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Research Article

Strong and Δ Convergence Theorems for Multivalued Mappings in CAT(0) Spaces

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We show strong and Δ convergence for Mann iteration of a multivalued nonexpansive mapping whose domain is a nonempty closed convex subset of a CAT(0) space. The results we obtain are analogs of Banach space results by Song and Wang [2009, 2008]. Strong convergence of Ishikawa iteration are also included.

1. Introduction

Let K be a nonempty subset of a Banach space X. We shall denote by $\mathcal{CB}(K)$ the family of nonempty closed bounded subsets of K, by $\mathcal{P}(K)$ the family of nonempty bounded proximinal subsets of K, and by $\mathcal{K}(K)$ the family of nonempty compact subsets of K. Let $H(\cdot,\cdot)$ be the Hausdorff distance on $\mathcal{CB}(X)$, that is,

$$H(A,B)=\max\left\{\sup_{a\in A}\operatorname{dist}(a,B),\sup_{b\in B}\operatorname{dist}(b,A)\right\},\quad A,B\in\mathcal{CB}(X), \tag{1.1}$$

where $\operatorname{dist}(a,B)=\inf\{d(a,b):b\in B\}$ is the distance from the point a to the set B.

A multivalued mapping $T:K\to\mathcal{CB}(X)$ is said to be a *nonexpansive* if

$$H(Tx,Ty)\le d(x,y)\quad\forall x,y\in K. \tag{1.2}$$

A point x is called a *fixed point* of T if $x\in Tx$. We denote by $F(T)$ the set of all fixed points of T.

In 2005, Sastry and Babu [1] introduced the Mann and Ishikawa iterations for multivalued mappings as follows: let X be a real Hilbert space and $T : X \to \mathcal{P}(X)$ be a multivalued mapping for which $F(T) \neq \emptyset$. Fix $p \in F(T)$ and define

(A) the sequence of Mann iterates by $x_0 \in X$,

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n, \quad \alpha_n \in [0,1],\ n \geq 0 \tag{1.3}$$

where $y_n \in Tx_n$ is such that $\|y_n - p\| = \operatorname{dist}(p, Tx_n)$,

(B) the sequence of Ishikawa iterates by $x_0 \in X$,

$$y_n = (1-\beta_n) x_n + \beta_n z_n, \quad \beta_n \in [0,1],\ n \geq 0 \tag{1.4}$$

where $z_n \in Tx_n$ is such that $\|z_n - p\| = \operatorname{dist}(p, Tx_n)$, and

$$x_{n+1} = (1-\alpha_n) x_n + \alpha_n z_n', \quad \alpha_n \in [0,1], \tag{1.5}$$

where $z_n' \in Ty_n$ is such that $\|z_n' - p\| = \operatorname{dist}(p, Ty_n)$.

They proved the following results.

Theorem 1.1. *Let K be a nonempty compact convex subset of a Hilbert space X. Suppose $T : K \to \mathcal{P}(K)$ is nonexpansive and has a fixed point p. Assume that* (i) $0 \leq \alpha_n < 1$ *and* (ii) $\sum \alpha_n = \infty$.*Then the sequence of Mann iterates defined by* (A) *converges to a fixed point q of T.*

Theorem 1.2. *Let K be a nonempty compact convex subset of a Hilbert space X. Suppose that a nonexpansive map $T : K \to \mathcal{P}(K)$ has a fixed point p. Assume that* (i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\lim_n \beta_n = 0$, *and* (iii)$\sum \alpha_n \beta_n = \infty$. *Then the sequence of Ishikawa iterates defined by (B) converges to a fixed point q of T.*

In 2007, Panyanak [2] extended Sastry-Babu's results to uniformly convex Banach spaces as the following results.

Theorem 1.3. *Let K be a nonempty compact convex subset of a uniformly convex Banach spaces X. Suppose that a nonexpansive map $T : K \to \mathcal{P}(K)$ has a fixed point p. Let $\{x_n\}$ be the sequence of Mann iterates defined by (A). Assume that* (i) $0 \leq \alpha_n < 1$ *and* (ii) $\sum \alpha_n = \infty$.*Then the sequence $\{x_n\}$ converges to a fixed point of T.*

Theorem 1.4. *Let K be a nonempty compact convex subset of a uniformly convex Banach spaces X. Suppose that a nonexpansive map $T : K \to \mathcal{P}(K)$ has a fixed point p. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by (B). Assume that* (i) $0 \leq \alpha_n, \beta_n < 1$, (ii) $\lim_n \beta_n = 0$, *and* (iii) $\sum \alpha_n \beta_n = \infty$. *Then the sequence $\{x_n\}$ converges to a fixed point of T.*

Recently, Song and Wang [3, 4] pointed out that the proof of Theorem 1.4 contains a gap. Namely, the iterative sequence $\{x_n\}$ defined by (B) depends on the fixed point p. Clearly, if $q \in F(T)$ and $q \neq p$, then the sequence $\{x_n\}$ defined by q is different from the one defined by p. Thus, for $\{x_n\}$ defined by p, we cannot obtain that $\{\|x_n - q\|\}$ is a decreasing sequence from the monotony of $\{\|x_n - p\|\}$. Hence, the conclusion of Theorem 1.4 (also Theorem 1.3) is very dubious.

