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Research Article

A Hilbert-Type Linear Operator with the Norm and Its Applications

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A Hilbert-type linear operator $T : \ell_{\phi}^{p} \to \ell_{\psi}^{p}$ is defined. As for applications, a more precise operator inequality with the norm and its equivalent forms are deduced. Moreover, three equivalent reverses from them are given as well. The constant factors in these inequalities are proved to be the best possible.

1. Introduction

In 1925, Hardy [1] extended Hilbert inequality as follows.

If $p > 1$, $1/p + 1/q = 1$, $a_n, b_n \ge 0$, $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{n=1}^{\infty}a_n^p\right)^{1/p}\left(\sum_{n=1}^{\infty}b_n^q\right)^{1/q}, \tag{1.1}$$

$$\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right]^p < \left[\frac{\pi}{\sin(\pi/p)}\right]^p\sum_{n=1}^{\infty}a_n^p, \tag{1.2}$$

where (p,q) is a pair of conjugate exponents. The constant factors $\pi/\sin(\pi/p)$ and $[\pi/\sin(\pi/p)]^p$ are the best possible. The expression (1.1) is the famous Hardy-Hilbert's inequality.

Under the same conditions, there are the classic inequalities [2]:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln(m/n)a_m b_n}{m-n} < \left[\frac{\pi}{\sin(\pi/p)}\right]^2 \left(\sum_{n=1}^{\infty}a_n^p\right)^{1/p}\left(\sum_{n=1}^{\infty}b_n^q\right)^{1/q}, \tag{1.3}$$

$$\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{\ln(m/n)a_m}{m-n}\right]^p < \left[\frac{\pi}{\sin(\pi/p)}\right]^{2p}\sum_{n=1}^{\infty}a_n^p, \tag{1.4}$$

where the constant factors $[\pi/\sin(\pi/p)]^2$ and $[\pi/\sin(\pi/p)]^{2p}$ are also the best possible. The expression (1.3) is well known as a Hilbert-type inequality.

By setting a real space of sequences: $\ell^p := \{a;\ a = \{a_n\}_{n=0}^{\infty},\ \|a\|_p = \{\sum_{n=1}^{\infty}|a_n|^p\}^{1/p} < \infty\}$ and defining a linear operator $T : \ell^p \to \ell^p$, $(Ta)(n) = C_n = \sum_{n=1}^{\infty}(\ln(m/n)a_m/(m-n))$ $(n \in N_0)$, the expressions (1.3) and (1.4) can be rewritten as

$$(Ta, b) < \|T\|\|a\|_p\|b\|_q, \tag{1.5}$$

$$\|Ta\|_p < \|T\|\|a\|_p, \tag{1.6}$$

respectively, where $\|T\| = [\pi/\sin(\pi/p)]^2$, $b \in \ell^q$. (Ta, b) is the formal inner product of Ta and b.

The inequalities (1.1)–(1.4) play important roles in theoretical analysis and applications [3]. These inequalities and their integral forms have been recently extended or strengthened in [4–8]. Zhao and Debnath [9] obtained a Hilbert-Pachpatte's reverse inequality. Zhong and Yang [10, 11] have given some reverses concerning some extensions of (1.1). Papers in [12–15] studied some multiple Hardy-Hilbert-type or Hilbert-type inequalities. Articles in [16, 17] got some Hilbert-type linear operator inequalities. In 2006, Yang [18] deduced a new Hilbert-type inequality as follows.

Set (p, q) as a pair of conjugate exponents, and $p > 1$, $1/2 \le \alpha \le 1$, $a_n, b_n \ge 0$, such that $0 < \sum_{n=1}^{\infty}a_n^p < \infty$, $0 < \sum_{n=1}^{\infty}b_n^q < \infty$, then one has

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\ln((m+\alpha)/(n+\alpha))a_m b_n}{m-n} < \left[\frac{\pi}{\sin(\pi/p)}\right]^2 \left(\sum_{n=0}^{\infty}a_n^p\right)^{1/p}\left(\sum_{n=0}^{\infty}b_n^q\right)^{1/q}, \tag{1.7}$$

$$\sum_{n=0}^{\infty}\left[\sum_{m=0}^{\infty}\frac{\ln((m+\alpha)/(n+\alpha))a_m}{m-n}\right]^p < \left[\frac{\pi}{\sin(\pi/p)}\right]^{2p}\sum_{n=0}^{\infty}a_n^p. \tag{1.8}$$

It has been proved that (1.7) and (1.8) are two equivalent inequalities and their constant factors $[\pi/\sin(\pi/p)]^2$ and $[\pi/\sin(\pi/p)]^{2p}$ are the best possible. When $\alpha = 1$, the expressions (1.7) and (1.8) can be reduced to (1.3) and (1.4), respectively.

This paper reports the studies on a Hilbert-type linear operator $T : \ell_\phi^p \to \ell_\psi^p$. As for the applications, a more precise linear operator's general form of Hilbert-type inequality (1.3) incorporating the norm and its equivalent form are deduced. Moreover, three equivalent reverses of the new general forms are deduced as well. The constant factors in these inequalities are all the best possible.

At first, two known results are introduced.

(1) If $s > 1$, (r, s) is a pair of conjugate exponents, then the Beta function is defined as follows (cf. [2, Theorem 342]),

$$\int_0^\infty \frac{\ln u}{u-1} u^{1/s-1} du = \left[\frac{\pi}{\sin(\pi/s)}\right]^2 = \left[B\left(\frac{1}{s}, \frac{1}{r}\right)\right]^2 = \left[B\left(\frac{1}{r}, \frac{1}{s}\right)\right]^2. \tag{1.9}$$

(2) (Euler-Maclaurin's summation formula). Set $f \in C^3[0, \infty)$, if $(-1)^i f^{(i)}(x) > 0$, $f^{(i)}(\infty) = 0$ $(i = 0, 1, 2, 3)$, then (cf. [19, Lemma 1])

$$\sum_{n=0}^{\infty} f(n) < \int_0^\infty f(x)dx + \frac{1}{2} f(0) - \frac{1}{12} f'(0), \tag{1.10}$$

$$\sum_{n=0}^{\infty} f(n) > \int_0^\infty f(x)dx + \frac{1}{2} f(0). \tag{1.11}$$

2. Lemmas

Lemma 2.1. *Set* (r, s) *as a pair of conjugate exponents,* $s > 1$, $\alpha > 0$, $0 < \lambda \leq 1$, *and define*

$$g(u) := \begin{cases} \dfrac{\ln u}{u-1}, & u \in (0,1) \cup (1, \infty), \\ 1, & u = 1, \end{cases} \tag{2.1}$$

$$f_s(x) := h_{m,\lambda}\left(x, \frac{1}{s}\right) := g\left(\left(\frac{x+\alpha}{m+\alpha}\right)^\lambda\right)\left[\left(\frac{x+\alpha}{m+\alpha}\right)^\lambda\right]^{1/s-1/\lambda}, \quad x \in (-\alpha, \infty),\ m \in N_0. \tag{2.2}$$

Then, one has the following:

(1) *the function* $f_s(x)$ *satisfies the conditions of* (1.10) *and* (1.11). *This means*

$$(-1)^i f_s^{(i)}(x) > 0 \quad (x > -\alpha), \qquad f_s^{(i)}(\infty) = 0 \quad (i = 0, 1, 2, 3), \tag{2.3}$$

(2)

$$k_\lambda(s) := \frac{1}{\lambda(m+\alpha)} \int_{-\alpha}^\infty f_s(x)dx = \left[\frac{B(1/s, 1/r)}{\lambda}\right]^2 = \left[\frac{\pi}{\lambda \sin(\pi/s)}\right]^2. \tag{2.4}$$

