Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 124953, 16 pages
doi:10.1155/2009/124953

Research Article

Generalized Strongly Nonlinear Implicit Quasivariational Inequalities

Salahuddin and M. K. Ahmad

Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India

Correspondence should be addressed to Salahuddin, salahuddin12@mailcity.com

Received 11 February 2009; Accepted 17 June 2009

Recommended by Ram U. Verma

We prove an existence theorem for solution of generalized strongly nonlinear implicit quasivariational inequality problems and convergence of iterative sequences with errors, involving Lipschitz continuous, generalized pseudocontractive and generalized g-pseudocontractive mappings in Hilbert spaces.

1. Introduction

Variational inequality was initially studied by Stampacchia [1] in 1964. Since then, it has been extensively studied because of its crucial role in the study of mechanics, physics, economics, transportation and engineering sciences, and optimization and control. Thanks to its wide applications, the classical variational inequality has been well studied and generalized in various directions. For details, readers are referred to [2–5] and the references therein.

It is known that one of the most important and difficult problems in variational inequality theory is the development of an efficient and implementable approximation schemes for solving various classes of variational inequalities and variational inclusions. Recently, Huang [6–8] and Cho et al. [9] constructed some new perturbed iterative algorithms for approximation of solutions of some generalized nonlinear implicit quasi-variational inclusions (inequalities), which include many iterative algorithms for variational and quasi-variational inclusions (inequalities) as special cases. Inspired and motivated by recent research works [1, 9–19], we prove an existence theorem for solution of generalized strongly nonlinear implicit quasi-variational inequality problems and convergence of iterative sequences with errors, involving Lipschitzian, generalized pseudocontractivity and generalized g-pseudocontractive mappings in Hilbert spaces.

2. Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. For a nonempty closed convex subset $K \subset H$, let P_K be the projection of H onto K. Let $K : H \to 2^H$ be a set valued mapping with nonempty closed convex values, $F, g, G, A : H \to H$ and $N : H\times H\times H \to H$ be the mappings. We consider the following problem.

Find $x \in H$, such that $g(x) \in K(x)$ and

$$\left\langle g(x) - N(Ax, Gx, Fx), y - g(x)\right\rangle \geq 0, \quad \forall y \in K(x). \tag{2.1}$$

The problem (2.1) is called the *generalized strongly nonlinear implicit quasi-variational inequality problem*.

Special Cases

(i) If $K(x) = m(x) + K$, for all $x \in H$, where K is a nonempty closed convex subset of H and $m : H \to H$ is a mapping, then the problem (2.1) is equivalent to finding $x \in H$ such that $g(x) - m(x) \in K$ and

$$\left\langle g(x) - N(Ax, Gx, Fx), y - g(x)\right\rangle \geq 0, \quad \forall y \in K + m(x), \tag{2.2}$$

the problem (2.2) is called generalized nonlinear quasi-variational inequality problem.

(ii) If we assume A, G, F as identity mappings, then (2.1) reduces to the problem of finding $x \in H$ such that $g(x) \in K(x)$ and

$$\left\langle g(x) - N(x, x, x), y - g(x)\right\rangle \geq 0, \quad \forall y \in K(x), \tag{2.3}$$

which is known as general implicit nonlinear quasi-variational inequality problem.

(iii) If we assume $N(x, x, x) = N(x, x)$, then (2.3) reduces to the following problem of finding $x \in H$ such that $g(x) \in K(x)$ and

$$\left\langle g(x) - N(x, x), y - g(x)\right\rangle \geq 0, \quad \forall y \in K(x), \tag{2.4}$$

which is known as generalized implicit nonlinear quasi-variational inequality problem, a variant form as can be seen in [20, equation (2.6)].

(iv) If we assume $g(x) - N(x, x) = x - N(x, x)$, then (2.4) reduces to the following problem of finding $x \in H$ such that $g(x) \in K(x)$ and

$$\left\langle x - N(x, x), y - g(x)\right\rangle \geq 0, \quad \forall y \in K(x). \tag{2.5}$$

The problem (2.5) is called the generalized strongly nonlinear implicit quasi-variational inequality problem, considered and studied by Cho et al. [9].

(v) If $g \equiv I$, I an identity mapping, then (2.5) is equivalent to finding $x \in K(x)$ such that

$$\langle x - N(x, x), y - x\rangle \geq 0, \quad \forall y \in K(x). \tag{2.6}$$

Problem (2.6) is called generalized strongly nonlinear quasi-variational inequality problem, see special cases of Cho et al. [9].

(vi) If $K(x) = K$, K a nonempty closed convex subset of H and $N(x,x) = Tx$ for all $x \in H$, where $T : H \to H$ a nonlinear mapping, then the problem (2.6) is equivalent to finding $x \in H$ such that

$$\langle x - T(x), y - x \rangle \geq 0, \quad \forall y \in K, \tag{2.7}$$

which is a nonlinear variational inequality, considered by Verma [17].

(vii) If $x - Tx = Tx$, for all $x \in H$, then (2.7) reduces to the following problem for finding $x \in H$ such that

$$\langle Tx, y - x \rangle \geq 0, \quad \forall y \in K, \tag{2.8}$$

which is a classical variational inequality considered by [1, 4, 5].

Now, we recall the following iterative process due to Ishikawa [13], Mann [14], Noor [15] and Liu [21].

(1) Let K be a nonempty convex subset of H and $T : K \to X$ a mapping. The sequence $\{x_n\}$, defined by

$$x_0 \in K, \tag{2.9}$$

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n \\ y_n &= (1 - \beta_n)x_n + \beta_n T z_n \\ z_n &= (1 - \gamma_n)x_n + \gamma_n T x_n, \end{aligned} \tag{2.10}$$

$n \geq 0$, is called the three-step iterative process, where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three real sequences in [0,1] satisfying some conditions.