Motivated by solving the above gap, they defined the modified Mann and Ishikawa iterations as follows.

Let K be a nonempty convex subset of a Banach space $(X, \|\cdot\|)$ and $T : K \to \mathcal{CB}(K)$ be a multivalued mapping. *The sequence of Mann iterates* is defined as follows: let $\alpha_n \in [0,1]$ and $\gamma_n \in (0, +\infty)$ such that $\lim_{n\to\infty}\gamma_n = 0$. Choose $x_0 \in K$ and $y_0 \in Tx_0$. Let

$$x_1 = (1-\alpha_0)x_0 + \alpha_0 y_0. \tag{1.6}$$

There exists $y_1 \in Tx_1$ such that $d(y_1, y_0) \leq H(Tx_1, Tx_0) + \gamma_0$ (see [5, 6]). Take

$$x_2 = (1-\alpha_1)x_1 + \alpha_1 y_1. \tag{1.7}$$

Inductively, we have

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n, \tag{1.8}$$

where $y_n \in Tx_n$ such that $d(y_{n+1}, y_n) \leq H(Tx_{n+1}, Tx_n) + \gamma_n$.

The sequence of Ishikawa iterates is defined as follows: let $\beta_n \in [0,1]$, $\alpha_n \in [0,1]$ and $\gamma_n \in (0, +\infty)$ such that $\lim_{n\to\infty}\gamma_n = 0$. Choose $x_0 \in K$ and $z_0 \in Tx_0$. Let

$$y_0 = (1-\beta_0)x_0 + \beta_0 z_0. \tag{1.9}$$

There exists $z_0' \in Ty_0$ such that $d(z_0, z_0') \leq H(Tx_0, Ty_0) + \gamma_0$. Let

$$x_1 = (1-\alpha_0)x_0 + \alpha_0 z_0'. \tag{1.10}$$

There is $z_1 \in Tx_1$ such that $d(z_1, z_0') \leq H(Tx_1, Ty_0) + \gamma_1$. Take

$$y_1 = (1-\beta_1)x_1 + \beta_1 z_1. \tag{1.11}$$

There exists $z_1' \in Ty_1$ such that $d(z_1, z_1') \leq H(Tx_1, Ty_1) + \gamma_1$. Let

$$x_2 = (1-\alpha_1)x_1 + \alpha_1 z_1'. \tag{1.12}$$

Inductively, we have

$$y_n = (1-\beta_n)x_n + \beta_n z_n, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n z_n', \tag{1.13}$$

where $z_n \in Tx_n$ and $z_n' \in Ty_n$ such that $d(z_n, z_n') \leq H(Tx_n, Ty_n) + \gamma_n$ and $d(z_{n+1}, z_n') \leq H(Tx_{n+1}, Ty_n) + \gamma_n$.

They obtained the following results.

Theorem 1.5 (see [3, Theorem 2.3]). *Let K be a nonempty compact convex subset of a Banach space X. Suppose that $T : K \to \mathcal{CB}(K)$ is a multivalued nonexpansive mapping for which $F(T) \neq \emptyset$ and $T(y) = \{y\}$ for each $y \in F(T)$. Let $\{x_n\}$ be the sequence of Mann iteration defined by* (1.8). *Assume that*

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1. \tag{1.14}$$

Then the sequence $\{x_n\}$ strongly converges to a fixed point of T.

Recall that a multivalued mapping $T : K \to \mathcal{CB}(\mathrm{K})$ is said to satisfy Condition I ([7]) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that

$$\operatorname{dist}(x, Tx) \geq f(\operatorname{dist}(x, F(T))) \quad \forall x \in K. \tag{1.15}$$

Theorem 1.6 (see [3, Theorem 2.4]). *Let K be a nonempty closed convex subset of a Banach space X. Suppose that $T : K \to \mathcal{CB}(K)$ is a multivalued nonexpansive mapping that satisfies Condition I. Let $\{x_n\}$ be the sequence of Mann iteration defined by* (1.8). *Assume that $F(T) \neq \emptyset$ and satisfies $T(y) = \{y\}$ for each $y \in F(T)$ and*

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1. \tag{1.16}$$

Then the sequence $\{x_n\}$ strongly converges to a fixed point of T.