Proof. (1) For $\alpha > 0$, $x > -\alpha$, $m \in N_0$, $0 < \lambda \leq 1$ and $s > 1$, set $z(x) = g(((x+\alpha)/(m+\alpha))^\lambda)$, $t(x) = [((x+\alpha)/(m+\alpha))^\lambda]^{1/s-1/\lambda} = ((x+\alpha)/(m+\alpha))^{\lambda/s-1}$ and $u = ((x+\alpha)/(m+\alpha))^\lambda$. These show that $z(x) = g(u)$ and $f_s(x) = z(x)t(x) = g(u)t(x)$ when $u > 0$. With the settings,

$(-1)^i g^{(i)}(u) > 0$, $g^{(i)}(\infty) = 0\,(u > 0,\ i = 0,1,2,3)$ (cf. [16, Lemma 2.2]), one has $z(x) > 0$, $t(x) > 0$,

$$\begin{aligned}
z'(x) &= g'(u)\frac{\lambda}{m+\alpha}\left(\frac{x+\alpha}{m+\alpha}\right)^{\lambda-1} < 0,\\
z''(x) &= g''(u)\left[\frac{\lambda}{m+\alpha}\left(\frac{x+\alpha}{m+\alpha}\right)^{\lambda-1}\right]^2 + g'(u)\frac{\lambda(\lambda-1)}{(m+\alpha)^2}\left(\frac{x+\alpha}{m+\alpha}\right)^{\lambda-2} > 0,\\
z'''(x) &= g'''(u)\left[\frac{\lambda}{m+\alpha}\left(\frac{x+\alpha}{m+\alpha}\right)^{\lambda-1}\right]^3 + 3g''(u)\frac{\lambda^2(\lambda-1)}{(m+\alpha)^3}\left(\frac{x+\alpha}{m+\alpha}\right)^{2\lambda-3}\\
&\quad + g'(u)\frac{\lambda(\lambda-1)(\lambda-2)}{(m+\alpha)^3}\left(\frac{x+\alpha}{m+\alpha}\right)^{\lambda-3} < 0,\\
t'(x) &= \frac{\lambda/s-1}{m+\alpha}\left(\frac{x+\alpha}{m+\alpha}\right)^{\lambda/s-2} < 0,\\
t''(x) &= \frac{(\lambda/s-1)(\lambda/s-2)}{(m+\alpha)^2}\left(\frac{x+\alpha}{m+\alpha}\right)^{\lambda/s-3} > 0,\\
t'''(x) &= \frac{(\lambda/s-1)(\lambda/s-2)(\lambda/s-3)}{(m+\alpha)^3}\left(\frac{x+\alpha}{m+\alpha}\right)^{\lambda/s-4} < 0.
\end{aligned} \tag{2.5}$$

These are followed by

$$\begin{aligned}
f_s(x) &= z(x)t(x) > 0, \quad f_s(\infty) = 0,\\
f_s'(x) &= z'(x)t(x) + z(x)t'(x) < 0, \quad f_s'(\infty) = 0,
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
f_s''(x) &= z''(x)t(x) + 2z'(x)t'(x) + z(x)t''(x) > 0, \quad f_s''(\infty) = 0,\\
f_s'''(x) &= z'''(x)t(x) + 3z''(x)t'(x) + 3z'(x)t''(x) + z(x)t'''(x) < 0, \quad f_s'''(\infty) = 0.
\end{aligned} \tag{2.7}$$

Then inequality (2.3) holds.

(2) For $x > -\alpha$, $m \in N_0$ and $\lambda > 0$, $s > 1$, set $u = ((x+\alpha)/(m+\alpha))^\lambda$, then one has

$$\begin{aligned}
\frac{1}{\lambda(m+\alpha)}\int_{-\alpha}^{\infty} f_s(x)dx &= \frac{1}{\lambda}\int_{-\alpha}^{\infty}\frac{\ln((x+\alpha)/(m+\alpha))^\lambda}{((x+\alpha)/(m+\alpha))^\lambda - 1}\left[\left(\frac{x+\alpha}{m+\alpha}\right)^\lambda\right]^{1/s-1/\lambda} d\left(\frac{x+\alpha}{m+\alpha}\right)\\
&= \frac{1}{\lambda^2}\int_0^\infty \frac{\ln u}{u-1}u^{1/s-1}du.
\end{aligned} \tag{2.8}$$

By (1.9), then (2.4) holds. Lemma 2.1 is proved. □

Lemma 2.2. *Set (r, s) as a pair of conjugate exponents, $s > 1$, $\alpha \geq 1/2, 0 < \lambda \leq 1$ and define*

$$\omega_\lambda(m, s) := \sum_{n=0}^{\infty} \frac{\ln((n+\alpha)/(m+\alpha))}{(n+\alpha)^\lambda - (m+\alpha)^\lambda} \cdot \frac{(m+\alpha)^{\lambda/r}}{(n+\alpha)^{1-\lambda/s}} \quad (m \in N_0), \tag{2.9}$$

then, one has

$$0 < \omega_\lambda(m, s) < k_\lambda(s), \tag{2.10}$$

$$0 < \omega_\lambda(n, r) < k_\lambda(r) = k_\lambda(s) \quad (n \in N_0), \tag{2.11}$$

where $k_\lambda(s)$ is defined by (2.4).

Proof. By (2.9) and (2.2), it is evident that

$$\begin{aligned} 0 < \omega_\lambda(m, s) &= \frac{1}{\lambda(m+\alpha)} \sum_{n=0}^{\infty} \frac{\ln((n+\alpha)/(m+\alpha))^\lambda}{((n+\alpha)/(m+\alpha))^\lambda - 1} \left[\left(\frac{n+\alpha}{m+\alpha}\right)^\lambda\right]^{1/s-1/\lambda} \\ &= \frac{1}{\lambda(m+\alpha)} \sum_{n=0}^{\infty} h_{m,\lambda}\left(n, \frac{1}{s}\right) = \frac{1}{\lambda(m+\alpha)} \sum_{n=0}^{\infty} f_s(n). \end{aligned} \tag{2.12}$$

In view of (2.3), (1.10), and (2.4), one has

$$\begin{aligned} \omega_\lambda(m, s) &< \frac{1}{\lambda(m+\alpha)} \left[\int_0^\infty f_s(x)dx + \frac{1}{2} f_s(0) - \frac{1}{12} f_s'(0)\right] \\ &= \frac{1}{\lambda(m+\alpha)} \left\{\int_{-\alpha}^\infty f_s(x)dx - \left[\int_{-\alpha}^0 f_s(x)dx - \frac{1}{2} f_s(0) + \frac{1}{12} f_s'(0)\right]\right\} \\ &= k_\lambda(s) - \frac{1}{\lambda(m+\alpha)} R(s, m), \end{aligned} \tag{2.13}$$

where $R(s, m) := \int_{-\alpha}^0 f_s(x)dx - (1/2) f_s(0) + (1/12) f_s'(0)$ $(m \in N_0)$. With (2.6), it follows that

$$f_s(0) = z(0)t(0) = g\left(\left(\frac{\alpha}{m+\alpha}\right)^\lambda\right)\left(\frac{\alpha}{m+\alpha}\right)^{\lambda/s-1}, \tag{2.14}$$

$$\begin{aligned} f_s'(0) &= z'(0)t(0) + z(0)t'(0) \\ &= \frac{\lambda - s}{s\alpha} g\left(\left(\frac{\alpha}{m+\alpha}\right)^\lambda\right)\left(\frac{\alpha}{m+\alpha}\right)^{\lambda/s-1} + \frac{\lambda}{\alpha} g'\left(\left(\frac{\alpha}{m+\alpha}\right)^\lambda\right)\left(\frac{\alpha}{m+\alpha}\right)^{\lambda/s+\lambda-1}. \end{aligned} \tag{2.15}$$

