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Research Article

Certain Integral Operators on the Classes $\mathcal{M}(\beta_i)$ and $\mathcal{N}(\beta_i)$

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We consider the classes $\mathcal{M}(\beta_i)$ and $\mathcal{N}(\beta_i)$ of the analytic functions and two general integral operators. We prove some properties for these operators on these classes.

1. Introduction

Let $\mathbf{U} = \{z \in \mathbf{C},\ |z| < 1\}$ be the open unit disk and let $\mathcal{A}$ denote the class of the functions $f(z)$ of the form

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots, \quad z \in \mathbf{U}, \tag{1.1}$$

which are analytic in the open disk $\mathbf{U}$.

Let $\mathcal{M}(\beta)$ be the subclass of $\mathcal{A}$, consisting of the functions $f(z)$, which satisfy the inequality

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \beta, \quad z \in \mathbf{U},\ \beta > 1, \tag{1.2}$$

and let $\mathcal{N}(\beta)$ be the subclass of $\mathcal{A}$, consisting of functions $f(z)$, which satisfy the inequality

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} < \beta, \quad z \in \mathbf{U}. \tag{1.3}$$

These classes are studied by Uralegaddi et al. in [1], and Owa and Srivastava in [2].

Consider the integral operator F_n introduced by D. Breaz and N. Breaz in [3], having the form

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdots \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt, \tag{1.4}$$

where $f_i(z) \in \mathcal{A}$ and $\alpha_i > 0$, for all $i \in \{1, \ldots, n\}$.

Remark 1.1. This operator extends the integral operator of Alexander given by $F(z) = \int_0^z (f(t)/t)dt$.

Also, we consider the next integral operator denoted by $F_{\alpha_1,\ldots,\alpha_n}$ that was introduced by Breaz et al. in [4], having the form

$$F_{\alpha_1,\ldots,\alpha_n}(z) = \int_0^z \left[f_1'(t)\right]^{\alpha_1} \cdots \left[f_n'(t)\right]^{\alpha_n} dt, \tag{1.5}$$

where $f_i(z) \in \mathcal{A}$ and $\alpha_i > 0$ for all $i \in \{1,\ldots,n\}$.

It is easy to see that these integral operators are analytic operators.

2. Main results

Theorem 2.1. *Let $f_i \in \mathcal{M}(\beta_i)$, for each $i = 1,2,3,\ldots,n$ with $\beta_i > 1$. Then $F_n(z) \in \mathcal{N}(\mu)$ with $\mu = 1 + \sum_{i=1}^n \alpha_i(\beta_i - 1)$ and $\alpha_i > 0$, $(i = 1,2,3,\ldots,n)$.*

Proof. After some calculi, we obtain that

$$\frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^n \alpha_i \frac{zf_i'(z)}{f_i(z)} - \sum_{i=1}^n \alpha_i. \tag{2.1}$$

The relation (2.1) is equivalent to

$$\operatorname{Re}\left(\frac{zF_n''(z)}{F_n'(z)} + 1\right) = \sum_{i=1}^n \alpha_i \operatorname{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right) - \sum_{i=1}^n \alpha_i + 1. \tag{2.2}$$

Since $f_i \in \mathcal{M}(\beta_i)$, we have

$$\operatorname{Re}\left(\frac{zF_n''(z)}{F_n'(z)} + 1\right) < \sum_{i=1}^n \alpha_i\beta_i - \sum_{i=1}^n \alpha_i + 1 = \sum_{i=1}^n \alpha_i(\beta_i - 1) + 1. \tag{2.3}$$

Because $\sum_{i=1}^n \alpha_i(\beta_i - 1) > 0$, we obtain that $F_n \in \mathcal{N}(\mu)$, where $\mu = 1 + \sum_{i=1}^n \alpha_i(\beta_i - 1)$. □

Corollary 2.2. *Let $f_i \in \mathcal{M}(\beta)$ for each $i = 1,2,3,\ldots,n$ with $\beta > 1$. Then $F_n(z) \in \mathcal{N}(\gamma)$ with $\gamma = 1 + (\beta - 1)\sum_{i=1}^n \alpha_i$ and $\alpha_i > 0$, $(i = 1,2,3,\ldots,n)$.*

Proof. In Theorem 2.1, we consider $\beta_1 = \beta_2 = \cdots = \beta_n = \beta$. □

Corollary 2.3. *Let $f \in \mathcal{M}(\beta)$ with $\beta > 1$. Then the integral operator $F(z) = \int_0^z (f(t)/t)^\alpha dt \in \mathcal{N}(\delta)$ with $\delta = \alpha(\beta - 1) + 1$ and $\alpha > 0$.*

Proof. In Corollary 2.2, we consider $n = 1$ and $\alpha_1 = \alpha$. □

Corollary 2.4. *Let $f \in \mathcal{M}(\beta)$ with $\beta > 1$. Then the integral operator of Alexander $F(z) = \int_0^z (f(t)/t)dt \in \mathcal{N}(\beta)$.*