Theorem 1.7 (see [3, Theorem 2.5]). *Let X be a Banach space satisfying Opial's condition and K be a nonempty weakly compact convex subset of X. Suppose that $T : K \to \mathcal{K}(K)$ is a multivalued nonexpansive mapping. Let $\{x_n\}$ be the sequence of Mann iteration defined by* (1.8). *Assume that $F(T) \neq \emptyset$ and satisfies $T(y) = \{y\}$ for each $y \in F(T)$ and*

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1. \tag{1.17}$$

Then the sequence $\{x_n\}$ weakly converges to a fixed point of T.

Theorem 1.8 (see [4, Theorem 1]). *Let K be a nonempty compact convex subset of a uniformly convex Banach space X. Suppose that $T : K \to \mathcal{CB}(K)$ is a multivalued nonexpansive mapping and $F(T) \neq \emptyset$ satisfying $T(y) = \{y\}$ for any fixed point $y \in F(T)$. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by* (1.13). *Assume that* (i) $\alpha_n, \beta_n \in [0,1)$; (ii) $\lim_{n\to\infty}\beta_n = 0$ *and* (iii) $\sum_{n=0}^{\infty}\alpha_n\beta_n = \infty$.*Then the sequence $\{x_n\}$ strongly converges to a fixed point of T.*

Theorem 1.9 (see [4, Theorem 2]). *Let K be a nonempty closed convex subset of a uniformly convex Banach space X. Suppose that $T : K \to \mathcal{CB}(K)$ is a multivalued nonexpansive mapping that satisfy Condition I. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by* (1.13). *Assume that $F(T) \neq \emptyset$ satisfying $T(y) = \{y\}$ for any fixed point $y \in F(T)$ and $\alpha_n, \beta_n \in [a,b] \subset (0,1)$. Then the sequence $\{x_n\}$ strongly converges to a fixed point of T.*

In this paper, we study the iteration processes defined by (1.8) and (1.13) in a CAT(0) space and give analogs of Theorems 1.5–1.9 in this setting.

2. CAT(0) Spaces

A metric space X is a CAT(0) space if it is geodesically connected, and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane. The precise definition is given below. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces, $\mathbb{R}$-trees (see [8]), Euclidean buildings (see [9]), the complex Hilbert ball with a hyperbolic metric (see [10]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry (see Bridson and Haefliger [8]). Burago, et al. [11] contains a somewhat more elementary treatment, and Gromov [12] a deeper study.

Fixed point theory in a CAT(0) space was first studied by Kirk (see [13] and [14]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed and many of papers have appeared (see, e.g., [15–24]). It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(κ) space with $\kappa \leq 0$ since any CAT(κ) space is a CAT(κ') space for every $\kappa' \geq \kappa$ (see [8], page 165).

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic space (X, d) consists of three points x_1, x_2, x_3 in X (the *vertices* of Δ) and a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom.

CAT(0): let Δ be a geodesic triangle in X and let $\overline{\Delta}$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) *inequality* if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}). \tag{2.1}$$

Let $x, y \in X$, by [24, Lemma 2.1(iv)] for each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = t d(x, y), \qquad d(y, z) = (1 - t) d(x, y). \tag{2.2}$$

From now on we will use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (2.2). By using this notation Dhompongsa and Panyanak [24] obtained the following lemma which will be used frequently in the proof of our main theorems.

Lemma 2.1. *Let* X *be a* CAT (0) *space . Then*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z) \tag{2.3}$$

for all $x, y, z \in X$ *and* $t \in [0,1]$.

If x, y_1, y_2 are points in a CAT(0) space and if $y_0 = 1/2y_1 \oplus 1/2y_2$ then the CAT(0) inequality implies

$$d(x,y_0)^2 \leq \frac{1}{2}d(x,y_1)^2 + \frac{1}{2}d(x,y_2)^2 - \frac{1}{4}d(y_1,y_2)^2. \tag{2.4}$$

This is the (CN) inequality of Bruhat and Tits [25]. In fact (cf. [8, page 163]), a geodesic metric space is a CAT(0) space if and only if it satisfies (CN).

The following lemma is a generalization of the (CN) inequality which can be found in [24].

Lemma 2.2. *Let* (X,d) *be a* CAT(0) *space. Then*

$$d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2 \tag{2.5}$$

for all $t \in [0,1]$ *and* $x, y, z \in X$.

The preceding facts yield the following result.

Proposition 2.3. *Let* X *be a geodesic space. Then the following are equivalent:*

(i) X *is a* CAT (0) *space;*

(ii) X *satisfies (CN);*

(iii) X *satisfies* (2.5).