Set $u = ((x+\alpha)/(m+\alpha))^{\lambda}$, with the partial integration, by the strictly monotonic increase of $g'(u)(g''(u) > 0)$ and $s = r/(r-1)$, it gives

$$\begin{aligned}
&\int_{-\alpha}^{0} f(x)dx \\
&= (m+\alpha)\int_{-\alpha}^{0} \frac{\ln((x+\alpha)/(m+\alpha))^{\lambda}}{((x+\alpha)/(m+\alpha))^{\lambda}-1}\left[\left(\frac{x+\alpha}{m+\alpha}\right)^{\lambda}\right]^{1/s-1/\lambda} d\frac{x+\alpha}{m+\alpha} \\
&= \frac{m+\alpha}{\lambda}\int_{0}^{(\alpha/(m+\alpha))^{\lambda}} g(u)u^{1/s-1}du \\
&= \frac{s(m+\alpha)}{\lambda}\int_{0}^{(\alpha/(m+\alpha))^{\lambda}} g(u)du^{1/s} \\
&= \frac{s(m+\alpha)}{\lambda} g\left(\left(\frac{\alpha}{m+\alpha}\right)^{\lambda}\right)\left(\frac{\alpha}{m+\alpha}\right)^{\lambda/s} - \frac{s(m+\alpha)}{\lambda}\int_{0}^{(\alpha/(m+\alpha))^{\lambda}} u^{1/s}g'(u)du \\
&> \frac{s\alpha}{\lambda} g\left(\left(\frac{\alpha}{m+\alpha}\right)^{\lambda}\right)\left(\frac{\alpha}{m+\alpha}\right)^{\lambda/s-1} - \frac{r(m+\alpha)}{\lambda(r-1)} g'\left(\left(\frac{\alpha}{m+\alpha}\right)^{\lambda}\right)\int_{0}^{(\alpha/(m+\alpha))^{\lambda}} u^{1-1/r}du \\
&= \frac{s\alpha}{\lambda} g\left(\left(\frac{\alpha}{m+\alpha}\right)^{\lambda}\right)\left(\frac{\alpha}{m+\alpha}\right)^{\lambda/s-1} - \frac{r^2(m+\alpha)}{\lambda(r-1)(2r-1)} g'\left(\left(\frac{\alpha}{m+\alpha}\right)^{\lambda}\right)\left[\left(\frac{\alpha}{m+\alpha}\right)^{\lambda}\right]^{2-1/r} \\
&= \frac{s\alpha}{\lambda} g\left(\left(\frac{\alpha}{m+\alpha}\right)^{\lambda}\right)\left(\frac{\alpha}{m+\alpha}\right)^{\lambda/s-1} - \frac{r^2\alpha}{\lambda(r-1)(2r-1)} g'\left(\left(\frac{\alpha}{m+\alpha}\right)^{\lambda}\right)\left(\frac{\alpha}{m+\alpha}\right)^{\lambda/s+\lambda-1}.
\end{aligned} \tag{2.16}$$

In view of (2.13)–(2.16), one has

$$\begin{aligned}
R(s,m) > {}&\left(\frac{s\alpha}{\lambda} - \frac{1}{2} + \frac{\lambda - s}{12s\alpha}\right) g\left(\left(\frac{\alpha}{m+\alpha}\right)^{\lambda}\right)\left(\frac{\alpha}{m+\alpha}\right)^{\lambda/s-1} \\
&- \left[\frac{r^2\alpha}{\lambda(r-1)(2r-1)} - \frac{\lambda}{12\alpha}\right] g'\left(\left(\frac{\alpha}{m+\alpha}\right)^{\lambda}\right)\left(\frac{\alpha}{m+\alpha}\right)^{\lambda/s+\lambda-1}.
\end{aligned} \tag{2.17}$$

If $\alpha \geq 1/2$, $s > 1(r > 1)$, $0 < \lambda \leq 1$, $g(u) > 0$, $-g'(u) > 0$, one has

$$\begin{aligned}
\frac{s\alpha}{\lambda} - \frac{1}{2} + \frac{\lambda - s}{12s\alpha} &= \frac{12s^2\alpha^2 - 6s\alpha\lambda + \lambda(\lambda - s)}{12s\alpha\lambda} = \frac{6s\alpha(2s\alpha - \lambda) - \lambda(s - \lambda)}{12s\alpha\lambda} \\
&\geq \frac{6s\alpha(s-\lambda) - \lambda(s-\lambda)}{12s\alpha\lambda} = \frac{(6s\alpha - \lambda)(s-\lambda)}{12s\alpha\lambda} > 0, \\
\frac{r^2\alpha}{\lambda(r-1)(2r-1)} - \frac{\lambda}{12\alpha} &= \frac{12r^2\alpha^2 - \lambda^2(r-1)(2r-1)}{12\lambda\alpha(r-1)(2r-1)} \\
&= \frac{2r^2(6\alpha^2 - \lambda^2) + \lambda^2(3r-1)}{12\lambda\alpha(r-1)(2r-1)} > 0.
\end{aligned} \tag{2.18}$$

This means that $R(s,m) > 0$. By (2.13) and (2.4), the inequalities (2.10) and (2.11) hold. Lemma 2.2 is proved. □

Lemma 2.3. *Set* (r,s) *as a pair of conjugate exponents,* $s > 1$, $\alpha \geq 1/2$, $0 < \lambda \leq 1$, *and* $\omega_\lambda(m,s)$, $k_\lambda(s)$ *are defined by* (2.9), (2.4), *respectively, then,*

$$(1)\quad \omega_\lambda(m,s) > k_\lambda(s)\left[1-\eta_\lambda(m)\right], \tag{2.19}$$

$$(2)\quad 0 < \eta_\lambda(m) < \theta_\lambda(r) < 1\left(\theta_\lambda(r) := \frac{1}{k_\lambda(s)\lambda^2}\int_0^1 \frac{\ln u}{u-1}u^{-1/r}du\right), \tag{2.20}$$

$$(3)\quad \eta_\lambda(m) = O\left(\left(\frac{1}{m+\alpha}\right)^{\lambda/2s}\right) \quad (m \longrightarrow \infty), \tag{2.21}$$

where $\eta_\lambda(m) := (1/k_\lambda(s)\lambda(m+\alpha))[\int_{-\alpha}^0 f_s(x)dx - (1/2)f_s(0)]$, $f_s(x)$ *is defined by* (2.2).

Proof. By (2.12), (1.11), and (2.4),

$$\begin{aligned}
\omega_\lambda(m,s) &= \frac{1}{\lambda(m+\alpha)}\sum_{n=0}^{\infty} f_s(n) > \frac{1}{\lambda(m+\alpha)}\left[\int_0^\infty f_s(x)dx + \frac{1}{2}f_s(0)\right] \\
&= \frac{1}{\lambda(m+\alpha)}\left[\int_{-\alpha}^\infty f_s(x)dx - \int_{-\alpha}^0 f_s(x)dx + \frac{1}{2}f_s(0)\right] \\
&= k_\lambda(s)\left\{1 - \frac{1}{k_\lambda(s)\lambda(m+\alpha)}\left[\int_{-\alpha}^0 f_s(x)dx - \frac{1}{2}f_s(0)\right]\right\} \\
&= k_\lambda(s)\left[1-\eta_\lambda(m)\right].
\end{aligned} \tag{2.22}$$

This implies that (2.19) holds.

