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Research Article

On Generalized Paranormed Statistically Convergent Sequence Spaces Defined by Orlicz Function

Metin Başarir and Selma Altundağ

Department of Mathematics, Faculty of Science and Arts, Sakarya University, 54187 Sakarya, Turkey

Correspondence should be addressed to Metin Başarir, basarir@sakarya.edu.tr

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We define generalized paranormed sequence spaces $\overline{c}(\sigma, M, p, q, s)$, $\overline{c}_0(\sigma, M, p, q, s)$, $m(\sigma, M, p, q, s)$, and $m_0(\sigma, M, p, q, s)$ defined over a seminormed sequence space (X, q). We establish some inclusion relations between these spaces under some conditions.

1. Introduction

$w(X)$, $c(X)$, $c_0(X)$, $\overline{c}(X)$, $\overline{c_0}(X)$, $l_\infty(X)$, $m(X)$, $m_0(X)$ will represent the spaces of all, convergent, null, statistically convergent, statistically null, bounded, bounded statistically convergent, and bounded statistically null X-valued sequence spaces throughout the paper, where (X, q) is a seminormed space, seminormed by q. For $X = \mathbb{C}$, the space of complex numbers, these spaces represent the w, c, c_0, $\overline{c}$, $\overline{c_0}$, l_∞, m, m_0 which are the spaces of all, convergent, null, statistically convergent, statistically null, bounded, bounded statistically convergent, and bounded statistically null sequences, respectively. The zero sequence is denoted by $\overline{\theta} = (\theta, \theta, \theta, \ldots)$, where θ is the zero element of X.

The idea of statistical convergence was introduced by Fast [1] and studied by various authors (see [2–4]). The notion depends on the density of subsets of the set $\mathbb{N}$ of natural numbers. A subset E of $\mathbb{N}$ is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \text{ exists}, \tag{1.1}$$

where χ_E is the characteristic function of E.

A sequence $x = (x_k)$ is said to be statistically convergent to the number L (i.e., $(x_k) \in \overline{c}$) if for every $\varepsilon > 0$

$$\delta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0. \tag{1.2}$$

In this case, we write $x_k \overset{\text{stat}}{\rightarrow} L$ or stat $-\lim x = L$.

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional ϕ on l_∞, the space of real bounded sequences, is said to be an invariant mean or σ-mean if and only if

(1) $\phi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all $n \in \mathbb{N}$,

(2) $\phi(e) = 1$, where $e = (1, 1, \ldots)$,

(3) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in l_\infty$.

The mappings σ are one to one and such that $\sigma^k(n) \neq n$ for all positive integers n and k, where $\sigma^k(n)$ denotes the kth iterate of the mapping σ at n. Thus ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. In that case σ is translation mapping $n \rightarrow n + 1$, a σ-mean is often called a Banach limit, and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences [5].

If $x = (x_n)$, set $Tx = (Tx_n) = (x_{\sigma(n)})$. It can be shown [6] that

$$V_\sigma = \left\{ x = (x_n) : \lim_m t_{mn}(x) = L\, e \text{ uniformly in } n,\ L = \sigma - \lim x \right\} \tag{1.3}$$

where $t_{mn}(x) = (x_n + Tx_n + \ldots + T^m x_n)/(m+1)$.

Several authors including Schaefer [7], Mursaleen [6], Savas [8], and others have studied invariant convergent sequences.

An Orlicz function is a function $M : [0, \infty) \rightarrow [0, \infty)$, which is continuous, nondecreasing, and convex with $M(0) = 0, M(x) > 0$ for $x > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$. If the convexity of an Orlicz function M is replaced by

$$M(x + y) \leq M(x) + M(y), \tag{1.4}$$

then this function is called modulus function, introduced and investigated by Nakano [9] and followed by Ruckle [10], Maddox [11], and many others.

Lindenstrauss and Tzafriri [12] used the idea of Orlicz function to construct the sequence space

$$l_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\} \tag{1.5}$$

which is called an Orlicz sequence space.

The space l_M becomes a Banach space with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}. \tag{1.6}$$

The space l_M is closely related to the space l_p which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$. Orlicz sequence spaces were introduced and studied by Parashar and Choudhary [13], Bhardwaj and Singh [14], and many others.

It is well known that since M is a convex function and $M(0) = 0$ then $M(tx) \leq tM(x)$ for all t with $0 < t < 1$.