The existence of fixed points for multivalued nonexpansive mappings in a CAT(0) space was proved by S. Dhompongsa et al. [17], as follows.

Theorem 2.4. *Let* K *be a closed convex subset of a complete* CAT(0) *space* X, *and let* $T : K \to \mathcal{K}(X)$ *be a nonexpansive nonself-mapping. Suppose*

$$\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0 \tag{2.6}$$

for some bounded sequence $\{x_n\}$ *in* K. *Then* T *has a fixed point.*

3. The Setting

Let $(X, \|\cdot\|)$ be a Banach space, and let $\{x_n\}$ be a bounded sequence in X, for $x \in X$ we let

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \|x - x_n\|. \tag{3.1}$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}, \tag{3.2}$$

and the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}. \tag{3.3}$$

The notion of asymptotic centers in a Banach space $(X, \|\cdot\|)$ can be extended to a CAT(0) space (X, d) as well, simply replacing $\|\cdot\|$ with $d(\cdot,\cdot)$. It is known (see, e.g., [18, Proposition 7]) that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

Next we provide the definition and collect some basic properties of Δ-convergence.

Definition 3.1 (see [23]). A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case one must write $\Delta\text{-}\lim_n x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Remark 3.2. In a CAT(0) space X, strong convergence implies Δ-convergence and they are coincided when X is a Hilbert space. Indeed, we prove a much more general result. Recall that a Banach space is said to satisfy *Opial's condition* ([26]) if given whenever $\{x_n\}$ converges weakly to $x \in X$,

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\| \quad \text{for each } y \in X \text{ with } y \neq x. \tag{3.4}$$

Proposition 3.3. *Let X be a reflexive Banach space satisfying Opial's condition and let $\{x_n\}$ be a bounded sequence in X and let $x \in X$. Then $\{x_n\}$ converges weakly to x if and only if $A(\{u_n\}) = \{x\}$ for all subsequence $\{u_n\}$ of $\{x_n\}$.*

Proof. ($\Rightarrow$) Let $\{u_n\}$ be a subsequence of $\{x_n\}$. Then $\{u_n\}$ converges weakly to x. By Opial's condition $A(\{u_n\}) = \{x\}$. ($\Leftarrow$) Suppose $A(\{u_n\}) = \{x\}$ for all subsequence $\{u_n\}$ of $\{x_n\}$ and assume that $\{x_n\}$ does not converge weakly to x. Then there exists a subsequence $\{z_n\}$ of $\{x_n\}$ such that for each n, z_n is outside a weak neighborhood of x. Since $\{z_n\}$ is bounded, without loss of generality we may assume that $\{z_n\}$ converges weakly to $z \neq x$. By Opial's condition $A(\{z_n\}) = \{z\} \neq \{x\}$, a contradiction. □

Lemma 3.4. (i) *Every bounded sequence in X has a Δ-convergent subsequence* (see [23, page 3690]). (ii) *If C is a closed convex subset of X and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C* (see [17, Proposition 2.1]).

Now, we define the sequences of Mann and Ishikawa iterates in a CAT(0) space which are analogs of the two defined in Banach spaces by Song and Wang [3, 4].

Definition 3.5. Let K be a nonempty convex subset of a CAT(0) space X and $T : K \to \mathcal{CB}(K)$ be a multivalued mapping. *The sequence of Mann iterates* is defined as follows: let $\alpha_n \in [0,1]$ and $\gamma_n \in (0,+\infty)$ such that $\lim_{n\to\infty}\gamma_n = 0$. Choose $x_0 \in K$ and $y_0 \in Tx_0$. Let

$$x_1 = (1-\alpha_0)x_0 \oplus \alpha_0 y_0. \tag{3.5}$$

There exists $y_1 \in Tx_1$ such that $d(y_1, y_0) \le H(Tx_1, Tx_0) + \gamma_0$. Take

$$x_2 = (1-\alpha_1)x_1 \oplus \alpha_1 y_1. \tag{3.6}$$

Inductively, we have

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n y_n, \tag{3.7}$$

where $y_n \in Tx_n$ such that $d(y_{n+1}, y_n) \le H(Tx_{n+1}, Tx_n) + \gamma_n$.