From the monotonic decrease of the function $f_s(x)$ (see (2.3)), $f_s(0) > 0$ and $\alpha \geq 1/2$, one has $\eta_\lambda(m) > (1/k_\lambda(s)\lambda(m+\alpha))[\alpha f_s(0) - (1/2)f_s(0)] \geq 0$. On the other hand, if $f_s(0) > 0$ and by the computation as in (2.16),

$$\begin{aligned}
\eta_\lambda(m) &= \frac{1}{k_\lambda(s)\lambda(m+\alpha)}\left[\int_{-\alpha}^0 f_s(x)dx - \frac{1}{2}f_s(0)\right] \\
&< \frac{1}{k_\lambda(s)\lambda(m+\alpha)}\int_{-\alpha}^0 f_s(x)dx = \frac{1}{k_\lambda(s)\lambda^2}\int_0^{(\alpha/(m+\alpha))^\lambda} \frac{\ln u}{u-1}u^{1/s-1}du \leq \theta_\lambda(r) < 1.
\end{aligned} \tag{2.23}$$

Equation (2.20) is valid.

Since $\lim_{u\to 0^+}(\ln u/(u-1))u^{1/2s} = 0\,(s > 1)$, there exists a constant $L > 0$, such that $|(\ln u/(u-1))u^{1/2s}| \le L\,(u \in (0,(\alpha/(m+\alpha))^{\lambda}))$. Then,

$$0 < \eta_\lambda(m) < \frac{L}{k_\lambda(s)\lambda^2}\int_0^{(\alpha/(m+\alpha))^{\lambda}} u^{1/2s-1}du = \frac{2sL}{k_\lambda(s)\lambda^2}\left(\frac{\alpha}{m+\alpha}\right)^{\lambda/2s}. \tag{2.24}$$

This means that $\eta_\lambda(m) = O((1/(m+\alpha))^{\lambda/2s})\ (m \to \infty)$, the proof is finished. □

Lemma 2.4. *Set (p,q) and (r,s) as two pairs of conjugate exponents, $p > 1$, $r > 1$, $\alpha > 0$, $\lambda > 0$, $0 < \varepsilon < p\lambda/2r$, $\widetilde{a}_m := (m+\alpha)^{\lambda/r-\varepsilon/p-1}$, $\widetilde{b}_n := (n+\alpha)^{\lambda/s-\varepsilon/q-1}$, and $k_\lambda(s)$ is defined by (2.4). Defining*

$$I_1 := \varepsilon\left\{\sum_{m=0}^{\infty}(m+\alpha)^{p(1-\lambda/r)-1}\widetilde{a}_m^{p}\right\}^{1/p}\left\{\sum_{n=0}^{\infty}(n+\alpha)^{q(1-\lambda/s)-1}\widetilde{b}_n^{q}\right\}^{1/q},$$
$$I_2 := \varepsilon\iint_{1-\alpha}^{\infty}\frac{(x+\alpha)^{\lambda/r-\varepsilon/p-1}(y+\alpha)^{\lambda/s-\varepsilon/q-1}\ln((x+\alpha)/(y+\alpha))}{(x+\alpha)^{\lambda}-(y+\alpha)^{\lambda}}dx\,dy, \tag{2.25}$$

then

$$(1)\quad 0 < I_1 < \frac{\varepsilon}{\alpha^{1+\varepsilon}} + \frac{1}{\alpha^{\varepsilon}}, \tag{2.26}$$

$$(2)\quad I_2 \ge k_\lambda(s) + o(1) \quad (\varepsilon \longrightarrow 0^+). \tag{2.27}$$

Proof. (1) By $\alpha > 0$ and $\varepsilon > 0$, one has

$$\begin{aligned}0 < I_1 &= \varepsilon\left[\sum_{m=0}^{\infty}(m+\alpha)^{-1-\varepsilon}\right]^{1/p}\left[\sum_{n=0}^{\infty}(n+\alpha)^{-1-\varepsilon}\right]^{1/q}\\ &= \varepsilon\left[\frac{1}{\alpha^{1+\varepsilon}} + \sum_{n=1}^{\infty}\frac{1}{(n+\alpha)^{1+\varepsilon}}\right] < \varepsilon\left[\frac{1}{\alpha^{1+\varepsilon}} + \int_0^{\infty}\frac{1}{(x+\alpha)^{1+\varepsilon}}dx\right]\\ &= \varepsilon\left[\frac{1}{\alpha^{1+\varepsilon}} - \frac{1}{\varepsilon}(x+\alpha)^{-\varepsilon}\Big|_0^{\infty}\right],\end{aligned} \tag{2.28}$$

which implies that inequality (2.26) holds.

(2) By $y \geq 1-\alpha$, letting $0 < \varepsilon < p\lambda/2r$, one has $(y+\alpha)^{-1-\varepsilon} \leq (y+\alpha)^{-1}$. And setting $u = ((x+\alpha)/(y+\alpha))^{\lambda}$, with $\lim_{u\to 0^+}(\ln u/(u-1))u^{1/2r} = 0\,(r>1)$, $|(\ln u/(u-1))u^{1/2r}| \leq L_1$ $(u \in (0,1),\ L_1 > 0)$, one has

$$\begin{aligned}
I_2 &= \frac{\varepsilon}{\lambda^2}\int_{1-\alpha}^{\infty}(y+\alpha)^{-1-\varepsilon}\left[\int_{(1/(y+\alpha))^{\lambda}}^{\infty}\frac{\ln u}{u-1}u^{1/r-\varepsilon/p\lambda-1}du\right]dy \\
&= \frac{\varepsilon}{\lambda^2}\int_{1-\alpha}^{\infty}(y+\alpha)^{-1-\varepsilon}\left[\int_{0}^{\infty}\frac{\ln u}{u-1}u^{1/r-\varepsilon/p\lambda-1}du - \int_{0}^{(1/(y+\alpha))^{\lambda}}\frac{\ln u}{u-1}u^{1/r-\varepsilon/p\lambda-1}du\right]dy \\
&= \frac{B^2(1/r-\varepsilon/p\lambda, 1/s+\varepsilon/p\lambda)}{\lambda^2} - \frac{\varepsilon}{\lambda^2}\int_{1-\alpha}^{\infty}(y+\alpha)^{-1-\varepsilon}\left[\int_{0}^{(1/(y+\alpha))^{\lambda}}\frac{\ln u}{u-1}u^{1/r-\varepsilon/p\lambda-1}du\right]dy \\
&\geq \frac{B^2(1/r-\varepsilon/p\lambda, 1/s+\varepsilon/p\lambda)}{\lambda^2} - \frac{\varepsilon L_1}{\lambda^2}\int_{1-\alpha}^{\infty}(y+\alpha)^{-1}\left[\int_{0}^{(1/(y+\alpha))^{\lambda}}u^{1/2r-\varepsilon/p\lambda-1}du\right]dy \\
&= \left[\frac{B(1/r-\varepsilon/p\lambda, 1/s+\varepsilon/p\lambda)}{\lambda}\right]^2 - \frac{\varepsilon L_1}{\lambda^2(1/2r-\varepsilon/p\lambda)}\int_{1-\alpha}^{\infty}(y+\alpha)^{-\lambda(1/2r-\varepsilon/p\lambda)-1}dy \\
&= \left[\frac{B(1/r-\varepsilon/p\lambda, 1/s+\varepsilon/p\lambda)}{\lambda}\right]^2 + \frac{\varepsilon L_1}{\lambda^3(1/2r-\varepsilon/p\lambda)^2}.
\end{aligned} \tag{2.29}$$

Set $\varepsilon \to 0^+$, then the inequality (2.27) holds. Lemma 2.4 is proved. □

3. Main Results

Firstly, the following notations are given.