(2) In particular, if $\gamma_n = 0$ for all $n \geq 0$, then $\{x_n\}$, defined by

$$\begin{aligned} &x_0 \in K, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n, \end{aligned} \tag{2.11}$$

$n \geq 0$, is called the Ishikawa iterative process, where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in [0,1] satisfying some conditions.

(3) In particular, if $\beta_n = 0$ for all $n \geq 0$, then $\{x_n\}$ defined by

$$\begin{aligned} &x_0 \in K \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T x_n, \end{aligned} \tag{2.12}$$

for $n \geq 0$, is called the Mann iterative process.

Recently Liu [21] introduced the concept of three-step iterative process with errors which is the generalization of Ishikawa [13] and Mann [14] iterative process, for nonlinear strongly accretive mappings as follows.

(4) For a nonempty subset K of a Banach spaces X and a mapping $T : K \to X$, the sequence $\{x_n\}$, defined by

$$\begin{aligned} &x_0 \in K, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n + u_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T z_n + v_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n T x_n + w_n, \end{aligned} \tag{2.13}$$

$n \geq 0$, is called the three-step iterative process with errors. Here $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ are three summable sequences in X (i.e., $\sum_{n=0}^{\infty} \|u_n\| < +\infty$, $\sum_{n=0}^{\infty} \|v_n\| < +\infty$ and $\sum_{n=0}^{\infty} \|w_n\| < +\infty$), and $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in [0,1] satisfying certain restrictions.

(5) In particular, if $\gamma_n = 0$ for $n \geq 0$ and $w_n = 0$. The sequence $\{x_n\}$ defined by

$$\begin{aligned} &x_0 \in K, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n + u_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T z_n + v_n, \end{aligned} \tag{2.14}$$

$n = 0, 1, 2, \ldots$, is called the Ishikawa iterative process with errors. Here $\{u_n\}$ and $\{v_n\}$ are two summable sequences in X (i.e., $\sum_{n=0}^{\infty} \|u_n\| < +\infty$ and $\sum_{n=0}^{\infty} \|v_n\| < +\infty$); $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in [0,1] satisfying certain restrictions.

(6) In particular, if $\beta_n = 0$ and $v_n = 0$ for all $n \geq 0$. The sequence $\{x_n\}$, defined by

$$x_0 \in K, \tag{2.15}$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + u_n \tag{2.16}$$

for $n = 0, 1, 2, \ldots$, is called the Mann iterative process with errors, where $\{u_n\}$ is a summable sequence in X and $\{\alpha_n\}$ a sequence in [0,1] satisfying certain restrictions.

However, in a recent paper [19] Xu pointed out that the definitions of Liu [21] are against the randomness of the errors and revised the definitions of Liu [21] as follows.

(7) Let K be a nonempty convex subset of a Banach space X and $T : K \to X$ a mapping. For any given $x_0 \in K$, the sequence $\{x_n\}$, defined by

$$x_0 \in K, \tag{2.17}$$

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \\ y_n &= \widehat{\alpha} x_n + \widehat{\beta} T z_n + \widehat{\gamma}_n v_n, \\ z_n &= \overline{\alpha} x_n + \overline{\beta}_n T x_n + \overline{\gamma}_n w_n \end{aligned} \tag{2.18}$$

for $n = 0, 1, 2, \ldots$, is called the three-step iterative process with errors, where $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ are three bounded sequences in K and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\overline{\alpha}_n\}$, $\{\overline{\beta}_n\}$, $\{\overline{\gamma}_n\}$, $\{\widehat{\alpha}_n\}$, $\{\widehat{\beta}_n\}$, and $\{\widehat{\gamma}_n\}$ are nine sequences in [0,1] satisfying the conditions

$$\alpha_n + \beta_n + \gamma_n = 1, \quad \overline{\alpha}_n + \overline{\beta}_n + \overline{\gamma}_n = 1, \quad \widehat{\alpha}_n + \widehat{\beta}_n + \widehat{\gamma}_n = 1 \quad \text{for } n \geq 0. \tag{2.19}$$

(8) If $\overline{\beta}_n = \overline{\gamma}_n = 0$ for $n = 0, 1, 2 \ldots$ the sequence $\{x_n\}$, defined by

$$\begin{gathered} x_0 \in K, \\ x_{n+1} = \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \\ y_n = \widehat{\alpha} x_n + \widehat{\beta} T x_n + \widehat{\gamma}_n v_n \end{gathered} \tag{2.20}$$

for $n = 0, 1, 2, \ldots$, is called the Ishikawa iterative process with errors, where $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\widehat{\alpha}_n\}$, $\{\widehat{\beta}_n\}$, and $\{\widehat{\gamma}_n\}$ are six sequences in [0,1] satisfying the conditions

$$\alpha_n + \beta_n + \gamma_n = 1, \quad \widehat{\alpha}_n + \widehat{\beta}_n + \widehat{\gamma}_n = 1 \quad \text{for } n \geq 0. \tag{2.21}$$

(9) If $\widehat{\beta}_n = \widehat{\gamma}_n = 0$ for $n = 0, 1, 2 \ldots$, the sequence $\{x_n\}$ defined by

$$\begin{gathered} x_0 \in K, \\ x_{n+1} = \alpha_n x_n + \beta_n T x_n + \gamma_n u_n, \end{gathered} \tag{2.22}$$

for $n = 0, 1, 2, \ldots$, is called the Mann iterative process with errors.

For our main results, we need the following lemmas.

Lemma 2.1 (see [3]). *If $K \subset H$ is a closed convex subset and $x \in H$ a given point, then $z \in K$ satisfies the inequality*

$$\langle x - z, y - x \rangle \geq 0, \quad \forall y \in K, \tag{2.23}$$

if and only if

$$x = P_K(z), \tag{2.24}$$

where P_K is the projection of H onto K.