Definition 3.6. Let K be a nonempty convex subset of a CAT(0) space X and $T : K \to \mathcal{CB}(K)$ be a multivalued mapping. *The sequence of Ishikawa iterates* is defined as follows: let $\beta_n \in [0,1]$, $\alpha_n \in [0,1]$ and $\gamma_n \in (0,+\infty)$ such that $\lim_{n\to\infty}\gamma_n = 0$. Choose $x_0 \in K$ and $z_0 \in Tx_0$. Let

$$y_0 = (1-\beta_0)x_0 \oplus \beta_0 z_0. \tag{3.8}$$

There exists $z'_0 \in Ty_0$ such that $d(z_0, z'_0) \le H(Tx_0, Ty_0) + \gamma_0$. Let

$$x_1 = (1-\alpha_0)x_0 \oplus \alpha_0 z'_0. \tag{3.9}$$

There is $z_1 \in Tx_1$ such that $d(z_1, z'_0) \le H(Tx_1, Ty_0) + \gamma_1$. Take

$$y_1 = (1-\beta_1)x_1 \oplus \beta_1 z_1. \tag{3.10}$$

There exists $z'_1 \in Ty_1$ such that $d(z_1, z'_1) \le H(Tx_1, Ty_1) + \gamma_1$. Let

$$x_2 = (1-\alpha_1)x_1 \oplus \alpha_1 z'_1. \tag{3.11}$$

Inductively, we have

$$y_n = (1-\beta_n)x_n \oplus \beta_n z_n, \qquad x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n z'_n, \tag{3.12}$$

where $z_n \in Tx_n$ and $z'_n \in Ty_n$ such that $d(z_n, z'_n) \le H(Tx_n, Ty_n) + \gamma_n$ and $d(z_{n+1}, z'_n) \le H(Tx_{n+1}, Ty_n) + \gamma_n$.

Lemma 3.7. *Let K be a nonempty compact convex subset of a complete* CAT (0) *space X, and let $T : K \to \mathcal{CB}(X)$ be a nonexpansive nonself-mapping. Suppose that*

$$\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0 \tag{3.13}$$

for some sequence $\{x_n\}$ *in* K. *Then* T *has a fixed point. Moreover, if* $\{d(x_n, y)\}$ *converges for each* $y \in F(T)$, *then* $\{x_n\}$ *strongly converges to a fixed point of* T.

Proof. By the compactness of K, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to q \in K$. Thus

$$\operatorname{dist}(q, Tq) \le d(q, x_{n_k}) + \operatorname{dist}(x_{n_k}, Tx_{n_k}) + H(Tx_{n_k}, Tq) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{3.14}$$

This implies that q is a fixed point of T. Since the limit of $\{d(x_n, q)\}$ exists and $\lim_{k\to\infty} d(x_{n_k}, q) = 0$, we have $\lim_{n\to\infty} d(x_n, q) = 0$. This show that the sequence $\{x_n\}$ strongly converges to $q \in F(T)$. □

Before proving our main results we state a lemma which is an analog of Lemma 2.2 of [27]. The proof is metric in nature and carries over to the present setting without change.

Lemma 3.8. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a* CAT (0)*space* X *and let* $\{\alpha_n\}$ *be a sequence in* $[0, 1]$ *with* $0 < \liminf_n \alpha_n \le \limsup_n \alpha_n < 1$. *Suppose that* $x_{n+1} = \alpha_n y_n \oplus (1-\alpha_n) x_n$ *for all* $n \in \mathbb{N}$ *and*

$$\limsup_{n\to\infty} \left(d(y_{n+1}, y_n) - d(x_{n+1}, x_n) \right) \le 0. \tag{3.15}$$

Then $\lim_n d(x_n, y_n) = 0$.

4. Strong and Δ Convergence of Mann Iteration

Theorem 4.1. *Let* K *be a nonempty compact convex subset of a complete* CAT (0)*space* X. *Suppose that* $T : K \to \mathcal{CB}(K)$ *is a multivalued nonexpansive mapping and* $F(T) \neq \emptyset$ *satisfying* $Ty = \{y\}$ *for any fixed point* $y \in F(T)$. *If* $\{x_n\}$ *is the sequence of Mann iterates defined by* (3.7) *such that one of the following two conditions is satisfied:*

(i) $\alpha_n \in [0, 1)$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_n \alpha_n \le \limsup_n \alpha_n < 1$.

Then the sequence $\{x_n\}$ *strongly converges to a fixed point of* T.