An Orlicz funtion M is said to satisfy Δ_2-condition for all values of u, if there exists constant $K > 0$, such that $M(2u) \leq KM(u)$ $(u \geq 0)$. The Δ_2-condition is equivalent to the inequality $M(Lu) \leq KLM(u)$ for all values of u and for $L > 1$ being satisfied [15].

The notion of paranormed space was introduced by Nakano [16] and Simons [17]. Later on it was investigated by Maddox [18], Lascarides [19], Rath and Tripathy [20], Tripathy and Sen [21], Tripathy [22], and many others.

The following inequality will be used throughout this paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \leq \sup p_k = G$ and let $D = \max(1, 2^{G-1})$. Then for $a_k, b_k \in \mathbb{C}$, the set of complex numbers for all $k \in \mathbb{N}$, we have [23]

$$|a_k + b_k|^{p_k} \leq D\left\{|a_k|^{p_k} + |b_k|^{p_k}\right\}. \tag{1.7}$$

2. Definitions and Notations

A sequence space E is said to be solid (or normal) if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$ and for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.

A sequence space E is said to be symmetric if $(x_k) \in E$ implies $(x_{\pi(k)}) \in E$, where $\pi(k)$ is a permutation of $\mathbb{N}$.

A sequence space E is said to be monotone if it contains the canonical preimages of its step spaces.

Throughout the paper $p = (p_k)$ will represent a sequence of positive real numbers and (X, q) a seminormed space over the field $\mathbb{C}$ of complex numbers with the seminorm q. We define the following sequence spaces:

$$\begin{aligned}
\overline{c}(\sigma, M, p, q, s) = \Bigg\{ &(x_k) \in l_\infty(X) : k^{-s}\left[M\left(q\left(\frac{x_{\sigma^k(n)} - L}{\rho}\right)\right)\right]^{p_k} \xrightarrow{\text{stat}} 0, \\
&\text{as } k \longrightarrow \infty, \text{ uniformly in } n, s \geq 0, \text{ for some } \rho > 0,\ L \in X \Bigg\}, \\
\overline{c}_0(\sigma, M, p, q, s) = \Bigg\{ &(x_k) \in l_\infty(X) : k^{-s}\left[M\left(q\left(\frac{x_{\sigma^k(n)}}{\rho}\right)\right)\right]^{p_k} \xrightarrow{\text{stat}} 0, \\
&\text{as } k \longrightarrow \infty, \text{ uniformly in } n, s \geq 0, \text{ for some } \rho > 0 \Bigg\},
\end{aligned}$$

$$
\begin{aligned}
l_\infty(\sigma,M,p,q,s) = \Bigg\{ &(x_k)\in l_\infty(X) : \sup_{k,n} k^{-s}\left[M\left(q\left(\frac{x_{\sigma^k(n)}}{\rho}\right)\right)\right]^{p_k} < \infty, \\
&s\geq 0,\ \text{ for some } \rho>0\Bigg\}, \\
W(\sigma,M,p,q,s) = \Bigg\{ &(x_k)\in l_\infty(X) : \lim_j \frac{1}{j}\sum_{k=1}^{j} k^{-s}\left[M\left(q\left(\frac{x_{\sigma^k(n)}-L}{\rho}\right)\right)\right]^{p_k} = 0, \\
&\text{uniformly in } n, s\geq 0,\ \text{ for some } \rho>0\Bigg\}.
\end{aligned}
\tag{2.1}
$$

We write

$$
\begin{aligned}
m(\sigma,M,p,q,s) &= \bar{c}(\sigma,M,p,q,s)\cap l_\infty(\sigma,M,p,q,s), \\
m_0(\sigma,M,p,q,s) &= \overline{c_0}(\sigma,M,p,q,s)\cap l_\infty(\sigma,M,p,q,s).
\end{aligned}
\tag{2.2}
$$

If $M(x)=x, q(x)=|x|, s=0, \sigma(n)=n+1$ for each n and $k=0$ then these spaces reduce to the spaces

$$
\begin{aligned}
\bar{c}(p) &= \left\{(x_k)\in w : |x_k-L|^{p_k} \xrightarrow{\text{stat}} 0,\ \text{as } k\longrightarrow\infty,\ L\in X\right\}, \\
\overline{c_0}(p) &= \left\{(x_k)\in w : |x_k|^{p_k} \xrightarrow{\text{stat}} 0,\ \text{as } k\longrightarrow\infty\right\}, \\
l_\infty(p) &= \left\{(x_k)\in w : \sup_k |x_k|^{p_k} < \infty\right\}, \\
m(p) &= \bar{c}(p)\cap l_\infty(p), \\
m_0(p) &= \overline{c_0}(p)\cap l_\infty(p),
\end{aligned}
\tag{2.3}
$$

defined by Tripathy and Sen [21].