Proof. We have

$$\frac{zF''(z)}{F'(z)} = \frac{zf'(z)}{f(z)} - 1. \tag{2.4}$$

From (2.4), we have

$$\operatorname{Re}\left(\frac{zF''(z)}{F'(z)} + 1\right) = \operatorname{Re}\frac{zf'(z)}{f(z)} < \beta. \tag{2.5}$$

So relation (2.5) implies that Alexander operator is in $\mathcal{N}(\beta)$. □

Theorem 2.5. *Let $f_i \in \mathcal{N}(\beta_i)$ for each $i = 1, 2, 3, \ldots, n$, with $\beta_i > 1$. Then $F_{\alpha_1,\ldots,\alpha_n}(z) \in \mathcal{N}(\rho)$ with $\rho = 1 + \sum_{i=1}^{n} \alpha_i(\beta_i - 1)$ and $\alpha_i > 0$, $(i = 1, 2, 3, \ldots, n)$.*

Proof. After some calculi, we have

$$\frac{zF''_{\alpha_1,\ldots,\alpha_n}(z)}{F'_{\alpha_1,\ldots,\alpha_n}(z)} = \alpha_1 \frac{zf''_1(z)}{f'_1(z)} + \cdots + \alpha_n \frac{zf''_n(z)}{f'_n(z)} \tag{2.6}$$

that is equivalent to

$$\frac{zF''_{\alpha_1,\ldots,\alpha_n}(z)}{F'_{\alpha_1,\ldots,\alpha_n}(z)} + 1 = \alpha_1\left(\frac{zf''_1(z)}{f'_1(z)} + 1\right) + \cdots + \alpha_n\left(\frac{zf''_n(z)}{f'_n(z)} + 1\right) - \sum_{i=1}^{n} \alpha_i + 1. \tag{2.7}$$

Since $f_i \in \mathcal{N}(\beta_i)$, for all $i \in \{1, \ldots, n\}$, we have

$$\operatorname{Re}\left(\frac{zf''_n(z)}{f'_n(z)} + 1\right) < \beta_i. \tag{2.8}$$

So we obtain

$$\operatorname{Re}\left(\frac{zF''_{\alpha_1,\ldots,\alpha_n}(z)}{F'_{\alpha_1,\ldots,\alpha_n}(z)} + 1\right) < \sum_{i=1}^{n} \alpha_i\beta_i - \sum_{i=1}^{n} \alpha_i + 1 = \sum_{i=1}^{n} \alpha_i(\beta_i - 1) + 1 \tag{2.9}$$

which implies that $F_{\alpha_1,\ldots,\alpha_n} \in \mathcal{N}(\rho)$, where $\rho = 1 + \sum_{i=1}^{n} \alpha_i(\beta_i - 1)$. □

Corollary 2.6. *Let $f_i \in \mathcal{N}(\beta)$ for each $i = 1, 2, 3, \ldots, n$ with $\beta > 1$. Then $F_{\alpha_1,\ldots,\alpha_n}(z) \in \mathcal{N}(\eta)$ with $\eta = 1 + \sum_{i=1}^{n} \alpha_i(\beta - 1)$ and $\alpha_i > 0$, $(i = 1, 2, 3, \ldots, n)$.*

Proof. In Thorem 2.5, we consider $\beta_1 = \beta_2 = \cdots = \beta_n = \beta$. □

Corollary 2.7. *Let $f \in \mathcal{N}(\beta)$ with $\beta > 1$. Then the integral operator*

$$F_\alpha(z) = \int_0^z \left[f'(t)\right]^\alpha dt \tag{2.10}$$

is in the class $\mathcal{N}(\alpha(\beta - 1) + 1)$ and $\alpha > 0$.

Proof. We have

$$\frac{zF_\alpha''(z)}{F_\alpha'(z)} = \alpha\frac{zf''(z)}{f'(z)}. \tag{2.11}$$

From (2.11) we have

$$\mathrm{Re}\left(\frac{zF_\alpha''(z)}{F_\alpha'(z)}+1\right) = \alpha\,\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}+1\right)+1-\alpha < \alpha\beta+1-\alpha = \alpha(\beta-1)+1. \tag{2.12}$$

So the relation (2.12) implies that the operator F_α is in $\mathcal{N}(\alpha(\beta-1)+1)$. □

Example 2.8. Let $f(z) = (1/(2\beta-1))\{1-(1-z)^{2\beta-1}\} \in \mathcal{N}(\beta)$. After some calculi, we obtain that

$$F_\alpha(z) = \int_0^z \left[f'(t)\right]^\alpha dt = \frac{1}{2\alpha(1-\beta)-1}(1-z)^{2\alpha(\beta-1)+1} \in \mathcal{N}\left(\alpha(\beta-1)+1\right). \tag{2.13}$$

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