Proof

Case 1. Suppose that (i) is satisfied. Let $p \in F(T)$, by Lemma 2.2 and the nonexpansiveness of T, we have

$$\begin{aligned} d(x_{n+1}, p)^2 &\le (1-\alpha_n) d(x_n, p)^2 + \alpha_n d(y_n, p)^2 - \alpha_n (1-\alpha_n) d(x_n, y_n)^2 \\ &\le (1-\alpha_n) d(x_n, p)^2 + \alpha_n (H(Tx_n, Tp))^2 - \alpha_n (1-\alpha_n) d(x_n, y_n)^2 \\ &\le (1-\alpha_n) d(x_n, p)^2 + \alpha_n d(x_n, p)^2 - \alpha_n (1-\alpha_n) d(x_n, y_n)^2 \\ &= d(x_n, p)^2 - \alpha_n (1-\alpha_n) d(x_n, y_n)^2. \end{aligned} \tag{4.1}$$

This implies

$$d(x_{n+1}, p)^2 \le d(x_n, p)^2, \tag{4.2}$$

$$\alpha_n(1-\alpha_n)d(x_n, y_n)^2 \le d(x_n, p)^2 - d(x_{n+1}, p)^2. \tag{4.3}$$

It follows from (4.2) that $d(x_n, p) \le d(x_1, p)$ for all $n \ge 1$. This implies that $\{d(x_n, p)\}_{n=1}^{\infty}$ is bounded and decreasing. Hence $\lim_n d(x_n, p)$ exists for all $p \in F(T)$. On the other hand, (4.3) implies

$$\sum_{n=0}^{\infty}\alpha_n(1-\alpha_n)d(x_n, y_n)^2 \le d(x_1, p)^2 < \infty. \tag{4.4}$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges, we have $\liminf_n d(x_n, y_n)^2 = 0$ and hence $\liminf_n d(x_n, y_n) = 0$. Then there exists a subsequence $\{d(x_{n_k}, y_{n_k})\}$ of $\{d(x_n, y_n)\}$ such that

$$\lim_{k\to\infty} d(x_{n_k}, y_{n_k}) = 0. \tag{4.5}$$

This implies

$$\lim_{k\to\infty} \operatorname{dist}(x_{n_k}, Tx_{n_k}) = 0. \tag{4.6}$$

By Lemma 3.7, $\{x_{n_k}\}$ converges to a point $q \in F(T)$. Since the limit of $\{d(x_n, q)\}$ exists, it must be the case

that $\lim_{n\to\infty} d(x_n, q) = 0$, and hence the conclusion follows.

Case 2. If (ii) is satisfied. As in the Case 1, $\lim_n d(x_n, p)$ exists for each $p \in F(T)$. It follows from the definition of Mann iteration (3.7) that

$$\begin{aligned} d(y_{n+1}, y_n) &\le H(Tx_{n+1}, Tx_n) + \gamma_n \\ &\le d(x_{n+1}, x_n) + \gamma_n. \end{aligned} \tag{4.7}$$

Therefore,

$$\limsup_{n\to\infty}\left(d(y_{n+1}, y_n) - d(x_{n+1}, x_n)\right) \le \limsup_{n\to\infty}\gamma_n = 0. \tag{4.8}$$

By Lemma 3.8, we obtain

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{4.9}$$

This implies

$$\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0, \tag{4.10}$$

so the conclusion follows from Lemma 3.7.

Theorem 4.2. *Let K be a nonempty closed convex subset of a complete* CAT(0) *space X. Suppose that $T : K \to \mathcal{CB}(K)$ is a multivalued nonexpansive mapping that satisfies Condition I. Let $\{x_n\}$ be the sequence of Mann iterates defined by* (3.7)*. Assume that $F(T) \neq \emptyset$ satisfying $Ty = \{y\}$ for any fixed point $y \in F(T)$ and $\alpha_n \in [a, b] \subset (0, 1)$. Then the sequence $\{x_n\}$ strongly converges to a fixed point of T.*

Proof. It follows from the proof of the Case 1 in Theorem 4.1 that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F(T)$ and

$$\alpha_n(1-\alpha_n)d(x_n, y_n)^2 \le d(x_n, p)^2 - d(x_{n+1}, p)^2. \tag{4.11}$$

Then

$$a(1-b)d(x_n, y_n)^2 \le \alpha_n(1-\alpha_n)d(x_n, y_n)^2 \le d(x_n, p)^2 - d(x_{n+1}, p)^2. \tag{4.12}$$

This implies

$$\sum_{n=0}^{\infty} a(1-b)d(x_n, y_n)^2 \le d(x_1, p)^2 < \infty. \tag{4.13}$$

Thus, $\lim_{n\to\infty} d(x_n, y_n)^2 = 0$ and hence $\lim_{n\to\infty} d(x_n, y_n) = 0$. Since $y_n \in Tx_n$,

$$\operatorname{dist}(x_n, Tx_n) \le d(x_n, y_n). \tag{4.14}$$

Therefore, $\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0$. Furthermore Condition I implies

$$\lim_{n\to\infty} \operatorname{dist}(x_n, F(T)) = 0. \tag{4.15}$$

The proof of remaining part closely follows the proof of of [2, Theorem 3.8], simply replacing $\|\cdot\|$ with $d(\cdot,\cdot)$. □

Next we show a Δ-convergence theorem of Mann iteration in a CAT(0) space setting which is an analog of Theorem 1.7. For this we need more lemmas.