Firstly, we give some results; those will help in establishing the results of this paper.

Lemma 2.1 ([21]). *For two sequences (p_k) and (t_k) one has $m_0(p)\supseteq m_0(t)$ if and only if $\liminf_{k\in K}(p_k/t_k)>0$, where $K\subseteq\mathbb{N}$ such that $\delta(K)=1$.*

Lemma 2.2 ([21]). *Let $h=\inf p_k$ and $G=\sup p_k$, then the followings are equivalent:*

(i) *$G<\infty$ and $h>0$,*

(ii) *$m(p)=m$.*

Lemma 2.3 ([21]). *Let $K=\{n_1,n_2,n_3,\ldots\}$ be an infinite subset of $\mathbb{N}$ such that $\delta(K)=0$. Let*

$$
T=\{(x_k) : x_k = 0 \text{ or } 1 \text{ for } k=n_i,\ i\in\mathbb{N} \text{ and } x_k=0, \text{ otherwise}\}.
\tag{2.4}
$$

Then T is uncountable.

Lemma 2.4 ([24]). *If a sequence space E is solid then E is monotone.*

3. Main Results

Theorem 3.1. *$\overline{c}(\sigma, M, p, q, s)$, $\overline{c_0}(\sigma, M, p, q, s)$, $m(\sigma, M, p, q, s)$, $m_0(\sigma, M, p, q, s)$ are linear spaces.*

Proof. Let $(x_k), (y_k) \in \overline{c}(\sigma, M, p, q, s)$. Then there exist ρ_1, ρ_2 positive real numbers and K, $L \in X$ such that

$$\begin{gathered} k^{-s}\left[M\left(q\left(\frac{x_{\sigma^k(n)} - K}{\rho_1}\right)\right)\right]^{p_k} \xrightarrow{\text{stat}} 0, \quad \text{as } k \longrightarrow \infty, \text{ uniformly in } n, \\ k^{-s}\left[M\left(q\left(\frac{y_{\sigma^k(n)} - L}{\rho_2}\right)\right)\right]^{p_k} \xrightarrow{\text{stat}} 0, \quad \text{as } k \longrightarrow \infty, \text{ uniformly in } n. \end{gathered} \tag{3.1}$$

Let α, β be scalars and let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Then by (1.7) we have

$$\begin{aligned} & k^{-s}\left[M\left(q\left(\frac{\alpha x_{\sigma^k(n)} + \beta y_{\sigma^k(n)} - (\alpha K + \beta L)}{\rho_3}\right)\right)\right]^{p_k} \\ & \quad \le k^{-s}\left[M\left(q\left(\frac{x_{\sigma^k(n)} - K}{2\rho_1}\right) + q\left(\frac{y_{\sigma^k(n)} - L}{2\rho_2}\right)\right)\right]^{p_k} \\ & \quad \le k^{-s}\frac{1}{2^{p_k}}\left[M\left(q\left(\frac{x_{\sigma^k(n)} - K}{\rho_1}\right)\right) + M\left(q\left(\frac{y_{\sigma^k(n)} - L}{\rho_2}\right)\right)\right]^{p_k} \\ & \quad \le D\left\{k^{-s}\left[M\left(q\left(\frac{x_{\sigma^k(n)} - K}{\rho_1}\right)\right)\right]^{p_k}\right. \\ & \qquad\qquad \left. + k^{-s}\left[M\left(q\left(\frac{y_{\sigma^k(n)} - L}{\rho_2}\right)\right)\right]^{p_k}\right\} \xrightarrow{\text{stat}} 0 \text{ as } k \longrightarrow \infty, \text{ uniformly in } n. \end{aligned} \tag{3.2}$$

Hence $\overline{c}(\sigma, M, p, q, s)$ is a linear space.

The rest of the cases will follow similarly. □

Theorem 3.2. *The spaces $m(\sigma, M, p, q, s)$ and $m_0(\sigma, M, p, q, s)$ are paranormed spaces, paranormed by*

$$g(x) = \inf\left\{\rho^{p_m/H} : \sup_k k^{-s} M\left(q\left(\frac{x_{\sigma^k(n)}}{\rho}\right)\right) \le 1, \textit{ uniformly in } n, s \ge 0,\ \rho > 0,\ m \in \mathbb{N}\right\}, \tag{3.3}$$

where $H = \max(1, \sup p_k)$.