Lemma 2.2 (see [10]). *The mapping P_K defined by* (2.24) *is nonexpansive, that is,*

$$\|P_K(u) - P_K(v)\| \leq \|u - v\|, \quad \forall u, v \in H. \tag{2.25}$$

Lemma 2.3 (see [10]). *If $K(u) = m(u) + K$ and $K \subset H$ is a closed convex subset, then for any $u, v \in H$, one has*

$$P_{K(u)}(v) = m(u) + P_K(v - m(u)). \tag{2.26}$$

Lemma 2.4 (see [21]). *Let a_n, b_n and c_n be three nonnegative real sequences satisfying*

$$\begin{gathered} a_{n+1} = (1 - t_n)a_n + b_n + c_n, \quad \text{for } n \geq 0, \\ t_n \in [0,1], \quad \sum_{n=0}^{\infty} t_n = \infty, \quad b_n = O(t_n), \sum_{n=0}^{+\infty} c_n < +\infty. \end{gathered} \tag{2.27}$$

Then

$$\lim_{n \to \infty} a_n = 0. \tag{2.28}$$

By Lemma 2.1, we know that the generalized strongly nonlinear implicit quasi-variational inequality (2.1) has a unique solution if and only if the mapping $Q : H \to H$ by

$$Q(x) = x - g(x) + P_{K(x)}\left[g(x) - t\left(g(x) - N(Ax, Gx, Fx)\right)\right] \tag{2.29}$$

has a unique fixed point, where $t > 0$ is a constant.

3. Main Results

In this section, we establish an existence theorem for solution of generalized strongly nonlinear implicit quasi-variational inequality problems and convergence of the iterative sequences generated by (2.18). First, we give some definitions.

Definition 3.1. A mapping $T : H \to H$ is said to be generalized pseudo-contractive if there exists a constant $r > 0$ such that

$$\|Tx - Ty\|^2 \leq r^2\|x - y\|^2 + \|Tx - Ty - r(x - y)\|^2, \quad \forall x, y \in H. \tag{3.1}$$

It is easy to check that (3.1) is equivalent to

$$\langle Tx - Ty, x - y \rangle \leq r\|x - y\|^2. \tag{3.2}$$

For $r = 1$ in (3.1), we get the usual concept of pseudo-contractive of T, introduced by Browder and Petryshyn [10], that is,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - Ty - (x - y)\|^2. \tag{3.3}$$

Definition 3.2. Let $A : H \to H$ and $N : H \times H \times H \to H$ be the mappings. The mapping N is said to be as follows.

(i) Generalized pseudo-contractive with respect to A in the first argument of N, if there exists a constant $p > 0$ such that

$$\langle N(Ax, z, z) - N(Ay, z, z), x - y\rangle \le p\|x - y\|^2 \quad \forall x, y, z \in H. \tag{3.4}$$

(ii) Lipschitz continuous with respect to the first argument of N if there exists a constant $s > 0$ such that

$$\|N(x, z, z) - N(y, z, z)\| \le s\|x - y\| \quad \forall x, y, z \in H. \tag{3.5}$$

In a similar way, we can define Lipschitz continuity of N with respect to the second and third arguments.

(iii) A is also said to be Lipschitz continuous if there exists a constant $\eta > 0$ such that

$$\|Ax - Ay\| \le \eta\|x - y\| \quad \forall x, y \in H. \tag{3.6}$$

Definition 3.3. Let $g, G : H \to H$ be the mappings. A mapping $N : H \times H \times H \to H$ is said to be the generalized g-pseudo-contractive with respect to the second argument of N, if there exists a constant $q > 0$ such that

$$\langle N(z, Gx, z) - N(z, Gy, z), g(x) - g(y)\rangle \le q\|g(x) - g(y)\|^2 \quad \forall x, y, z \in H. \tag{3.7}$$

Definition 3.4. Let $K : H \to 2^H$ be a set-valued mapping such that for each $x \in H$, $K(x)$ is a nonempty closed convex subset of H. The projection $P_{K(x)}$ is said to be Lipschitz continuous if there exists a constant $\xi > 0$ such that

$$\|P_{K(x)}(z) - P_{K(y)}(z)\| \le \xi\|x - y\|, \quad \forall x, y, z \in H. \tag{3.8}$$

Remark 3.5. In many important applications, $K(u)$ has the following form:

$$K(x) = m(x) + K, \tag{3.9}$$

where $m : H \to H$ is a single-valued mapping and K a nonempty closed convex subset of H. If m is Lipschitz continuous with constant $\chi > 0$, then from Lemma 2.3, $P_{K(x)}$ is Lipschitz continuous with Lipschitz constant $\xi = 2\chi$.

Now, we give the main result of this paper.

Theorem 3.6. *Let H be a real Hilbert space and $K : H \to 2^H$ a set-valued mapping with nonempty closed convex values. Let $A, G, F : H \to H$ be the Lipschitz continuous mappings with positive constants η, σ, and d, respectively. Let $g : H \to H$ be the mapping such that $I - g$ and g are Lipschitz* *continuous with positive constants λ and μ, respectively. A trimapping $N : H \times H \times H \rightarrow H$ is generalized pseudo-contractive with respect to A in the first argument of N with constant $p > 0$ and generalized g-pseudo-contractive with respect to G in the second argument of N with constant $q > 0$, Lipschitz continuous with respect to the first, second, and third arguments with positive constants s, δ, ζ, respectively. Suppose that $P_{K(x)}$ is Lipschitz continuous with constant $\xi > 0$. Let $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ be the three bounded sequences in H and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\overline{\alpha}_n\}$, $\{\overline{\beta}_n\}$, $\{\overline{\gamma}_n\}$, $\{\widehat{\alpha}_n\}$, $\{\widehat{\beta}_n\}$, and $\{\widehat{\gamma}_n\}$ are sequences in $[0,1]$ satisfying the following conditions:*

(1) $\alpha_n + \beta_n + \gamma_n = \overline{\alpha}_n + \overline{\beta}_n + \overline{\gamma}_n = \widehat{\alpha}_n + \widehat{\beta}_n + \widehat{\gamma}_n = 1,\ n \geq 0$,

(2) $\lim_{n\to\infty}\gamma_n = \lim_{n\to\infty}\widehat{\gamma}_n = \lim_{n\to\infty}\overline{\gamma}_n = 0$,

(3) $\sum_{n=0}^{+\infty}\beta_n = \infty,\ \sum_{n=0}^{+\infty}\gamma_n < +\infty$.