Lemma 4.3 (see [24, Lemma 2.8]). *If $\{x_n\}$ is a bounded sequence in a complete* CAT (0)*space X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 4.4. *Let K be a nonempty closed convex subset of a complete* CAT (0) *space X, and let $T : K \to \mathcal{K}(X)$ be a nonexpansive nonself-mapping. Suppose that $\{x_n\}$ is a sequence in K which Δ-converges to x in X and*

$$\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0. \tag{4.16}$$

Then $x \in F(T)$.

Proof. Notice from Lemma 3.4(ii) that $x \in K$. Since T is compact-valued, for each $n \geq 1$ there exists $y_n \in Tx_n$ and $z_n \in Tx$ such that $d(x_n, y_n) = \operatorname{dist}(x_n, Tx_n)$ and $d(y_n, z_n) = \operatorname{dist}(y_n, Tx)$. It follows from (4.16) that

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{4.17}$$

By the compactness of Tx, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $\lim_{k\to\infty} z_{n_k} = z \in Tx$. Then

$$\begin{aligned} d(x_{n_k}, z) &\leq d(x_{n_k}, y_{n_k}) + d(y_{n_k}, z_{n_k}) + d(z_{n_k}, z) \\ &\leq d(x_{n_k}, y_{n_k}) + \operatorname{dist}(y_{n_k}, Tx) + d(z_{n_k}, z) \\ &\leq d(x_{n_k}, y_{n_k}) + H(Tx_{n_k}, Tx) + d(z_{n_k}, z) \\ &\leq d(x_{n_k}, y_{n_k}) + d(x_{n_k}, x) + d(z_{n_k}, z). \end{aligned} \tag{4.18}$$

This implies

$$\limsup_k d(x_{n_k}, z) \leq \limsup_k d(x_{n_k}, x). \tag{4.19}$$

Since $\Delta\text{-}\lim_n x_n = x$, $A(\{x_{n_k}\}) = \{x\}$ and hence $z = x$ by (4.19). Therefore x is a fixed point of T. □

Lemma 4.5. *Let K be a closed convex subset of a complete* CAT (0)*spaceX,and let $T : K \to \mathcal{K}(X)$be a nonexpansive mapping. Suppose$\{x_n\}$is a bounded sequence inKsuch that* $\lim_n \operatorname{dist}(x_n, Tx_n) = 0$*and$\{d(x_n, v)\}$converges for all$v \in F(T)$, then $\omega_w(x_n) \subset F(T)$. Here $\omega_w(x_n) := \cup A(\{u_n\})$where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$.Moreover, $\omega_w(x_n)$consists of exactly one point.*

Proof. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 3.4(i) and (ii) there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_n v_n = v \in K$. By Lemma 4.4, $v \in F(T)$. By Lemma 4.3, $u = v$. This shows that $\omega_w(x_n) \subset F(T)$. Next, we show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in \omega_w(x_n) \subset F(T)$, $\{d(x_n, u)\}$ is convergent by the assumption. By Lemma 4.3, $x = u$. This completes the proof. □

Theorem 4.6. *Let K be a nonempty closed convex subset of a complete* CAT (0) *space* X. *Suppose that $T : K \to \mathcal{K}(K)$ is a multivalued nonexpansive mapping. Let $\{x_n\}$ be the sequence of Mann iterates defined by* (3.7). *Assume that $F(T) \neq \emptyset$ satisfying $Ty = \{y\}$ for any fixed point $y \in F(T)$ and*

$$0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1. \tag{4.20}$$

Then the sequence $\{x_n\}$ Δ-converges to a fixed point of T.

Proof. Let $p \in F(T)$, it follows from (4.2) in the proof of Theorem 4.1 that $d(x_n, p) \leq d(x_1, p)$ for all $n \geq 1$. This implies that $\{d(x_n, p)\}_{n=1}^{\infty}$ is bounded and decreasing. Hence $\lim_n d(x_n, p)$ exists for all $p \in F(T)$. Since $y_n \in Tx_n$,

$$\operatorname{dist}(x_n, Tx_n) \leq d(x_n, y_n). \tag{4.21}$$

Thus $\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0$ by (4.9). By Lemma 4.5, $\omega_w(x_n)$ consists of exactly one point and is contained in $F(T)$. This shows that $\{x_n\}$ Δ-converges to an element of $F(T)$. □

5. Strong Convergence of Ishikawa Iteration

The following lemma can be found in [2].

Lemma 5.1. *Let $\{\alpha_n\}, \{\beta_n\}$ be two real sequences such that*

(i) $0 \leq \alpha_n, \beta_n < 1$;

(ii) $\beta_n \to 0$ *as* $n \to \infty$;

(iii) $\sum \alpha_n \beta_n = \infty$.