Proof. We prove the theorem for the space $m_0(\sigma, M, p, q, s)$. The proof for the other space can be proved by the same way. Clearly $g(x) = g(-x)$ for all $x \in m_0(\sigma, M, p, q, s)$ and $g(\theta) = 0$. Let $x, y \in m_0(\sigma, M, p, q, s)$. Then we have $\rho_1, \rho_2 > 0$ such that

$$\begin{aligned} &\sup_k k^{-s} M\left(q\left(\frac{x_{\sigma^k(n)}}{\rho_1}\right)\right) \leq 1, \quad \text{uniformly in } n, \\ &\sup_k k^{-s} M\left(q\left(\frac{y_{\sigma^k(n)}}{\rho_2}\right)\right) \leq 1, \quad \text{uniformly in } n. \end{aligned} \tag{3.4}$$

Let $\rho = \rho_1 + \rho_2$. Then by the convexity of M, we have

$$\begin{aligned} &\sup_k k^{-s} M\left(q\left(\frac{x_{\sigma^k(n)} + y_{\sigma^k(n)}}{\rho}\right)\right) \\ &\quad \leq \sup_k k^{-s} M\left(\frac{\rho_1}{\rho_1+\rho_2} q\left(\frac{x_{\sigma^k(n)}}{\rho_1}\right) + \frac{\rho_2}{\rho_1+\rho_2} q\left(\frac{y_{\sigma^k(n)}}{\rho_2}\right)\right) \\ &\quad \leq \frac{\rho_1}{\rho_1+\rho_2} \sup_k k^{-s} M\left(q\left(\frac{x_{\sigma^k(n)}}{\rho_1}\right)\right) \\ &\qquad + \frac{\rho_2}{\rho_1+\rho_2} \sup_k k^{-s} M\left(q\left(\frac{y_{\sigma^k(n)}}{\rho_2}\right)\right) \leq 1, \quad \text{uniformly in } n. \end{aligned} \tag{3.5}$$

Hence from above inequality, we have

$$\begin{aligned} &g(x+y) \\ &\quad = \inf\left\{\rho^{p_m/H} : \sup_k k^{-s} M\left(q\left(\frac{x_{\sigma^k(n)} + y_{\sigma^k(n)}}{\rho}\right)\right) \leq 1, \right. \\ &\qquad\qquad \left. \text{uniformly in } n, \rho > 0,\ m \in \mathbb{N}\right\} \\ &\quad \leq \inf\left\{\rho_1^{p_m/H} : \sup_k k^{-s} M\left(q\left(\frac{x_{\sigma^k(n)}}{\rho_1}\right)\right) \leq 1, \text{uniformly in } n,\ \rho_1 > 0\right\} \\ &\qquad + \inf\left\{\rho_2^{p_m/H} : \sup_k k^{-s} M\left(q\left(\frac{y_{\sigma^k(n)}}{\rho_1}\right)\right) \leq 1, \text{uniformly in } n,\ \rho_2 > 0\right\} \\ &\quad = g(x) + g(y). \end{aligned} \tag{3.6}$$

For the continuity of scalar multiplication let $\lambda \neq 0$ be any complex number. Then by the definition of g we have

$$\begin{aligned} g(\lambda x) &= \inf\left\{\rho^{p_m/H} : \sup_k k^{-s} M\left(q\left(\frac{\lambda x_{\sigma^k(n)}}{\rho}\right)\right) \leq 1, \text{ uniformly in } n,\ \rho > 0\right\} \\ &= \inf\left\{(r|\lambda|)^{p_m/H} : \sup_k k^{-s} M\left(q\left(\frac{x_{\sigma^k(n)}}{r}\right)\right) \leq 1, \text{ uniformly in } n,\ r > 0\right\}, \end{aligned} \tag{3.7}$$

where $r = \rho/|\lambda|$.

