Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 503782, 7 pages
doi:10.1155/2009/503782

Research Article

A Double Inequality for Gamma Function

Xiaoming Zhang[1] and Yuming Chu[2]

[1] *Haining Radio and TV University, Haining 314400, Zhejiang, China*
[2] *Department of Mathematics, Huzhou Teachers College, Huzhou 313000, Zhejiang, China*

Correspondence should be addressed to Yuming Chu, chuyuming2005@yahoo.com.cn

Received 12 June 2009; Revised 21 August 2009; Accepted 30 August 2009

Recommended by Ramm Mohapatra

Using the Alzer integral inequality and the elementary properties of the gamma function, a double inequality for gamma function is established, which is an improvement of Merkle's inequality.

1. Introduction

For real and positive values of x, the Euler gamma function Γ and its logarithmic derivative ψ, the so-called psi function, are defined by

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt, \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{1.1}$$

respectively. For extensions of these functions to complex variables and for basic properties, see [1].

Recently, the gamma function has been the subject of intensive research, many remarkable inequalities for Γ can be found in literature [2–21]. In particular, the ratio $(\Gamma(s)/\Gamma(r))(s > r > 0)$ have attracted the attention of many mathematicians and physicists. Gautschi [22] first proved that

$$n^{1-s} < \frac{\Gamma(n+1)}{\Gamma(n+s)} < \exp\left[(1-s)\psi(n+1)\right] \tag{1.2}$$

for $0 < s < 1$ and $n = 1, 2, 3 \ldots$.

A strengthened upper bound was given by Erber [23]:

$$\frac{\Gamma(n+1)}{\Gamma(n+s)} < \frac{4(n+s)(n+1)^{1-s}}{4n+(s+1)^2}. \tag{1.3}$$

In [24], Kečkić and Vasić established the following double inequality for $b > a > 0$:

$$\frac{b^{b-1}}{a^{a-1}}e^{a-b} < \frac{\Gamma(b)}{\Gamma(a)} < \frac{b^{b-(1/2)}}{a^{a-(1/2)}}e^{a-b}. \tag{1.4}$$

In [25], Kershaw obtained

$$\begin{gathered} \exp\left[(1-s)\psi\left(x+s^{1/2}\right)\right] < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp\left[(1-s)\psi\left(x+\frac{1}{2}(s+1)\right)\right], \\ \left(x+\frac{1}{2}s\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left[x-\frac{1}{2}+\left(s+\frac{1}{4}\right)^{1/2}\right]^{1-s} \end{gathered} \tag{1.5}$$

for $x > 0$ and $0 < s < 1$.

The generalized logarithmic mean $L_p(a,b)$ of order p of two positive numbers a and b with $a \neq b$ is defined by

$$L_p(a,b) = \begin{cases} \left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & p \neq -1,\ p \neq 0, \\ \dfrac{b-a}{\log b - \log a}, & p = -1, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & p = 0. \end{cases} \tag{1.6}$$

It is well known that $L_p(a,b)$ is strictly increasing with respect to p for fixed a and b. If we denote $A(a,b) = L_1(a,b) = (a+b)/2$, $I(a,b) = L_0(a,b) = (1/e)(b^b/a^a)^{1/(b-a)}$, $L(a,b) = L_{-1}(a,b) = (b-a)/(\log b - \log a)$, and $G(a,b) = L_{-2}(a,b) = \sqrt{ab}$ the arithmetic mean, identric mean, logarithmic mean, and geometric mean of a and b with $a \neq b$, respectively, then

$$\min\{a,b\} < G(a,b) < L(a,b) < I(a,b) < A(a,b) < \max\{a,b\}. \tag{1.7}$$

In 1996, Merkle [26] established

$$A(\psi(a),\psi(b)) < \frac{\log\Gamma(b) - \log\Gamma(a)}{b-a} < \psi(A(a,b)) \tag{1.8}$$

for $a, b > 0$ with $a \neq b$.

It is the aim of this paper to present the new upper and lower bounds of inequality (1.8) in terms of I and L.

2. Lemmas

In order to establish our main result we need several lemmas, which we present in this section.