If the following conditions hold:

$$\left|t - \frac{h\Omega - p - h}{s^2\eta^2 - h^2}\right| < \frac{\sqrt{(4\Omega - p - h)^2 - (s^2\eta^2 - h^2)\Omega(2-\Omega)}}{s^2\eta^2 - h^2},$$
$$h\Omega > p + h + \sqrt{(s\eta - h)(s\eta + h)\Omega(2-\Omega)}, \quad h\Omega > p + h,\ h < s\eta \tag{3.10}$$

where $\Omega = 2\lambda + \xi$, $h = \varphi + \zeta d$, and $\theta = \Omega + \sqrt{1 + 2pt + t^2 s^2\eta^2} + th < 1$.

Then there exists a unique $x \in H$ satisfying the generalized strongly nonlinear implicit quasi-variational inequality (2.1) *and $x_n \rightarrow x$ as $n \rightarrow \infty$, where $\{x_n\}$ is the three-step iteration process with errors defined as follows:*

$$\begin{gathered} x_0 \in H, \\ x_{n+1} = \alpha_n x_n + \beta_n\{y_n - g(y_n) + P_{K(y_n)}[g(y_n) - t(g(y_n) - N(Ay_n, Gy_n, Fy_n))]\} + \gamma_n u_n, \\ y_n = \widehat{\alpha}_n x_n + \widehat{\beta}_n\{z_n - g(z_n) + P_{K(z_n)}[g(z_n) - t(g(z_n) - N(Az_n, Gz_n, Fz_n))]\} + \widehat{\gamma}_n v_n, \\ z_n = \overline{\alpha}_n x_n + \overline{\beta}_n\{x_n - g(x_n) + P_{K(x_n)}[g(x_n) - t(g(x_n) - N(Ax_n, Gx_n, Fx_n))]\} + \overline{\gamma}_n w_n \end{gathered} \tag{3.11}$$

for $n = 0, 1, 2, \ldots$.

Proof. We first prove that the generalized strongly nonlinear implicit quasi-variational inequality (2.1) has a unique solution. By Lemma 2.1, it is sufficient to prove the mapping defined by

$$Q(x) = x - g(x) + P_{K(x)}[g(x) - t(g(x) - N(Ax, Gx, Fx))] \tag{3.12}$$

has a unique fixed point in H.

Let $x,\ y$ be two arbitrary points in H. From Lemma 2.2 and Lipschitz continuity of $P_{K(u)}$ and $I-g$, we have

$$\begin{aligned}
&\|Q(x)-Q(y)\| \\
&= \big\|x-g(x)+P_{K(x)}[g(x)-t(g(x)-N(Ax,Gx,Fx))] \\
&\qquad -y+g(y)-P_{K(y)}[g(y)-t(g(y)-N(Ay,Gy,Fy))]\big\| \\
&\leq \big\|x-g(x)-(y-g(y))\big\| \\
&\quad + \big\|P_{K(x)}[g(x)-t(g(x)-N(Ax,Gx,Fx))]-P_{K(x)}[g(y)-t(g(y)-N(Ay,Gy,Fy))]\big\| \\
&\quad + \big\|P_{K(x)}[g(y)-t(g(y)-N(Ay,Gy,Fy))]-P_{K(y)}[g(y)-t(g(y)-N(Ay,Gy,Fy))]\big\| \\
&\leq 2\big\|x-g(x)-(y-g(y))\big\| + \big\|x-y+t(N(Ax,Gx,Fx)-N(Ay,Gx,Fx))\big\| \\
&\quad + t\|g(x)-g(y)-(N(Ay,Gx,Fx)-N(Ay,Gy,Fx))\| \\
&\quad + t\big\|N(Ay,Gy,Fx)-N(Ay,Gy,Fy)\big\| + \xi\|x-y\| \\
&\leq 2\lambda\|x-y\| + \|x-y+t(N(Ax,Gx,Fx)-N(Ay,Gx,Fx))\| \\
&\quad + t\|g(x)-g(y)-(N(Ay,Gx,Fx)-N(Ay,Gy,Fx))\| \\
&\quad + t\|N(Ay,Gy,Fx)-N(Ay,Gy,Fy)\| + \xi\|x-y\| \\
&\leq (2\lambda+\xi)\|x-y\| + \|x-y+t(N(Ax,Gx,Fx)-N(Ay,Gx,Fx))\| \\
&\quad + t\|g(x)-g(y)-(N(Ay,Gx,Fx)-N(Ay,Gy,Fx))\| \\
&\quad + t\|N(Ay,Gy,Fx)-N(Ay,Gy,Fy)\|.
\end{aligned} \tag{3.13}$$

Since N is generalized pseudo-contractive with respect to A in the first argument of N and Lipschitz continuous with respect to first argument of N and also A is Lipschitz continuous, we have

$$\begin{aligned}
&\|x-y+t(N(Ax,Gx,Fx)-N(Ay,Gx,Fx))\|^2 \\
&\qquad = \|x-y\|^2 + 2t\langle x-y, N(Ax,Gx,Fx)-N(Ay,Gx,Fx)\rangle \\
&\qquad\quad + t^2\|N(Ax,Gx,Fx)-N(Ay,Gx,Fx)\|^2 \\
&\qquad \leq \|x-y\|^2 + 2tp\|x-y\|^2 + t^2s^2\|Ax-Ay\|^2 \\
&\qquad \leq \|x-y\|^2 + 2tp\|x-y\|^2 + t^2s^2\eta^2\|x-y\|^2 \\
&\qquad \leq \left(1+2tp+t^2s^2\eta^2\right)\|x-y\|^2.
\end{aligned} \tag{3.14}$$