Since $|\lambda|^{p_m} \leq \max(1, |\lambda|^H)$, we have $|\lambda|^{p_m/H} \leq (\max(1, |\lambda|^H))^{1/H}$. Then

$$\begin{aligned} g(\lambda x) \leq \left(\max\left(1, |\lambda|^H\right)\right)^{1/H} \inf\Bigg\{(r)^{p_m/H} &: \sup_k k^{-s} M\left(q\left(\frac{x_{\sigma^k(n)}}{r}\right)\right) \leq 1, \\ &\text{uniformly in } n, r > 0\Bigg\} = (\max(1, |\lambda|^H))^{1/H} \cdot g(x), \end{aligned} \tag{3.8}$$

and therefore $g(\lambda x)$ converges to zero when $g(x)$ converges to zero or λ converges to zero.

Hence the spaces $m(\sigma, M, p, q, s)$ and $m_0(\sigma, M, p, q, s)$ are paranormed by g. □

Theorem 3.3. *Let (X, q) be complete seminormed space, then the spaces $m(\sigma, M, p, q, s)$ and $m_0(\sigma, M, p, q, s)$ are complete.*

Proof. We prove it for the case $m_0(\sigma, M, p, q, s)$ and the other case can be established similarly. Let $x^s = (x^s_{\sigma^k(n)})$ be a Cauchy sequence in $m_0(\sigma, M, p, q, s)$ for all $k, n \in \mathbb{N}$. Then $g(x^i - x^j) \to 0$, as $i, j \to \infty$. For a given $\varepsilon > 0$, let $r > 0$ and $\delta > 0$ to be such that $(\varepsilon/r\delta) > 0$. Then there exists a positive integer N such that

$$g\left(x^i - x^j\right) < \frac{\varepsilon}{r\delta} \quad \forall i, j \geq N. \tag{3.9}$$

Using definition of paranorm we get

$$\inf\left\{\rho^{p_k/H} : \sup_k k^{-s} M\left(q\left(\frac{x^i_{\sigma^k(n)} - x^j_{\sigma^k(n)}}{\rho}\right)\right) \leq 1, \text{ uniformly in } n,\ \rho > 0\right\} < \frac{\varepsilon}{r\delta}. \tag{3.10}$$

Hence x^i is a Cauchy sequence in (X, q). Therefore for each ε $(0 < \varepsilon < 1)$ there exists a positive integer N such that

$$q\left(x^i - x^j\right) < \varepsilon \quad \forall i, j \geq N. \tag{3.11}$$

Using continuity of M, we find that

$$\sup_k k^{-s} M\left(q\left(\frac{x^i - \lim_j x^j}{\rho}\right)\right) \le 1. \tag{3.12}$$

Thus

$$\sup_k k^{-s} M\left(q\left(\frac{x^i - x}{\rho}\right)\right) \le 1. \tag{3.13}$$

Taking infimum of such ρ's we get

$$\inf\left\{\rho^{p_k/H} : \sup_k k^{-s} M\left(q\left(\frac{x^i - x}{\rho}\right)\right) \le 1\right\} < \varepsilon \tag{3.14}$$

for all $i \ge N$ and $j \to \infty$. Since $x^i \in m_0(\sigma, M, p, q, s)$ and M is continuous, it follows that $x \in m_0(\sigma, M, p, q, s)$. This completes the proof of the theorem. □

Theorem 3.4. *Let M_1 and M_2 be two Orlicz functions satisfying Δ_2-condition. Then*

(i) $Z(\sigma, M_1, p, q, s) \subseteq Z(\sigma, M_2 \circ M_1, p, q, s)$,

(ii) $Z(\sigma, M_1, p, q, s) \cap Z(\sigma, M_2, p, q, s) \subseteq Z(\sigma, M_1 + M_2, p, q, s)$,

where $Z = \overline{c}$, m, $\overline{c_0}$, and m_0.

Proof. (i) We prove this part for $Z = \overline{c_0}$ and the rest of the cases will follow similarly. Let $(x_k) \in \overline{c_0}(\sigma, M_1, p, q, s)$. Then for a given $0 < \varepsilon < 1$, there exists $\rho > 0$ such that there exists a subset K of $\mathbb{N}$ with $\delta(K) = 1$, where

$$\begin{gathered} K = \left\{k \in \mathbb{N} : k^{-s}\left[M_1\left(q\left(\frac{x_{\sigma^k(n)}}{\rho}\right)\right)\right]^{p_k} < \frac{\varepsilon}{B}\right\}, \\ B = \max\left(1, \sup\left[M_2\left(\frac{1}{(k^{-s})^{1/p_k}}\right)\right]^{p_k}\right). \end{gathered} \tag{3.15}$$