Lemma 2.1 (see [27, page 2670]). *If $x > 0$, then*

$$\psi'(x) > \frac{1}{x} + \frac{1}{2x^2}. \tag{2.1}$$

Lemma 2.2 (see [28]). *Let $f \in C[a,b]$ be a strictly increasing function. If $1/f^{-1}$ is strictly convex (or concave, resp.), then*

$$\frac{1}{b-a}\int_a^b f(t)dt > (\text{or} <, \text{resp.}) f(L(a,b)). \tag{2.2}$$

Here, f^{-1} is the inverse of f.

Lemma 2.3. *If $x > 0$, then*

$$0 < 2\psi'(x) + x\psi''(x) < \frac{1}{x}. \tag{2.3}$$

Proof. It is well known that $\log\Gamma(x) = -\gamma x + \sum_{k=1}^{\infty}[x/k - \log(1+(x/k))] - \log x$, where $\gamma = 0.577\,215\ldots$ is the Euler constant. Then, we have

$$\psi'(x) = \sum_{k=0}^{\infty}\frac{1}{(k+x)^2} \tag{2.4}$$

$$\psi''(x) = -2\sum_{k=0}^{\infty}\frac{1}{(k+x)^3}. \tag{2.5}$$

From (2.4) and (2.5), we get

$$2\psi'(x) + x\psi''(x) = \sum_{k=1}^{\infty} \frac{2k}{(k+x)^3} > 0,$$

$$\begin{aligned}
2\psi'(x) + x\psi''(x) &= \sum_{k=1}^{\infty} \frac{2k}{(k+x)^3} \\
&< \sum_{k=1}^{\infty} \frac{2k}{(k-1+x)(k+x)(k+1+x)} \\
&= \sum_{k=1}^{\infty} \left[\frac{k}{(k-1+x)(k+x)} - \frac{k}{(k+x)(k+1+x)} \right] \\
&= \sum_{k=1}^{\infty} \frac{1}{(k-1+x)(k+x)} \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{k-1+x} - \frac{1}{k+x} \right) \\
&= \frac{1}{x}.
\end{aligned} \tag{2.6}$$

□

Lemma 2.4. *Suppose that $b > a > 0$ and $f : [a,b] \to R$ is a twice differentiable function. If $f'(x) > 0$ and $2f'(x) + xf''(x) >$ (or $<$, resp.) 0 for $x \in [a,b]$, then there exists the inverse function f^{-1} of f and $1/f^{-1}$ is strictly convex (or concave, resp.).*

Proof. The existence of f^{-1} can be derived from $f'(x) > 0$ directly. Next, let $y = f(x)$, then simple computation yields

$$\begin{gathered}
f'(x)\left(f^{-1}(y)\right)' = 1, \\
f''(x)\left[\left(f^{-1}(y)\right)'\right]^2 + f'(x)\left(f^{-1}(y)\right)'' = 0, \\
\left(\frac{1}{f^{-1}(y)}\right)'' = \frac{2\left[\left(f^{-1}(y)\right)'\right]^2}{\left(f^{-1}(y)\right)^3} - \frac{\left(f^{-1}(y)\right)''}{\left(f^{-1}(y)\right)^2}.
\end{gathered} \tag{2.7}$$

From (2.7) and $x = f^{-1}(y)$, we get

$$\left(\frac{1}{f^{-1}(y)}\right)'' = \frac{2f'(x) + xf''(x)}{x^3\left(f'(x)\right)^3}. \tag{2.8}$$

Therefore, the strict convexity (or concavity, resp.) of $1/f^{-1}$ follows from (2.8) and the assumed condition $2f'(x) + xf''(x) >$ (or $<$, resp.) 0. □

3. Main Result

Theorem 3.1. *For all $a,b>0$ with $a\neq b$, one has*

$$\psi(L(a,b)) < \frac{\log\Gamma(b)-\log\Gamma(a)}{b-a} < \psi(L(a,b)) + \log\frac{I(a,b)}{L(a,b)}. \tag{3.1}$$

Proof. Without loss of generality, we assume that $b>a>0$. From (2.4) and Lemma 2.3, together with Lemma 2.4, we clearly see that ψ is strictly increasing and $1/\psi^{-1}$ is strictly convex on $[a,b]$. Then, Lemma 2.2 leads to

$$\frac{1}{b-a}\int_a^b \psi(t)dt > \psi(L(a,b)). \tag{3.2}$$

Therefore, the left-side inequality in (3.1) follows from (3.2).