Again since N is generalized g-pseudo-contractive with respect to G in the second argument of N and Lipschitz continuous with respect to second argument of N and G is Lipschitz continuous, we have

$$\begin{aligned}
&\|g(x) - g(y) - (N(Ay, Gx, Fx) - N(Ay, Gy, Fx))\|^2 \\
&\quad = \|g(x) - g(y)\|^2 - 2\langle g(x) - g(y), N(Ay, Gx, Fx) - N(Ay, Gy, Fx)\rangle \\
&\qquad + \|N(Ay, Gx, Fx) - N(Ay, Gy, Fx)\|^2 \\
&\quad \le \mu^2\|x - y\|^2 - 2q\|g(x) - g(y)\|^2 + \delta^2\|Gx - Gy\|^2 \\
&\quad \le \mu^2\|x - y\|^2 - 2q\mu^2\|x - y\|^2 + \delta^2\sigma^2\|x - y\|^2 \\
&\quad \le \left(\mu^2(1 - 2q) + \delta^2\sigma^2\right)\|x - y\|^2,
\end{aligned} \tag{3.15}$$

$$\|N(Ay, Gy, Fx) - N(Ay, Gy, Fy)\| \le \zeta\|Fx - Fy\| \le \zeta d\|x - y\|. \tag{3.16}$$

It follows from (3.13)–(3.16) that

$$\|Q(x) - Q(y)\| \le \theta\|x - y\|, \quad \forall x, y \in H, \tag{3.17}$$

where

$$\begin{aligned}
\theta &= \Omega + \sqrt{1 + 2pt + t^2s^2\eta^2} + th, \\
\varphi &= \sqrt{\mu^2(1 - 2q) + \delta^2\sigma^2}, \\
h &= \varphi + \zeta d.
\end{aligned} \tag{3.18}$$

From (3.10), we know that $0 < \theta < 1$ and so Q has a unique fixed point $x \in H$, which is a unique solution of the generalized strongly nonlinear implicit quasi-variational inequality (2.1).

Now we prove that $\{x_n\}$ converges to x. In fact, it follows from (3.11) and $x \in Q(x)$ that

$$\begin{aligned}
&\|x_{n+1} - x\| \\
&\quad = \left\|\alpha_n x_n + \beta_n[y_n - g(y_n) + P_{K(y_n)}\{g(y_n) - t(g(y_n) - N(Ay_n, Gy_n, Fy_n))\}] + \gamma_n u_n - x\right\| \\
&\quad \le \big\|\alpha_n x_n + \beta_n[y_n - g(y_n) + P_{K(y_n)}\{g(y_n) - t(g(y_n) - N(Ay_n, Gy_n, Fy_n))\}] + \gamma_n u_n \\
&\qquad -\alpha_n x + \beta_n\{x - g(x) + P_{K(x)}[g(x) - t(g(x) - N(Ax, Gx, Fx))]\} - \gamma_n x\big\| \\
&\quad \le \alpha_n\|x_n - x\| + \beta_n\|Q(y_n) - Q(x)\| + \gamma_n\|u_n - x\|.
\end{aligned} \tag{3.19}$$

From (3.17) and (3.19), it follows that

$$\|x_{n+1} - x\| \le \alpha_n \|x_n - x\| + \theta\beta_n \|y_n - x\| + \gamma_n \|u_n - x\|. \tag{3.20}$$

Similarly, we have

$$\begin{aligned}
&\|y_n - x\| \\
&\quad = \left\| \widehat{\alpha}_n x_n + \widehat{\beta}_n \left[z_n - g(z_n) + P_{K(z_n)} \{ g(z_n) - t\big(g(z_n) - N(Az_n, Gz_n, Fz_n)\big) \} \right] + \widehat{\gamma}_n v_n - x \right\| \\
&\quad = \Big\| \widehat{\alpha}_n (x_n - x) + \widehat{\beta}_n \left[z_n - g(z_n) + P_{K(z_n)} \{ g(z_n) - t\big(g(z_n) - N(Az_n, Gz_n, Fz_n)\big) \} \right] \\
&\qquad - \widehat{\beta} \left[x - g(x) + P_{K(x)} \{ g(x) - t\big(g(x) - N(Ax, Gx, Fx)\big) \} \right] + \widehat{\gamma}_n (v_n - x) \Big\| \\
&\quad \le \widehat{\alpha}_n \|x_n - x\| + \widehat{\beta}_n \|Q(z_n) - Q(x)\| + \widehat{\gamma}_n \|v_n - x\| \\
&\quad \le \widehat{\alpha}_n \|x_n - x\| + \widehat{\beta}_n \theta \|z_n - x\| + \widehat{\gamma}_n \|v_n - x\|.
\end{aligned} \tag{3.21}$$

Again,

$$\|z_n - x\| \le \overline{\alpha}_n \|x_n - x\| + \overline{\beta}_n \theta \|x_n - x\| + \overline{\gamma}_n \|w_n - x\|. \tag{3.22}$$

Let

$$M = \max \left\{ \sup_n \|u_n - x\| \ , \sup_n \|v_n - x\| \ , \sup_n \|w_n - x\| \ , n \ge 0 \right\}. \tag{3.23}$$

Then $M < \infty$ and

$$\|z_n - x\| \le \overline{\alpha}_n \|x_n - x\| + \overline{\beta}_n \theta \|x_n - x\| + \overline{\gamma}_n M. \tag{3.24}$$