If we take $a_k = (k^{-s})^{1/p_k} M_1(q(x_{\sigma^k(n)}/\rho))$ then $a_k^{p_k} < (\varepsilon/B) < 1$ implies that $a_k < 1$. Hence we have by convexity of M,

$$(M_2 \circ M_1)\left(q\left(\frac{x_{\sigma^k(n)}}{\rho}\right)\right) = M_2\left(\frac{a_k}{(k^{-s})^{1/p_k}}\right) \le a_k M_2\left(\frac{1}{(k^{-s})^{1/p_k}}\right). \tag{3.16}$$

Thus

$$k^{-s}[M_2(a_k)]^{p_k} \le k^{-s}\left[M_2\left(\frac{a_k}{(k^{-s})^{1/p_k}}\right)\right]^{p_k} \le k^{-s} B (a_k)^{p_k} \le B(a_k)^{p_k} < \varepsilon. \tag{3.17}$$

Hence by (3.15) it follows that for a given $\varepsilon > 0$, there exists $\rho > 0$ such that

$$\delta\left(\left\{k \in \mathbb{N} : k^{-s}\left[M_2\left(M_1\left(q\left(\frac{x_{\sigma^k(n)}}{\rho}\right)\right)\right)\right]^{p_k} < \varepsilon\right\}\right) = 1. \tag{3.18}$$

Therefore $(x_k) \in \overline{c_0}(\sigma, M_2 \circ M_1, p, q, s)$.

(ii) We prove this part for the case $Z = \overline{c_0}$ and the other cases will follow similarly.

Let $(x_k) \in \overline{c_0}(\sigma, M_1, p, q, s) \cap \overline{c_0}(\sigma, M_2, p, q, s)$. Then by using (1.7) it can be shown that $(x_k) \in \overline{c_0}(\sigma, M_1 + M_2, p, q, s)$. Hence

$$\overline{c_0}(\sigma, M_1, p, q, s) \cap \overline{c_0}(\sigma, M_2, p, q, s) \subseteq \overline{c_0}(\sigma, M_1 + M_2, p, q, s). \tag{3.19}$$

This completes the proof. □

Theorem 3.5. *For any sequence $p = (p_k)$ of positive real numbers and for any two seminorms q_1 and q_2 on X one has*

$$Z(\sigma, M, p, q_1, s) \cap Z(\sigma, M, p, q_2, s) \neq \emptyset, \tag{3.20}$$

where $Z = \overline{c}$, m, $\overline{c_0}$, and m_0.

Proof. The proof follows from the fact that the zero sequence belongs to each of the classes the sequence spaces involved in the intersection.

The proof of the following result is easy, so omitted. □

Proposition 3.6. *Let M be an Orlicz function which satisfies Δ_2–condition, and let q_1 and q_2 be two seminorms on X. Then*

(i) $\overline{c_0}(\sigma, M, p, q_1, s) \subseteq \overline{c}(\sigma, M, p, q_1, s)$,

(ii) $m_0(\sigma, M, p, q_1, s) \subseteq m(\sigma, M, p, q_1, s)$,

(iii) $Z(\sigma, M, p, q_1, s) \cap Z(\sigma, M, p, q_2, s) \subseteq Z(\sigma, M, p, q_1 + q_2, s)$ *where $Z = \overline{c}$, m, $\overline{c_0}$, and m_0,*

(iv) *if q_1 is stronger than q_2, then*

$$Z(\sigma, M, p, q_1, s) \subseteq Z(\sigma, M, p, q_2, s), \tag{3.21}$$

where $Z = \overline{c}$, m, $\overline{c_0}$, and m_0.

Theorem 3.7. *The spaces $Z(\sigma, M, p, q, s)$ are not solid, where $Z = \overline{c}$ and m.*

Proof. To show that the spaces are not solid in general, consider the following example. Let $M(x) = x^p (1 \leq p < \infty)$, $p_k = (1/p)$ for all k, $q(x) = \sup_i |x^i|$, where $x = (x^i) \in l_\infty$ and $\sigma(n) = n + 1$ for all $n \in \mathbb{N}$. Then we have $\sigma^k(n) = n + k$ for all $k, n \in \mathbb{N}$. Consider the sequence (x_k), where $x_k = (x_k^i) \in l_\infty$is defined by $(x_k^i) = (k, k, k, \ldots), k = i^2, i \in \mathbb{N}$ and $(x_k^i) = (2, 2, 2, \ldots), k \neq i^2, i \in \mathbb{N}$ for each fixed $k \in \mathbb{N}$. Hence $(x_k) \in Z(\sigma, M, p, q, s)$ for $Z = \overline{c}$ and m. Let $\alpha_k = (1, 1, 1, \ldots)$ if k is odd and $\alpha_k = \theta$, otherwise. Then $(\alpha_k x_k) \notin Z(\sigma, M, p, q, s)$ for $Z = \overline{c}$ and m. Thus $Z(\sigma, M, p, q, s)$ is not solid for $Z = \overline{c}$ and m. □

The proof of the following result is obvious in view of Lemma 2.4.