(1) Set $p > 0$, $p \neq 1$, $r > 1$, (p,q) and (r,s) are two pairs of conjugate exponents. Let

$$\begin{aligned}
\phi(x) &:= (x+\alpha)^{p(1-\lambda/r)-1}, \\
\varphi(x) &:= (x+\alpha)^{q(1-\lambda/s)-1}, \\
\psi(x) &:= [\varphi(x)]^{1-p} = (x+\alpha)^{p\lambda/s-1}, \quad (x \in [0,\infty)).
\end{aligned} \tag{3.1}$$

(2) Set $p > 1$, (p,q) is a pair of conjugate exponents. Let

$$\ell_{\phi}^{p} := \left\{a;\ a = \{a_n\}_{n=0}^{\infty},\ \|a\|_{p,\phi} := \left\{\sum_{n=0}^{\infty}\phi(n)|a_n|^p\right\}^{1/p} < \infty\right\}. \tag{3.2}$$

It is a real space of sequences, where

$$\|a\|_{p,\phi} = \left\{\sum_{n=0}^{\infty}\phi(n)|a_n|^p\right\}^{1/p} \tag{3.3}$$

is a norm of sequence a with the weight function ϕ. Similarly, it can define the real spaces of sequences: ℓ_{φ}^{q}, ℓ_{ψ}^{p} and the norm of sequence b with the weight function φ: $\|b\|_{q,\varphi}$ as well. (For $0 < p < 1$ or $q < 0$, the marks $\|a\|_{p,\phi}$ and $\|b\|_{q,\varphi}$ as two formal norms are still used in Theorem 3.3.)

(3) Set $p > 1$, (p,q) is a pair of conjugate exponents. Define a Hilbert-type linear operator T, for all $a \in \ell_{\phi}^{p}$, one has

$$(Ta)(n) := C_n := \sum_{m=0}^{\infty}\frac{\ln((m+\alpha)/(n+\alpha))}{(m+\alpha)^{\lambda}-(n+\alpha)^{\lambda}}a_m \quad (n \in N_0). \tag{3.4}$$

(4) For $a \in \ell_{\phi}^{p}$, $b \in \ell_{\varphi}^{q}$, define the formal inner product of Ta and b as

$$(Ta,b) := \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{\ln((m+\alpha)/(n+\alpha))a_m}{(m+\alpha)^{\lambda}-(n+\alpha)^{\lambda}}\right)b_n = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\ln((m+\alpha)/(n+\alpha))a_m b_n}{(m+\alpha)^{\lambda}-(n+\alpha)^{\lambda}}, \tag{3.5}$$

Then one will have some results in the following theorem.

Theorem 3.1. *Suppose that (p,q) and (r,s) are two pairs of conjugate exponents and $p > 1$, $r > 1$, $1/2 \le \alpha \le 1$, $0 < \lambda \le 1$, $a_n \ge 0$. Then for $\forall a \in \ell_{\phi}^{p}$, one has*

(1)

$$Ta = C = \{C_n\}_{n=0}^{\infty} \in \ell_{\psi}^{p}. \tag{3.6}$$

It means that $T : \ell_{\phi}^{p} \to \ell_{\psi}^{p}$.

(2) *T is a bounded linear operator and*

$$\|T\|_{p,\psi} := \sup_{a\in\ell_{\phi}^{p}\,(a\neq\theta)}\frac{\|Ta\|_{p,\psi}}{\|a\|_{p,\phi}} = k_{\lambda}(s), \tag{3.7}$$

where C_n, T are defined by (3.4)*, $\|Ta\|_{p,\psi} = \|C\|_{p,\psi}$ is defined as by* (3.3)*, and $k_{\lambda}(s)$ is a constant defined by* (2.4).

Proof. If $p > 1$, by using Hölder's inequality (cf. [20]) and the result (2.11), for $n \in N_0$, it is obvious that $C_n \geq 0$ and

$$\begin{aligned}
C_n^p &= \left\{ \sum_{m=0}^{\infty} \frac{\ln((m+\alpha)/(n+\alpha))}{(m+\alpha)^{\lambda} - (n+\alpha)^{\lambda}} \left[\frac{(m+\alpha)^{(1-\lambda/r)/q}}{(n+\alpha)^{(1-\lambda/s)/p}} a_m \right] \left[\frac{(n+\alpha)^{(1-\lambda/s)/p}}{(m+\alpha)^{(1-\lambda/r)/q}} \right] \right\}^p \\
&\leq \left\{ \sum_{m=0}^{\infty} \frac{\ln((m+\alpha)/(n+\alpha))}{(m+\alpha)^{\lambda} - (n+\alpha)^{\lambda}} \frac{(m+\alpha)^{(p-1)1-\lambda/r}}{(n+\alpha)^{1-\lambda/s}} a_m^p \right\} \\
&\quad \times \left\{ \sum_{m=0}^{\infty} \frac{\ln((m+\alpha)/(n+\alpha))}{(m+\alpha)^{\lambda} - (n+\alpha)^{\lambda}} \frac{(n+\alpha)^{(q-1)1-\lambda/s}}{(m+\alpha)^{1-\lambda/r}} \right\}^{p-1} \\
&= \left\{ \sum_{m=0}^{\infty} \frac{\ln((m+\alpha)/(n+\alpha))}{(m+\alpha)^{\lambda} - (n+\alpha)^{\lambda}} \frac{(m+\alpha)^{(p-1)1-\lambda/r}}{(n+\alpha)^{1-\lambda/s}} a_m^p \right\} \{\omega_{\lambda}(n,r)\varphi(n)\}^{p-1} \\
&\leq k_{\lambda}^{p-1}(s) \left\{ \sum_{m=0}^{\infty} \frac{\ln((m+\alpha)/(n+\alpha))}{(m+\alpha)^{\lambda} - (n+\alpha)^{\lambda}} \frac{(m+\alpha)^{(p-1)1-\lambda/r}}{(n+\alpha)^{1-\lambda/s}} a_m^p \right\} \left\{ \varphi^{p-1}(n) \right\}.
\end{aligned} \tag{3.8}$$

And if $\psi(n) = \varphi^{1-p}(n)$, by (2.9) and (2.10), it follows that

$$\begin{aligned}
\|Ta\|_{p,\psi}^p &= \sum_{n=0}^{\infty} \psi(n) C_n^p \\
&\leq k_{\lambda}^{p-1}(s) \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\ln((m+\alpha)/(n+\alpha))}{(m+\alpha)^{\lambda} - (n+\alpha)^{\lambda}} \frac{(m+\alpha)^{(p-1)1-\lambda/r}}{(n+\alpha)^{1-\lambda/s}} a_m^p \right\} \\
&= k_{\lambda}^{p-1}(s) \sum_{m=0}^{\infty} \omega_{\lambda}(m,s)\phi(m) a_m^p \leq k_{\lambda}^p(s) \|a\|_{p,\phi}^p < \infty.
\end{aligned} \tag{3.9}$$

This means that $C = \{C_n\}_{n=0}^{\infty} \in \ell_{\psi}^p$ and $\|T\|_{p,\psi} \leq k_{\lambda}(s)$.