Next, for $x\in[a,b]$, let $g(x)=\psi(x)-\log x$. Then, Lemmas 2.1 and 2.3 lead to

$$g'(x) = \psi'(x) - \frac{1}{x} > \frac{1}{2x^2} > 0, \tag{3.3}$$

$$2g'(x) + xg''(x) = 2\psi'(x) + x\psi''(x) - \frac{1}{x} < 0. \tag{3.4}$$

From (3.3) and (3.4), together with Lemma 2.4, we clearly see that $g(x)$ is strictly increasing and $1/g^{-1}$ is strictly concave on $[a,b]$. Then, Lemma 2.2 implies

$$\frac{1}{b-a}\int_a^b \big(\psi(t)-\log t\big)dt < \psi(L(a,b)) - \log L(a,b). \tag{3.5}$$

Therefore, the right-side inequality in (3.1) follows from (3.5).

To compare the bounds in Theorem 3.1 with that in (1.8), we have the following two remarks. □

Remark 3.2. The lower bound in Theorem 3.1 is greater than that in (1.8), that is, $\psi(L(a,b)) > A(\psi(a),\psi(b))$ for $a,b>0$ with $a\neq b$. In fact, for any $b>a>0$ and $x\in[a,b]$, Lemmas 2.1 and 2.3 lead to

$$\psi'(x) + x\psi''(x) < -\frac{1}{2x^2} < 0. \tag{3.6}$$

From (3.6) and [29], we know that $\psi(x)$ is a strictly geometric-arithmetic concave function on $[a,b]$, hence, we get

$$\psi(G(a,b)) > A\big(\psi(a),\psi(b)\big). \tag{3.7}$$

Since ψ is strictly increasing and $G(a,b)<L(a,b)$, so we have

$$\psi(L(a,b)) > \psi(G(a,b)). \tag{3.8}$$

Inequalities (3.7) and (3.8) show that $\psi(L(a,b)) > A(\psi(a),\psi(b))$ for $a,b>0$ with $a \neq b$.

Remark 3.3. The upper bound in Theorem 3.1 is less than that in (1.8), that is, $\psi(L(a,b)) + \log I(a,b) - \log L(a,b) < \psi(A(a,b))$. In fact, for any $b > a > 0$ and $x \in [a,b]$, (3.3) and $L(a,b) < I(a,b)$ imply

$$\psi(L(a,b)) - \log L(a,b) < \psi(I(a,b)) - \log I(a,b). \tag{3.9}$$

On the other hand, the monotonicity of ψ and $I(a,b) < A(a,b)$ leads to

$$\psi(I(a,b)) < \psi(A(a,b)). \tag{3.10}$$

From (3.9) and (3.10), we get

$$\psi(L(a,b)) + \log I(a,b) - \log L(a,b) < \psi(A(a,b)). \tag{3.11}$$

Acknowledgments

The authors wish to thank the anonymous referee for the very careful reading of the manuscript and fruitful comments and suggestions. This research is partly supported by N S Foundation of China under Grants 60850005 and 10771195, and N S Foundation of Zhejiang Province under Grants D7080080 and Y607128.

References

[1] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, UK, 1996.

[2] H. Alzer and G. Felder, "A Turán-type inequality for the gamma function," *Journal of Mathematical Analysis and Applications*, vol. 350, no. 1, pp. 276–282, 2009.

[3] T.-H. Zhao, Y.-M. Chu, and Y.-P. Jiang, "Monotonic and logarithmically convex properties of a function involving gamma functions," *Journal of Inequalities and Applications*, vol. 2009, Article ID 728612, 13 pages, 2009.

[4] H. Alzer, "Inequalities for Euler's gamma function," *Forum Mathematicum*, vol. 20, no. 6, pp. 955–1004, 2008.

[5] N. Batir, "Inequalities for the gamma function," *Archiv der Mathematik*, vol. 91, no. 6, pp. 554–563, 2008.

[6] A. Laforgia and P. Natalini, "Supplements to known monotonicity results and inequalities for the gamma and incomplete gamma functions," *Journal of Inequalities and Applications*, vol. 2006, Article ID 48727, 8 pages, 2006.