Similarly, we deduce from (3.21) the following:

$$\|y_n - x\| \le \widehat{\alpha}_n \|x_n - x\| + \widehat{\beta}_n \theta \overline{\alpha}_n \|x_n - x\| + \overline{\beta}_n \widehat{\beta}_n \theta^2 \|x_n - x\| + \widehat{\beta}_n \theta \overline{\gamma}_n M + \widehat{\gamma}_n M, \tag{3.25}$$

$$\|x_{n+1} - x\| \le \alpha_n \|x_n - x\| + \beta_n \theta \|y_n - x\| + M\gamma_n \quad \forall n \ge 0. \tag{3.26}$$

From the above inequalities, we get

$$\begin{aligned}
\|x_{n+1} - x\| &\le \alpha_n \|x_n - x\| + \beta_n \theta \widehat{\alpha}_n \|x_n - x\| + \beta_n \theta^2 \widehat{\beta}_n \overline{\alpha}_n \|x_n - x\| \\
&\quad + \beta_n \theta^3 \widehat{\beta}_n \overline{\beta}_n \|x_n - x\| + \beta_n \theta^2 \widehat{\beta}_n \overline{\gamma}_n M + \beta_n \theta M \widehat{\gamma}_n + M\gamma_n \\
&\le \left[\alpha_n + \theta\beta_n \left(\widehat{\alpha}_n + \theta \widehat{\beta}_n \left(\overline{\alpha}_n + \theta \overline{\beta}_n \right) \right) \right] \|x_n - x\| + M\theta\beta_n \left(\widehat{\gamma}_n + \widehat{\beta}_n \overline{\gamma}_n \theta \right) + M\gamma_n \\
&\le a_n \|x_n - x\| + b_n + c_n,
\end{aligned} \tag{3.27}$$

where

$$a_n = \alpha_n + \theta\beta_n\left(\widehat{\alpha}_n + \theta\widehat{\beta}_n\left(\overline{\alpha}_n + \theta\overline{\beta}_n\right)\right), \qquad b_n = \theta M\beta_n\left(\widehat{\gamma}_n + \widehat{\beta}_n\overline{\gamma}_n\theta\right), \qquad c = M\gamma_n. \tag{3.28}$$

Since $0 < \theta < 1$, it follows from conditions (1) and (3) that

$$a_n = \alpha_n + \theta\beta_n\left(\widehat{\alpha}_n + \widehat{\beta}_n\theta\left(\overline{\alpha}_n + \theta\overline{\beta}_n\right)\right) \leq 1 - \beta_n + \theta\beta_n \leq 1 - (1-\theta)\beta_n, \tag{3.29}$$

$$\theta M\beta_n\left(\widehat{\gamma}_n + \widehat{\beta}_n\overline{\gamma}_n\theta\right) \leq M\left(\widehat{\gamma}_n + \overline{\gamma}_n\right)\beta_n. \tag{3.30}$$

Therefore,

$$\|x_{n+1} - x\| \leq \left(1 - (1-\theta)\beta_n\right)\|x_n - x\| + M\left(\widehat{\gamma}_n + \overline{\gamma}_n\right) + \gamma_n M. \tag{3.31}$$

From (3.29)-(3.31) and Lemma 2.4, we know that $\{x_n\}$ converges to the solution x. This completes the proof. □

Remark 3.7. We now deduce Theorem 3.6 in the direction of Ishikawa iteration.

Theorem 3.8. *Let H be a real Hilbert space and $K : H \to 2^H$ a set-valued mapping with the nonempty closed convex values. Let A, G, F, g, and N be the same as in Theorem 3.6. Suppose that $P_K(x)$ is Lipschitz continuous with constant $\xi > 0$. Let $\{u_n\}$ and $\{v_n\}$ be the two bounded sequences in H and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\widehat{\alpha}_n\}$, $\{\widehat{\beta}_n\}$, and $\{\widehat{\gamma}_n\}$ be six sequences in $[0,1]$ satisfying the following conditions:*

(1) $\alpha_n + \beta_n + \gamma_n = 1, \quad \widehat{\alpha}_n + \widehat{\beta}_n + \widehat{\gamma}_n = 1, \ n \geq 0$

(2) $\lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\widehat{\beta}_n = \lim_{n\to\infty}\widehat{\gamma}_n = 0$,

(3) $\sum_{n=0}^{+\infty}\beta_n = \infty, \ \sum_{n=0}^{+\infty}\gamma_n < +\infty$.

If the following conditions holds:

$$\begin{gathered}\theta = \Omega + \sqrt{1 + 2pt + t^2s^2\eta^2} + th < 1, \quad h = \varphi + \zeta d, \ \Omega = 2\lambda + \zeta, \\ \varphi = \sqrt{\mu^2(1-2q) + \delta^2\sigma^2} \\ \left|t - \frac{h\Omega - p - h}{s^2\eta^2 - h^2}\right| < \frac{\sqrt{(h\Omega - p - h)^2 - (s^2\eta^2 - h^2)\Omega(2-\Omega)}}{s^2\eta^2 - h^2}, \\ h\Omega > p + h + \sqrt{(s\eta - h)(s\eta + h)\Omega(2-\Omega)}, \quad h\Omega > p + h, \ h < s\eta.\end{gathered} \tag{3.32}$$

Then there exists a unique $x \in H$ satisfying the generalized strongly nonlinear implicit quasi-variational inequality (2.1) *and $x_n \to x$ as $n \to \infty$, where $\{x_n\}$ is the Ishikawa iteration process with errors defined as follows:*

$$\begin{gathered} x_0 \in H, \\ x_{n+1} = \alpha_n x_n + \beta_n \Big\{ y_n - g(y_n) + P_{K(y_n)} \big[g(y_n) - t\big(g(y_n) - N(Ay_n, Gy_n, Fy_n)\big)\big] \Big\} + \gamma_n u_n, \\ y_n = \widehat{\alpha}_n x_n + \widehat{\beta}_n \big\{ x_n - g(x_n) + P_{K(x_n)} \big[g(x_n) - t\big(g(x_n) - N(Ax_n, Gx_n, Fx_n)\big)\big] \big\} + \widehat{\gamma}_n v_n \end{gathered} \tag{3.33}$$

for $n = 0, 1, 2, \ldots$.

