_n) \le \limsup_{n\to\infty}\phi(x_{n+1},x_n) - 2\liminf_{n\to\infty}\langle e_n, y_n\rangle \\ +2\limsup_{n\to\infty}\|x_{n+1}\|\,\|e_n\| \le 0,\end{aligned} \tag{3.36}$$

which implies that $\phi(y_n,x_n) \to 0$ as $n \to \infty$. From Proposition 2.6, we have $y_n - x_n \to 0$ as $n \to \infty$. Obviously, $\{y_n\}$ is bounded. Note that $J : E \to E^*$ is uniformly norm-to-norm continuous on any bounded subset of $E$. Thus we conclude that $Jy_n - Jx_n \to 0$ as $n \to \infty$. The remainder of the proof is similar to that in the proof of Theorem 3.5 which will be omitted. This completes the proof. □

## 4. Application

Finally, we consider an application of Algorithm (I) to the minimization of a convex function.

Fix a sequence $\{\Omega_n\}$ of closed convex subsets of $E$ and a sequence $\{\varepsilon_n\} \subset [0,\infty)$ verifying (i) and (ii). Let $\varphi : E \to (-\infty,\infty]$ be a proper convex lower semicontinuous function. The subdifferential $\partial\varphi$ (see [11]) of $\varphi$ is defined by

$$\partial\varphi(z) = \{v \in E^* : \varphi(y) \ge \varphi(z) + \langle y - z, v\rangle \ \forall y \in E\} \tag{4.1}$$

for all $z \in E$. It is known that whenever $T \equiv \partial\varphi$, the convex optimization problem

$$\min_{x\in\Omega}\varphi(x) \tag{4.2}$$

is equivalent to GVI $(T,\Omega)$. By means of Theorem 3.6, we obtain the following result for finding a minimizer of the function $\varphi$.

Theorem 4.1. *Let $E$ be a uniformly convex and uniformly smooth Banach space, and $T : E \to 2^{E^*}$ the subdifferential $\partial\varphi$ of a proper convex lower semicontinuous function $\varphi : E \to (-\infty,\infty]$ with $D(T) \subset \Omega$. Suppose that $(H_2)$ and $(H_3)$ hold, and that for an arbitrary $x_0 \in E$,*

*$\{x_n\}$ and $\{y_n\}$ are the sequences generated from an arbitrary $x_0 \in E$ by*

$$\begin{gathered}
x_0 \in E,\\
y_n = \arg\min\left\{\varphi(z) + \frac{1}{2r_n}\|z\|^2 - \frac{1}{r_n}\langle Jx_n, z\rangle + \frac{1}{r_n}\langle e_n, z\rangle \,:\, z \in \Omega_n\right\},\\
\exists v_n \in \partial\varphi(y_n) \ \textit{with} \left\langle v_n + \frac{1}{r_n}(Jy_n - Jx_n) + \frac{1}{r_n}\cdot e_n, y - y_n\right\rangle \geq -\varepsilon_n \quad \forall y \in \Omega_n,\\
H_n = \{z \in \Omega_n : \langle z - y_n, v_n\rangle \leq 0\},\\
W_n = \{z \in \Omega_n : \langle z - x_n, Jx_0 - Jx_n\rangle \leq 0\},\\
x_{n+1} = Q_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, \ldots, 1,
\end{gathered} \tag{4.3}$$

*where $\{r_n\}$ is a positive bounded sequence with $\liminf_{n\to\infty} r_n > 0$, and $\{e_n\}$ is an error sequence of $E^*$ with $\lim_{n\to\infty}\|e_n\| = 0$. If $\{x_n\}$ is an AF bounded orbit for GVI $(T,\Omega)$ and $\liminf_{n\to\infty}\langle e_n, y_n\rangle \geq 0$, then both $\{x_n\}$ and $\{y_n\}$ converge strongly to $Q_{S^*}x_0$, where $S^*$ is the set of all minimizers of $\varphi$.*

*Proof.* Since $\varphi : E \to (-\infty, \infty]$ is a proper convex lower semicontinuous function, by Rockafellar [11], the subdifferential $\partial\varphi$ of $\varphi$ is a maximal monotone operator. Also, it is known that

$$y_n = \arg\min\left\{\varphi(z) + \frac{1}{2r_n}\|z\|^2 - \frac{1}{r_n}\langle Jx_n, z\rangle + \frac{1}{r_n}\langle e_n, z\rangle : z \in \Omega_n\right\} \tag{4.4}$$

implies that

$$\exists v_n \in \partial\varphi(y_n) \quad \text{with} \left\langle v_n + \frac{1}{r_n}(Jy_n - Jx_n) + \frac{1}{r_n}\cdot e_n, y - y_n\right\rangle \geq -\varepsilon_n \quad \forall y \in \Omega_n. \tag{4.5}$$

Thus the conclusion now follows from Theorem 3.6. □

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