[7] Y. Yu, "An inequality for ratios of gamma functions," *Journal of Mathematical Analysis and Applications*, vol. 352, no. 2, pp. 967–970, 2009.

[8] R. P. Agarwal, N. Elezović, and J. Pečarić, "On some inequalities for beta and gamma functions via some classical inequalities," *Journal of Inequalities and Applications*, no. 5, pp. 593–613, 2005.

[9] M. Merkle, "Gurland's ratio for the gamma function," *Computers & Mathematics with Applications*, vol. 49, no. 2-3, pp. 389–406, 2005.

[10] H. Alzer, "On Ramanujan's double inequality for the gamma function," *The Bulletin of the London Mathematical Society*, vol. 35, no. 5, pp. 601–607, 2003.

[11] B.-N. Guo and F. Qi, "Inequalities and monotonicity for the ratio of gamma functions," *Taiwanese Journal of Mathematics*, vol. 7, no. 2, pp. 239–247, 2003.

[12] H. Alzer, "On a gamma function inequality of Gautschi," *Proceedings of the Edinburgh Mathematical Society*, vol. 45, no. 3, pp. 589–600, 2002.

[13] S. S. Dragomir, R. P. Agarwal, and N. S. Barnett, "Inequalities for beta and gamma functions via some classical and new integral inequalities," *Journal of Inequalities and Applications*, vol. 5, no. 2, pp. 103–165, 2000.

[14] H. Alzer, "A mean-value inequality for the gamma function," *Applied Mathematics Letters*, vol. 13, no. 2, pp. 111–114, 2000.

[15] H. Alzer, "Inequalities for the gamma function," *Proceedings of the American Mathematical Society*, vol. 128, no. 1, pp. 141–147, 2000.

[16] M. Merkle, "Convexity, Schur-convexity and bounds for the gamma function involving the digamma function," *The Rocky Mountain Journal of Mathematics*, vol. 28, no. 3, pp. 1053–1066, 1998.

[17] B. Palumbo, "A generalization of some inequalities for the gamma function," *Journal of Computational and Applied Mathematics*, vol. 88, no. 2, pp. 255–268, 1998.

[18] J. Dutka, "On some gamma function inequalities," *SIAM Journal on Mathematical Analysis*, vol. 16, no. 1, pp. 180–185, 1985.

[19] A. Laforgia, "Further inequalities for the gamma function," *Mathematics of Computation*, vol. 42, no. 166, pp. 597–600, 1984.

[20] J. B. Selliah, "An inequality satisfied by the gamma function," *Canadian Mathematical Bulletin*, vol. 19, no. 1, pp. 85–87, 1976.

[21] W. Gautschi, "A harmonic mean inequality for the gamma function," *SIAM Journal on Mathematical Analysis*, vol. 5, pp. 278–281, 1974.

[22] W. Gautschi, "Some elementary inequalities relating to the gamma and incomplete gamma function," *Journal of Mathematics and Physics*, vol. 38, pp. 77–81, 1960.

[23] T. Erber, "The gamma function inequalities of Gurland and Gautschi," *Scandinavian Aktuarietidskr*, vol. 1960, pp. 27–28, 1961.

[24] J. D. Kečkić and P. M. Vasić, "Some inequalities for the gamma function," *Institut Mathématique Publications*, vol. 11(25), pp. 107–114, 1971.

[25] D. Kershaw, "Some extensions of W. Gautschi's inequalities for the gamma function," *Mathematics of Computation*, vol. 41, no. 164, pp. 607–611, 1983.

[26] M. Merkle, "Logarithmic convexity and inequalities for the gamma function," *Journal of Mathematical Analysis and Applications*, vol. 203, no. 2, pp. 369–380, 1996.

[27] Á. Elbert and A. Laforgia, "On some properties of the gamma function," *Proceedings of the American Mathematical Society*, vol. 128, no. 9, pp. 2667–2673, 2000.

[28] H. Alzer, "On an integral inequality," *L'Analyse Numérique et la Théorie de l'Approximation*, vol. 18, no. 2, pp. 101–103, 1989.

[29] R. A. Satnoianu, "Improved *GA*-convexity inequalities," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 3, no. 5, article 82, pp. 1–6, 2002.