Remark 3.9. We can also deduce Theorem 3.6 in the direction of (2.16).

Theorem 3.10. *Let H, K, N, G, A, F, g, and $P_{K(x)}$ be the same as in Theorem 3.6. Let $\{u_n\}$ be a bounded sequence in H and $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three sequences in* $[0, 1]$ *satisfying the following conditions:*

(1) $\alpha_n + \beta_n + \gamma_n = 1$ *for $n \geq 0$,*

(2) $\lim_{n\to\infty} \beta_n = 0$,

(3) $\sum_{n=0}^{+\infty} \beta_n = \infty$ *and* $\sum_{n=0}^{+\infty} \gamma_n < \infty$.

If the conditions of (3.10) *hold, then there exists a unique $x \in H$ satisfying the generalized strongly nonlinear implicit quasi-variational inequality* (2.1) *and $x_n \to x$ as $n \to \infty$, where $\{x_n\}$ is the Mann iterative process with errors defined as follows:*

$$\begin{gathered} x_0 \in H \\ x_{n+1} = \alpha_n x_n + \beta_n \big\{ x_n - g(x_n) + P_{K(x_n)} \big[g(x_n) - t\big(g(x_n) - N(Ax_n, Gx_n, Fx_n)\big)\big] \big\} + \gamma_n u_n \end{gathered} \tag{3.34}$$

for $n = 0, 1, 2, \ldots$.

Our results can be further improved in the direction of (2.25).

Theorem 3.11. *Let H be a real Hilbert space and $K : H \to 2^H$ a set-valued mapping with nonempty closed convex values. Let $A, G, F : H \to H$ be the Lipschitz continuous mapping with respect to positive constants η, σ and d, respectively. Let $g : H \to H$ be the mapping such that $I - g$ and g be Lipschitz continuous with respect to positive constants λ and μ, respectively. A trimapping $N : H \times H \times H \to H$ is generalized pseudo-contractive with respect to map A in first argument of N with constant $p > 0$ and generalized g-pseudo-contractive with respect to G in the second argument of N with constant $q > 0$, Lipschitz continuous with respect to first, second, and third arguments with positive constants s, δ, ζ, respectively. Suppose that $m : H \to H$ is a Lipschitz continuous with positive constant $\chi > 0$. Let $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ be three bounded sequences in* $[0, 1]$ *satisfying the conditions (1)–(3) of Theorem 3.6. If the conditions of* (3.10) *hold for $\xi = 2\chi$, then there exists a* *unique $x \in H$ satisfying* (2.2) *and $x_n \to x$ as $n \to \infty$, where $\{x_n\}$ is the three step iteration process with errors defined as follows:*

$$\begin{aligned}
x_0 &\in H \\
x_{n+1} &= \alpha_n x_n \\
&\quad + \beta_n \left[y_n - g(y_n) + m(y_n) + P_K\{g(y_n) - t(g(y_n) - N(Ay_n, Gy_n, Fy_n)) - m(y_n)\}\right] + \gamma_n u_n \\
y_n &= \widehat{\alpha}_n x_n \\
&\quad + \widehat{\beta}_n \left[z_n - g(z_n) + m(z_n) + P_K\{g(z_n) - tg(z_n) + tN(Az_n, Gz_n, Fz_n) - m(z_n)\}\right] + \widehat{\gamma}_n v_n \\
z_n &= \overline{\alpha}_n x \\
&\quad + \overline{\beta}_n \left[x_n - g(x_n) + m(x_n) + P_K\{g(x_n) - tg(x_n) + tN(Ax_n, Gx_n, Fx_n) - m(x_n)\}\right] + \overline{\gamma}_n w_n
\end{aligned} \tag{3.35}$$

for $n = 0, 1, 2, \ldots$.

Now, we deduce Theorem 3.6 for three step iterative process in terms of (2.10).

Theorem 3.12. *Let H, K, N, G, A, F, g and $P_{K(x)}$ be the same as in Theorem 3.6. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\widehat{\alpha}_n\}$, $\{\widehat{\beta}_n\}$, $\{\overline{\alpha}_n\}$, and $\{\overline{\beta}_n\}$ be six sequences in* $[0,1]$ *satisfying conditions:*

(1) $\alpha_n + \beta_n = 1$, $\widehat{\alpha}_n + \widehat{\beta}_n = 1$, $\overline{\alpha}_n + \overline{\beta}_n = 1$, *for $n \geq 0$,*

(2) $\lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\overline{\beta}_n = \lim_{n\to\infty}\widehat{\beta}_n = 0$,

(3) $\sum_{n=0}^{+\infty} \beta_n = \infty$.

If the conditions of (3.10) *hold, then there exists $x \in H$ satisfying* (2.1) *and $x_n \to x$ as $n \to \infty$, where the three-step iteration process $\{x_n\}$ is defined by*

$$\begin{aligned}
x_0 &\in H, \\
x_{n+1} &= \alpha_n x_n + \beta_n \left[y_n - g(y_n) + P_{K(y_n)}\{g(y_n) - t(g(y_n) - N(Ay_n, Gy_n, Fy_n))\}\right], \\
y_n &= \widehat{\alpha}_n x_n + \widehat{\beta}_n \left[z_n - g(z_n) + P_{K(z_n)}\{g(z_n) - t(g(z_n) - N(Az_n, Gz_n, Fz_n))\}\right], \\
z_n &= \overline{\alpha}_n x_n + \overline{\beta}_n \left[x_n - g(x_n) + P_{K(x_n)}\{g(x_n) - t(g(x_n) - N(Ax_n, Gx_n, Fx_n))\}\right]
\end{aligned} \tag{3.36}$$

for $n = 0, 1, 2, \ldots$.