Proposition 3.8. *The space* $Z(\sigma, M, p, q, s)$ *is solid as well as monotone for* $Z = \overline{c_0}$ *and* m_0.

Theorem 3.9. *The spaces* $Z(\sigma, M, p, q, s)$ *are not symmetric, where* $Z = \overline{c}$, m, $\overline{c_0}$, *and* m_0.

Proof. To show that the spaces are not symmetric, consider the following examples. Let $M(x) = x^p (1 \le p < \infty)$, $p_k = (1/p)$ for all k, $q(x) = \sup_i |x^i|$, where $x = (x^i) \in l_\infty$ and $\sigma(n) = n + 1$ for all $n \in \mathbb{N}$. Then we have $\sigma^k(n) = n + k$ for all $k \in \mathbb{N}$. We consider the sequence (x_k) defined by $x_k = (1, 1, 1, \ldots)$ if $k = i^2$, $i \in \mathbb{N}$, and $x_k = \theta$, otherwise. Then $(x_k) \in Z(\sigma, M, p, q, s)$ for $Z = \overline{c_0}$ and m_0. Let (y_k) be a rearrangement of (x_k), which is defined as $y_k = (1, 1, 1, \ldots)$ if k is odd and $y_k = \theta$, otherwise. Then $(y_k) \notin Z(\sigma, M, p, q, s)$ for $Z = \overline{c_0}$ and m_0.

To show for $Z = \overline{c}$ and m, let $p_k = 1$ for all k odd and $p_k = 2^{-1}$ for all k even. Let $X = \mathbb{R}^3$ and $q(x) = \max\{|x^1|, |x^2|, |x^3|\}$, where $x = (x^1, x^2, x^3) \in \mathbb{R}^3$. Let $M(x) = x^4$ and $\sigma(n) = n + 1$ for all $n \in \mathbb{N}$. Then we have $\sigma^k(n) = n + k$ for all $k, n \in \mathbb{N}$. We consider

$$(x_k) = \begin{cases} (1,1,1), & i^2 \le k < i^2 + 2i - 1, \ i \in \mathbb{N}, \\ (3,-3,5), & \text{otherwise}. \end{cases} \tag{3.22}$$

Then $(x_k) \in Z(\sigma, M, p, q, s)$ for $Z = \overline{c}$ and m. We consider the rearrengement (y_k) of (x_k) as

$$(y_k) = \begin{cases} (1,1,1), & k \text{ is odd}, \\ (3,-3,5), & k \text{ is even}. \end{cases} \tag{3.23}$$

Then $(y_k) \notin Z(\sigma, M, p, q, s)$ for $Z = \overline{c}$ and m. Thus the spaces $Z(\sigma, M, p, q, s)$ are not symmetric in general, where $Z = \overline{c}$, m, $\overline{c_0}$ and m_0. □

Proposition 3.10. *For two sequences* (p_k) *and* (t_k) *one has* $m_0(\sigma, M, p, q, s) \supseteq m_0(\sigma, M, t, q, s)$ *if and only if* $\liminf_{k \in K} (p_k)/(t_k) > 0$, *where* $K \subseteq \mathbb{N}$ *such that* $\delta(K) = 1$.

Proof. The proof is obvious in view of Lemma 2.1. □

The following result is a consequence of the above result.

Corollary 3.11. *For two sequences* (p_k) *and* (t_k) *one has* $m_0(\sigma, M, p, q, s) = m_0(\sigma, M, t, q, s)$ *if and only if* $\liminf_{k \in K} (p_k)/(t_k) > 0$ *and* $\liminf_{k \in K} (t_k)/(p_k) > 0$, *where* $K \subseteq \mathbb{N}$ *such that* $\delta(K) = 1$.

The following result is obvious in view of Lemma 2.2.

Proposition 3.12. *Let* $h = \inf p_k$ *and* $G = \sup p_k$, *then the followings are equivalent:*

(i) $G < \infty$ *and* $h > 0$,

(ii) $m(\sigma, M, p, q, s) = m(\sigma, M, q, s)$.