If there exists a constant $K < k_{\lambda}(s)$, such that $\|T\|_{p,\psi} \leq K$, then for $0 < \varepsilon < p\lambda/2r$, by the definition (3.5), and by using Hölder's inequality and the result (2.26), one has

$$\varepsilon\left(T\tilde{a}, \tilde{b}\right) \leq \varepsilon \|T\|_{p,\psi} \|\tilde{a}\|_{p,\phi} \left\|\tilde{b}\right\|_{q,\varphi} \leq K I_1 < K\left(\frac{\varepsilon}{\alpha^{1+\varepsilon}} + \frac{1}{\alpha^{\varepsilon}}\right), \tag{3.10}$$

where $\tilde{a} = \{\tilde{a}_m\}_{m=0}^{\infty} \in \ell_{\phi}^p$, $\tilde{b} = \{\tilde{b}_n\}_{n=0}^{\infty} \in \ell_{\varphi}^q$ and $\tilde{a}_m$, $\tilde{b}_n$ are defined as in Lemma 2.4.

On the other hand, from the strictly monotonic decrease of the function $g(u) = \ln u/(u-1)$ and the exponents $\lambda/r - \varepsilon/p - 1 < 0$, $\lambda/s - \varepsilon/q - 1 < 0$ and $1 - \alpha \geq 0$, and by $\alpha > 0$, $\lambda > 0$, in view of (2.27), one has

$$\begin{aligned}
\varepsilon\left(T\tilde{a}, \tilde{b}\right) &\geq \varepsilon \iint_{1-\alpha}^{\infty} \frac{(x+\alpha)^{\lambda/r-\varepsilon/p-1}(y+\alpha)^{\lambda/s-\varepsilon/q-1} \ln((x+\alpha)/(y+\alpha))}{(x+\alpha)^{\lambda} - (y+\alpha)^{\lambda}} dx\,dy \\
&= I_2 \geq k_{\lambda}(s) + o(1) \quad (\varepsilon \longrightarrow 0^+).
\end{aligned} \tag{3.11}$$

In view of (3.10) and (3.11), one has $k_\lambda(s)+o(1)<K(\varepsilon/\alpha^{1+\varepsilon}+1/\alpha^{\varepsilon})$. Setting $\varepsilon\to 0^+$, one has $k_\lambda(s)\le K$. This means that $K=k_\lambda(s)$, that is, $\|T\|_{p,\psi}=k_\lambda(s)$. Theorem 3.1 is proved. □

Theorem 3.2. *Suppose that (p,q) and (r,s) are two pairs of conjugate exponents, $r>1$, $p>1$, $1/2\le\alpha\le 1$, $0<\lambda\le 1$, $a_n,b_n\ge 0$ $(n\in N_0)$. Then one has the following.*

(1) *If $a\in\ell^p_\phi$, $b\in\ell^q_\varphi$, and $\|a\|_{p,\phi}>0$, $\|b\|_{q,\varphi}>0$, then*

$$(Ta,b)<k_\lambda(s)\|a\|_{p,\phi}\|b\|_{q,\varphi}. \tag{3.12}$$

(2) *If $a\in\ell^p_\phi$ and $\|a\|_{p,\phi}>0$, then*

$$\|Ta\|_{p,\psi}<k_\lambda(s)\|a\|_{p,\phi}, \tag{3.13}$$

where the mark $\|Ta\|_{p,\psi}$ is defind as in Theorem 3.1. The inequality (3.13) *is equivalent to* (3.12), *and the constant factor $k_\lambda(s)=[(1/\lambda)B(1/s,1/r)]^2=k_\lambda(r)$ is the best possible.*

Proof. In view of (3.9) and $0<\|a\|_{p,\phi}<\infty$, one has

$$\|Ta\|^p_{p,\psi}<k^p_\lambda(s)\|a\|^p_{p,\phi}. \tag{3.14}$$

And by $p>1$, (3.13) holds.

By using Hölder's inequality and (3.13), one has

$$\begin{aligned}(Ta,b)&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\ln((m+\alpha)/(n+\alpha))}{(m+\alpha)^\lambda-(n+\alpha)^\lambda}\left[\frac{(m+\alpha)^{(1-\lambda/r)/q}}{(n+\alpha)^{(1-\lambda/s)/p}}a_m\right]\left[\frac{(n+\alpha)^{(1-\lambda/s)/p}}{(m+\alpha)^{(1-\lambda/r)/q}}b_n\right]\\&\le\|Ta\|_{p,\psi}\|b\|_{q,\varphi}<k_\lambda(s)\|a\|_{p,\phi}\|b\|_{q,\varphi}.\end{aligned} \tag{3.15}$$

The inequality (3.12) is obtained.

From (3.12) and $\|a\|_{p,\phi}>0$, there exists $k_0\in N$, such that $\sum_{m=0}^{K}\phi(m)a_m^p>0$ and $b_n(K)=\psi(n)[\sum_{m=0}^{K}(\ln((m+\alpha)/(n+\alpha))a_m)/((m+\alpha)^\lambda-(n+\alpha)^\lambda)]^{p-1}>0$ when $K>k_0$. By a combination as in (3.15) and by $1/2\le\alpha\le 1$, $0<\lambda\le 1$, and with (2.10) and (2.11), then,

$$\begin{aligned}0&<\sum_{n=0}^{K}\varphi(n)b_n^q(K)=\sum_{n=0}^{K}\psi(n)\left[\sum_{m=0}^{K}\frac{\ln((m+\alpha)/(n+\alpha))a_m}{(m+\alpha)^\lambda-(n+\alpha)^\lambda}\right]^p\\&=\sum_{n=0}^{K}\sum_{m=0}^{K}\frac{\ln((m+\alpha)/(n+\alpha))a_mb_n(K)}{(m+\alpha)^\lambda-(n+\alpha)^\lambda}<k_\lambda(s)\left[\sum_{n=0}^{K}\phi(n)a_n^p\right]^{1/p}\left[\sum_{n=0}^{K}\varphi(n)b_n^q(K)\right]^{1/q}<\infty.\end{aligned} \tag{3.16}$$

By $p>1$ and $q>1$, it follows that

$$0<\sum_{n=0}^{K}\varphi(n)b_n^q(K)<k^p_\lambda(s)\sum_{n=0}^{\infty}\phi(n)a_n^p<\infty. \tag{3.17}$$

Letting $K \to \infty$ in (3.17), this means $0 < \sum_{n=0}^{\infty}\varphi(n)b_n^q(\infty) < \infty$, that is, $b = \{b_n(\infty)\}_{n=0}^{\infty} \in \ell_{\varphi}^q$ and $\|b\|_{q,\varphi} > 0$. Therefore the inequality (3.16) keeps the form of the strict inequality when $K \to \infty$. So does (3.17). In view of $\sum_{n=0}^{\infty}\varphi(n)b_n^q(\infty) = \|Ta\|_{p,\psi}^p$, the inequality (3.13) holds, and (3.12) is equivalent to (3.13). By $\|T\|_{p,\psi} = k_\lambda(s)$, it is obvious that the constant factor $k_\lambda(s) = k_\lambda(r)$ is the best possible. This completes the proof of Theorem 3.2. □

Theorem 3.3. *Set (p,q) and (r,s) as two pairs of conjugate exponents, $0 < p < 1 (q < 0)$, $r > 1$, $1/2 \le \alpha \le 1$, $0 < \lambda \le 1$, $a_n, b_n \ge 0$. Let*

$$H(a,b) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\ln((m+\alpha)/(n+\alpha))a_m b_n}{(m+\alpha)^\lambda - (n+\alpha)^\lambda}. \tag{3.18}$$

Then the reverse inequalities can be established as follows.