Next, we state the results in terms of iterations (2.10) and (2.25).

Theorem 3.13. *Let H, K, N, g, G, A, F, and m be the same as in the Theorem 3.11. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\widehat{\alpha}_n\}$, $\{\widehat{\beta}_n\}$, $\{\overline{\alpha}_n\}$, and $\{\overline{\beta}_n\}$ be six sequences in* $[0,1]$ *satisfying conditions (1)–(3) of Theorem 3.6. If the conditions of* (3.10) *hold for $\xi = 2\chi$, then there exists $x \in H$ satisfying the generalized strongly* *nonlinear implicit quasi-variational inequality* (2.2) *and* $x_n \to x$ *as* $n \to \infty$*, where the three-step iteration process* $\{x_n\}$ *is defined by*

$$
\begin{aligned}
&x_0 \in H \\
x_{n+1} &= \alpha_n x_n + \beta_n [y_n - g(y_n) + m(y_n) + P_K\{g(y_n) - tg(y_n) + tN(Ay_n, Gy_n, Fy_n)\} - m(y_n)] \\
y_n &= \widehat{\alpha}_n x_n + \widehat{\beta}_n [z_n - g(z_n) + m(z_n) + P_K\{g(z_n) - tg(z_n) + tN(Az_n, Gz_n, Fz_n)\} - m(z_n)] \\
z_n &= \overline{\alpha}_n x_n + \overline{\beta}_n [x_n - g(x_n) + m(x_n) + P_K\{g(x_n) - tg(x_n) + tN(Ax_n, Gx_n, Fx_n)\} - m(x_n)]
\end{aligned} \tag{3.37}
$$

for $n = 0, 1, 2, \ldots$.

Remark 3.14. Theorem 3.13 can also be deduce for Ishikawa and Mann iterative process.

Acknowledgment

The authors thank the editor Professor R. U. Verma and anonymous referees for their valuable useful suggestions that improved the paper.

References

[1] G. Stampacchia, "Formes bilinéaires coercitives sur les ensembles convexes," *Comptes Rendus de l'Académie des Sciences*, vol. 258, pp. 4413–4416, 1964.

[2] A. Bensoussan and J.-L. Lions, *Impulse Control and Quasi-Variational Inequalities*, Gauthier-Villars, Montrouge, France, 1984.

[3] S. S. Chang, *Variational Inequality and Complementarity Problem Theory with Application*, Shanghai Scientific and Technological Literature Publishing House, Shanghai, China, 1991.

[4] F. Giannessi and A. Maugeri, *Variational Inequalities and Network Equilibrium Problems*, Plenum Press, New York, NY, USA, 1995.

[5] R. Glowinski, J. L. Lions, and R. Tremolieres, *Numerical Analysis of Variational Inequalities*, Studies in Mathematics and Its Applications, North-Holland, Amsterdam, The Netherlands, 1981.

[6] N.-J. Huang, "Mann and Ishikawa type perturbed iterative algorithms for generalized nonlinear implicit quasi-variational inclusions," *Computers & Mathematics with Applications*, vol. 35, no. 10, pp. 1–7, 1998.

[7] N.-J. Huang, "On the generalized implicit quasi-variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 216, pp. 197–210, 1997.

[8] N.-J. Huang, "A new completely general class of variational inclusions with noncompact valued mappings," *Computers & Mathematics with Applications*, vol. 35, no. 10, pp. 9–14, 1998.

[9] Y. J. Cho, J. H. Kim, N.-J. Huang, and S. M. Kang, "Ishikawa and Mann iterative processes with errors for generalized strongly nonlinear implicit quasi-variational inequalities," *Publicationes Mathematicae Debrecen*, vol. 58, no. 4, pp. 635–649, 2001.

[10] F. E. Browder and W. V. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.

[11] C. E. Chidume and E. U. Ofoedu, "A new iteration process for finite families of generalized Lipschitz pseudo-contractive and generalized Lipschitz accretive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 4, pp. 1200–1207, 2008.

[12] X. P. Ding, "Perturbed proximal point algorithms for generalized quasivariational inclusions," *Journal of Mathematical Analysis and Applications*, vol. 210, no. 1, pp. 88–101, 1997.

[13] S. Ishikawa, "Fixed points by a new iteration method," *Proceedings of the American Mathematical Society*, vol. 44, no. 1, pp. 147–150, 1974.

[14] W. R. Mann, “Mean value methods in iteration,” *Proceedings of the American Mathematical Society*, vol. 4, pp. 506–510, 1953.
[15] M. A. Noor, “New approximation schemes for general variational inequalities,” *Journal of Mathematical Analysis and Applications*, vol. 251, no. 1, pp. 217–229, 2000.
[16] Y. Song, “On a Mann type implicit iteration process for continuous pseudo-contractive mappings,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 11, pp. 3058–3063, 2007.
[17] R. U. Verma, “Generalized pseudo-contractions and nonlinear variational inequalities,” *Publicationes Mathematicae Debrecen*, vol. 53, no. 1-2, pp. 23–28, 1998.
[18] C. H. Xiang, “Fixed point theorem for generalized ϕ-pseudocontractive mappings,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 6, pp. 2277–2279, 2009.
[19] Y. Xu, “Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations,” *Journal of Mathematical Analysis and Applications*, vol. 224, no. 1, pp. 91–101, 1998.
[20] Y. A.-P. Fang, N.-J. Huang, J. I. M. Kang, and Y. J. E. Cho, “Generalized nonlinear implicit quasivariational inclusions,” *Journal of Inequalities and Applications*, vol. 2005, no. 3, pp. 261–275, 2005.
[21] Q. Liu, “Convergence theorems of the sequence of iterates for asymptotically demicontractive and hemicontractive mappings,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 26, no. 11, pp. 1835–1842, 1996.