Theorem 3.13. *Let* $p = (p_k)$ *be a sequence of nonnegative bounded real numbers such that* $\inf p_k > 0$. *Then*

$$m(\sigma, M, p, q, s) = W(\sigma, M, p, q, s) \cap l_\infty(\sigma, M, p, q, s). \tag{3.24}$$

Proof. Let $(x_k) \in W(\sigma, M, p, q, s) \cap l_\infty(\sigma, M, p, q, s)$. Then for a given $\varepsilon > 0$, we have

$$\sum_{k=1}^{j} k^{-s}\left[M\left(q\left(\frac{x_{\sigma^k(n)} - L}{\rho}\right)\right)\right]^{p_k} \geq \left|\left\{k \leq j : k^{-s}\left[M\left(q\left(\frac{x_{\sigma^k(n)} - L}{\rho}\right)\right)\right]^{p_k} \geq \varepsilon\right\}\right| \cdot \varepsilon, \tag{3.25}$$

where the vertical bar indicates the number of elements in the enclosed set.

From the above inequality it follows that $(x_k) \in m(\sigma, M, p, q, s)$.

Conversely let $(x_k) \in m(\sigma, M, p, q, s)$. Let $\rho > 0$ such that

$$k^{-s}\left[M\left(q\left(\frac{x_{\sigma^k(n)} - L}{\rho}\right)\right)\right]^{p_k} \xrightarrow{\text{stat}} 0, \quad \text{as } k \longrightarrow \infty, \text{uniformly in } n. \tag{3.26}$$

For a given $\varepsilon > 0$, let $B = \sup_k (k^{-s}[M(q(x_{\sigma^k(n)} - L/\rho))]^{p_k})^{1/h} < \infty$.

Let $L_j = \{k \leq j : k^{-s}[M(q(x_{\sigma^k(n)} - L/\rho))]^{p_k} \geq \varepsilon/2\}$.

Since $(x_k) \in m(\sigma, M, p, q, s)$, so $|\{L_j\}|/j \to 0$, uniformly in n, as $j \to \infty$. There exits a positive integer n_0 such that $|\{L_j\}|/j < \varepsilon/2B^h$ for all $j > n_0$. Then for all $j > n_0$, we have

$$\begin{aligned} \frac{1}{j}\sum_{k=1}^{j} k^{-s}&\left[M\left(q\left(\frac{x_{\sigma^k(n)} - L}{\rho}\right)\right)\right]^{p_k} \\ &= \frac{1}{j}\sum_{k \notin L_j} k^{-s}\left[M\left(q\left(\frac{x_{\sigma^k(n)} - L}{\rho}\right)\right)\right]^{p_k} \\ &\quad + \frac{1}{j}\sum_{k \in L_j} k^{-s}\left[M\left(q\left(\frac{x_{\sigma^k(n)} - L}{\rho}\right)\right)\right]^{p_k} \\ &\leq \frac{j - |\{L_j\}|}{j}\frac{\varepsilon}{2} + \frac{|\{L_j\}|}{j}B^h < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \tag{3.27}$$

Hence $(x_k) \in W(\sigma, M, p, q, s) \cap l_\infty(\sigma, M, p, q, s)$.

This completes the proof of the theorem. □

The following result is a consequence of the above theorem.

Corollary 3.14. *Let* (p_k) *and* (t_k) *be two bounded sequences of real numbers such that* $\inf p_k > 0$ *and* $\inf t_k > 0$. *Then*

$$W(\sigma, M, p, q, s) \cap l_\infty(\sigma, M, p, q, s) = W(\sigma, M, t, q, s) \cap l_\infty(\sigma, M, t, q, s). \tag{3.28}$$

Since the inclusion relations $m(\sigma, M, p, q, s) \subset l_\infty(\sigma, M, p, q, s)$ and $m_0(\sigma, M, p, q, s) \subset l_\infty(\sigma, M, p, q, s)$ are strict, we have the following result.

Corollary 3.15. *The spaces $m(\sigma, M, p, q, s)$ and $m_0(\sigma, M, p, q, s)$ are nowhere dense subsets of $l_\infty(\sigma, M, p, q, s)$.*

The following result is obvious in view of Lemma 2.3.

Proposition 3.16. *The spaces $m(\sigma, M, p, q, s)$ and $m_0(\sigma, M, p, q, s)$ are not separable.*

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