(1) *If $0 < \|a\|_{p,\phi} < \infty$ and $0 < \|b\|_{q,\varphi} < \infty$, then*

$$H(a,b) > k_\lambda(s)\left\{\sum_{m=0}^{\infty}[1-\eta_\lambda(m)]\phi(m)a_m^p\right\}^{1/p}\|b\|_{q,\varphi}. \tag{3.19}$$

(2) *If $0 < \|a\|_{p,\phi} < \infty$, then*

$$\sum_{n=0}^{\infty}\psi(n)\left[\sum_{m=0}^{\infty}\frac{\ln((m+\alpha)/(n+\alpha))a_m}{(m+\alpha)^\lambda - (n+\alpha)^\lambda}\right]^p > k_\lambda^p(s)\sum_{m=0}^{\infty}[1-\eta_\lambda(m)]\phi(m)a_m^p. \tag{3.20}$$

(3) *If $0 < \|b\|_{q,\varphi} < \infty$, then*

$$\sum_{m=0}^{\infty}\left[\frac{\phi^{-1}(m)}{1-\eta_\lambda(m)}\right]^{q-1}\left[\sum_{n=0}^{\infty}\frac{\ln((m+\alpha)/(n+\alpha))b_n}{(m+\alpha)^\lambda - (n+\alpha)^\lambda}\right]^q < k_\lambda^q(s)\|b\|_{q,\varphi}^q, \tag{3.21}$$

where the marks $\|a\|_{p,\phi}$ and $\|b\|_{q,\varphi}$ $(0 < p < 1)$ as two formal norms are still defined like in (3.3) *and the factor $\eta_\lambda(m)$ in* (3.19)–(3.21) *is defined in Lemma 2.3. The inequalities* (3.20) *and* (3.21) *are equivalent to* (3.19)*. The constant factors $k_\lambda(s)$, $k_\lambda^p(s)$ and $k_\lambda^q(s)$ in* (3.19), (3.20), *and* (3.21) *are all the best possible.*

Proof. By $0 < p < 1$ $(q < 0)$, with the reverse Hölder's inequality, one has the following.

(1) By the combination as in (3.15) for (3.18), then one has

$$H(a,b) \ge \left\{\sum_{m=0}^{\infty}\omega_\lambda(m,s)\phi(m)a_m^p\right\}^{1/p}\left\{\sum_{n=0}^{\infty}\omega_\lambda(n,r)\varphi(n)b_n^q\right\}^{1/q}. \tag{3.22}$$

If $1/2 \le \alpha \le 1$, $0 < \lambda \le 1$, the expressions (2.19) and (2.11) are established for $\omega_\lambda(m,s)$ and $\omega_\lambda(n,r)$, respectively. And by $q < 0$, (3.19) holds.

Setting a constant $\widetilde{K} \geq k_\lambda(r)$, (3.19) is still valid if we replace $k_\lambda(r)$ by $\widetilde{K}$, then for $0 < \varepsilon < -q\lambda/r$, by (3.19) and (2.21), one will have

$$\begin{aligned} H\left(\widetilde{a},\widetilde{b}\right) &> \widetilde{K}\left\{\sum_{m=0}^{\infty}[1-\eta_\lambda(m)]\phi(m)\widetilde{a}_m^p\right\}^{1/p}\|b\|_{q,\varphi} \\ &= \widetilde{K}\left\{\sum_{m=0}^{\infty}\frac{1-\eta_\lambda(m)}{(m+\alpha)^{1+\varepsilon}}\right\}^{1/p}\left\{\sum_{n=0}^{\infty}\frac{1}{(n+\alpha)^{1+\varepsilon}}\right\}^{1/q} \\ &= \widetilde{K}\sum_{n=0}^{\infty}\frac{1}{(n+\alpha)^{1+\varepsilon}}\left\{1-\frac{\left[\sum_{m=0}^{\infty}O\left(1/(m+\alpha)^{1+\varepsilon+\lambda/2s}\right)\right]}{\left[\sum_{m=0}^{\infty}1/(m+\alpha)^{1+\varepsilon}\right]}\right\}^{1/p}, \end{aligned} \tag{3.23}$$

where $\widetilde{a} = \{\widetilde{a}_m\}_{m=0}^{\infty}$, $\widetilde{b} = \{\widetilde{b}_n\}_{n=0}^{\infty}$, $\widetilde{a}_m = (m+\alpha)^{\lambda/r-\varepsilon/p-1}$, $\widetilde{b}_n = (n+\alpha)^{\lambda/s-\varepsilon/q-1}$ and it is apparent that $0 < \|\widetilde{a}\|_{p,\phi} < \infty$, $0 < \|\widetilde{b}\|_{q,\varphi} < \infty$.

On the other hand, by (3.18), (2.2), (2.12), and (2.10),

$$\begin{aligned} H\left(\widetilde{a},\widetilde{b}\right) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\ln((n+\alpha)/(m+\alpha))(m+\alpha)^{\lambda/r-\varepsilon/p-1}(n+\alpha)^{\lambda/s-\varepsilon/q-1}}{(n+\alpha)^{\lambda}-(m+\alpha)^{\lambda}} \\ &= \sum_{m=0}^{\infty}(m+\alpha)^{-1-\varepsilon}\left[\frac{1}{\lambda(m+\alpha)}\sum_{n=0}^{\infty}\frac{\ln((n+\alpha)/(m+\alpha))^{\lambda}((n+\alpha)/(m+\alpha))^{\lambda/s-\varepsilon/q-1}}{((n+\alpha)/(m+\alpha))^{\lambda}-1}\right] \\ &= \sum_{m=0}^{\infty}(m+\alpha)^{-1-\varepsilon}\left[\frac{1}{\lambda(m+\alpha)}\sum_{n=0}^{\infty}h_m\left(n,\frac{1}{s}-\frac{\varepsilon}{q\lambda}\right)\right] \\ &< \frac{B^2(1/s-\varepsilon/q\lambda,1/r+\varepsilon/q\lambda)}{\lambda^2}\sum_{n=0}^{\infty}\frac{1}{(n+\alpha)^{1+\varepsilon}}. \end{aligned} \tag{3.24}$$

In view of (3.23) and (3.24), one has

$$\widetilde{K}\left\{1-\frac{\left[\sum_{m=0}^{\infty}O\left(1/(m+\alpha)^{1+\varepsilon+\lambda/2s}\right)\right]}{\left[\sum_{m=0}^{\infty}1/(m+\alpha)^{1+\varepsilon}\right]}\right\}^{1/p} < \frac{B^2(1/s-\varepsilon/q\lambda,1/r+\varepsilon/q\lambda)}{\lambda^2}. \tag{3.25}$$

Setting $\varepsilon \to 0^+$, one has $\widetilde{K} \leq k_\lambda(s)$, which means $\widetilde{K} = k_\lambda(s)$. The constant factor $k_\lambda(s)$ in (3.19) is the best possible.

(2) By $0 < \|a\|_{p,\phi} < \infty$, one has $\sum_{n=0}^{\infty}\varphi(n)[\sum_{m=0}^{\infty}(\ln((m+\alpha)/(n+\alpha))a_m)/((m+\alpha)^{\lambda}-(n+\alpha)^{\lambda})]^p > 0$. Setting $b_n := \varphi(n)[\sum_{m=0}^{\infty}(\ln((m+\alpha)/(n+\alpha))a_m)/((m+\alpha)^{\lambda}-(n+\alpha)^{\lambda})]^{p-1}$, making the calculations as in (3.8) and (3.9), one has $0 < \|b\|_{q,\varphi}^q < \infty$. By using (3.19) in the following:

$$\begin{aligned}\|b\|_{q,\varphi}^q &= \sum_{n=0}^{\infty}\varphi(n)b_n^q = \sum_{n=0}^{\infty}\varphi(n)\left[\sum_{m=0}^{\infty}\frac{\ln((m+\alpha)/(n+\alpha))a_m}{(m+\alpha)^{\lambda}-(n+\alpha)^{\lambda}}\right]^p \\ &= H(a,b) > k_{\lambda}(s)\left\{\sum_{m=0}^{\infty}[1-\eta_{\lambda}(m)]\phi(m)a_m^p\right\}^{1/p}\|b\|_{q,\varphi},\end{aligned} \tag{3.26}$$

one has

$$\sum_{n=0}^{\infty}\varphi(n)\left[\sum_{m=0}^{\infty}\frac{\ln((m+\alpha)/(n+\alpha))a_m}{(m+\alpha)^{\lambda}-(n+\alpha)^{\lambda}}\right]^p > k_{\lambda}^p(s)\sum_{m=0}^{\infty}[1-\eta_{\lambda}(m)]\phi(m)a_m^p. \tag{3.27}$$

Therefore, (3.20) holds.

On the other hand, if (3.20) is valid, by $0 < p < 1\,(q < 0)$ and by using the reverse Hölder's inequality, it has

$$\begin{aligned}H_{\llcorner}(a,b) &= \sum_{n=0}^{\infty}\left[(n+\alpha)^{\lambda/s-1/p}\sum_{m=0}^{\infty}\frac{\ln((m+\alpha)/(n+\alpha))a_m}{(m+\alpha)^{\lambda}-(n+\alpha)^{\lambda}}\right]\left[(n+\alpha)^{1/p-\lambda/s}b_n\right] \\ &\geq \left\{\sum_{n=0}^{\infty}\varphi(n)\left[\sum_{m=0}^{\infty}\frac{\ln((m+\alpha)/(n+\alpha))a_m}{(m+\alpha)^{\lambda}-(n+\alpha)^{\lambda}}\right]^p\right\}^{1/p}\left\{\sum_{n=0}^{\infty}(n+\alpha)^{q(1-\lambda/s)-1}b_n^q\right\}^{1/q} \\ &> k_{\lambda}(s)\left[\sum_{m=0}^{\infty}[1-\eta_{\lambda}(m)]\phi(m)a_m^p\right]^{1/p}\|b\|_{q,\varphi}.\end{aligned} \tag{3.28}$$

Then (3.19) holds. It means that (3.20) is equivalent to (3.19).

(3) By $0 < \|b\|_{q,\varphi} < \infty$, it is obvious that there exist $n_0 \in \mathbb{N}$, such that

$$\sum_{m=0}^{K}\left[\frac{\phi^{-1}(m)}{1-\eta_{\lambda}(m)}\right]^{q-1}\left[\sum_{n=0}^{K}\frac{\ln((m+\alpha)/(n+\alpha))b_n}{\left((m+\alpha)^{\lambda}-(n+\alpha)^{\lambda}\right)}\right]^q > 0, \quad \left\{\sum_{n=0}^{K}\varphi(n)b_n^q\right\} > 0 \quad \text{when } K > n_0. \tag{3.29}$$

Setting $a_m(K) = [(\phi^{-1}(m)/(1-\eta_\lambda(m)))\sum_{n=0}^{K}(\ln((m+\alpha)/(n+\alpha))b_n)/((m+\alpha)^\lambda-(n+\alpha)^\lambda)]^{q-1}$ (> 0), one has $0 < \sum_{m=0}^{K}\phi(m)a_m^p(K) < \infty$. By (3.19),

$$\begin{aligned}
&\sum_{m=0}^{K}[1-\eta_\lambda(m)]\phi(m)a_m^p(K)\\
&\quad=\sum_{m=0}^{K}\left[\frac{\phi^{-1}(m)}{1-\eta_\lambda(m)}\right]^{q-1}\left[\sum_{n=0}^{K}\frac{\ln((m+\alpha)/(n+\alpha))b_n}{(m+\alpha)^\lambda-(n+\alpha)^\lambda}\right]^q\\
&\quad=\sum_{m=0}^{K}\sum_{n=0}^{K}\frac{\ln((m+\alpha)/(n+\alpha))b_n a_m(K)}{(m+\alpha)^\lambda-(n+\alpha)^\lambda}\\
&\quad> k_\lambda(s)\left\{\sum_{m=0}^{K}[1-\eta_\lambda(m)]\phi(m)a_m^p(K)\right\}^{1/p}\left\{\sum_{n=0}^{K}\varphi(n)b_n^q\right\}^{1/q}.
\end{aligned}\tag{3.30}$$

Further one has

$$\left\{\sum_{m=0}^{K}[1-\eta_\lambda(m)]\phi(m)a_m^p(K)\right\}^{1/q} > k_\lambda(s)\left\{\sum_{n=0}^{K}\varphi(n)b_n^q\right\}^{1/q} > 0.\tag{3.31}$$

By $q < 0$, one has

$$0 < \sum_{m=0}^{K}[1-\eta_\lambda(m)]\phi(m)a_m^p(K) < k_\lambda^q(s)\sum_{n=0}^{K}\varphi(n)b_n^q < k_\lambda^q(s)\sum_{n=0}^{\infty}\varphi(n)b_n^q < \infty.\tag{3.32}$$

Setting $K \to \infty$ in (3.32), via (2.20), one has

$$0 < \sum_{m=0}^{\infty}\phi(m)a_m^p(\infty) < \frac{1}{1-\theta_\lambda(r)}\sum_{m=0}^{\infty}[1-\eta_\lambda(m)]\phi(m)a_m^p(\infty) < \infty.\tag{3.33}$$

It means that $0 < \|a\|_{p,\phi} < \infty(a := \{a_m(\infty)\}_{m=0}^{\infty})$. The conditions for (3.19) are satisfied. Equation (3.30) keeps a strict form when $K \to \infty$. So does (3.31). By $q < 0$, the inequality (3.21) holds.

Also, from (3.21), by $q < 0$ and by using the reverse Hölder's inequality,

$$\begin{aligned} H(a,b) &= \sum_{m=0}^{\infty}\left\{\left[\frac{1-\eta_\lambda(m)}{\phi^{-1}(m)}\right]^{1/p} a_m\right\}\left\{\left[\frac{\phi^{-1}(m)}{1-\eta_\lambda(m)}\right]^{1/p}\sum_{n=0}^{\infty}\frac{\ln((m+\alpha)/(n+\alpha))b_n}{(m+\alpha)^\lambda-(n+\alpha)^\lambda}\right\} \\ &\geq \left\{\sum_{m=0}^{\infty}[1-\eta_\lambda(m)]\phi(m)a_m^p\right\}^{1/p}\left\{\sum_{m=0}^{\infty}\left[\frac{\phi^{-1}(m)}{1-\eta_\lambda(m)}\right]^{q-1}\left[\sum_{n=0}^{\infty}\frac{\ln((m+\alpha)/(n+\alpha))b_n}{(m+\alpha)^\lambda-(n+\alpha)^\lambda}\right]^q\right\}^{1/q} \\ &> k_\lambda(s)\left\{\sum_{m=0}^{\infty}[1-\eta_\lambda(m)]\phi(m)a_m^p\right\}^{1/p}\|b\|_{q,\varphi}. \end{aligned} \tag{3.34}$$

Therefore, (3.19) holds. Equation (3.21) is equivalent to (3.19).

If the constant factor $k_\lambda^p(s)$ (or $k_\lambda^q(s)$) in (3.20) (or in (3.21)) is not the best possible, by (3.28) (or by (3.34)), then it leads to a contradiction in which the constant factor $k_\lambda(s)$ in (3.19) is not the best possible. Theorem 3.3 is proved. □

Remark 3.4. Set $r = q$, $s = p$, $\lambda = 1$, the inequalities (3.12) and (3.13) can be reduced to (1.7) and (1.8), respectively. So (3.12) (or (3.13)) is an extension of (1.7) (or (1.8)).

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