Let $\{\gamma_n\}$ be a nonnegative real sequence such that $\sum \alpha_n \beta_n (1 - \beta_n) \gamma_n$ is bounded. Then $\{\gamma_n\}$ has a subsequence which converges to zero.

The following theorem is an analog of Theorem 1.8.

Theorem 5.2. *Let K be a nonempty compact convex subset of a complete* CAT (0) *space* X. *Suppose that $T : K \to \mathcal{CB}(K)$ is a multivalued nonexpansive mapping and $F(T) \neq \emptyset$ satisfying $Ty = \{y\}$ for any fixed point $y \in F(T)$. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by* (3.12). *Assume that*

(i) $\alpha_n, \beta_n \in [0, 1)$;

(ii) $\lim_{n\to\infty} \beta_n = 0$;

(iii) $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$.

Then the sequence $\{x_n\}$ strongly converges to a fixed point of T.

Proof. Let $p \in F(T)$, by Lemma 2.2 and the nonexpansiveness of T we have

$$\begin{aligned} d(x_{n+1},p)^2 &\le (1-\alpha_n)d(x_n,p)^2 + \alpha_n d(z_n',p)^2 - \alpha_n(1-\alpha_n)d(x_n,z_n')^2 \\ &\le (1-\alpha_n)d(x_n,p)^2 + \alpha_n (H(Ty_n,Tp))^2 \\ &\le (1-\alpha_n)d(x_n,p)^2 + \alpha_n d(y_n,p)^2 \\ &\le (1-\alpha_n)d(x_n,p)^2 \\ &\quad + \alpha_n\left[(1-\beta_n)d(x_n,p)^2 + \beta_n d(z_n,p)^2 - \beta_n(1-\beta_n)d(x_n,z_n)^2\right] \\ &\le (1-\alpha_n)d(x_n,p)^2 \\ &\quad + \alpha_n\left[(1-\beta_n)d(x_n,p)^2 + \beta_n (H(Tx_n,Tp))^2 - \beta_n(1-\beta_n)d(x_n,z_n)^2\right] \\ &\le d(x_n,p)^2 - \alpha_n\beta_n(1-\beta_n)d(x_n,z_n)^2. \end{aligned} \tag{5.1}$$

This implies

$$d(x_{n+1},p)^2 \le d(x_n,p)^2, \tag{5.2}$$

$$\alpha_n\beta_n(1-\beta_n)d(x_n,z_n)^2 \le d(x_n,p)^2 - d(x_{n+1},p)^2. \tag{5.3}$$

It follows from (5.2) that the sequence $\{d(x_n,p)\}$ is decreasing and hence $\lim_n d(x_n,p)$ exists for each $p \in F(T)$. On the other hand, (5.3) implies

$$\sum_{n=0}^{\infty}\alpha_n\beta_n(1-\beta_n)d(x_n,z_n)^2 \le d(x_1,p)^2 < \infty. \tag{5.4}$$

By Lemma 5.1, there exists a subsequence $\{d(x_{n_k},z_{n_k})\}$ of $\{d(x_n,z_n)\}$ such that

$$\lim_{k\to\infty} d(x_{n_k},z_{n_k}) = 0. \tag{5.5}$$

This implies

$$\lim_{k\to\infty} \operatorname{dist}(x_{n_k},Tx_{n_k}) = 0. \tag{5.6}$$

By Lemma 3.7, $\{x_{n_k}\}$ converges to a point $q \in F(T)$. Since the limit of $\{d(x_n,q)\}$ exists, it must be the case that $\lim_{n\to\infty} d(x_n,q) = 0$, and hence the conclusion follows. □

The following theorem is an analog of Theorem 1.9.

Theorem 5.3. *Let K be a nonempty closed convex subset of a complete* CAT(0) *space X. Suppose that $T : K \to \mathcal{CB}(K)$ is a multivalued nonexpansive mapping that satisfies Condition I. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by* (3.12). *Assume that $F(T) \ne \emptyset$ satisfying $Ty = \{y\}$ for any fixed point $y \in F(T)$ and $\alpha_n, \beta_n \in [a,b] \subset (0,1)$. Then the sequence $\{x_n\}$ strongly converges to a fixed point of T.*

Proof. Similar to the proof of Theorem 5.2, we obtain $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F(T)$ and

$$\alpha_n \beta_n (1-\beta_n) d(x_n, z_n)^2 \le d(x_n, p)^2 - d(x_{n+1}, p)^2. \tag{5.7}$$

Then

$$a^2(1-b) d(x_n, z_n)^2 \le \alpha_n \beta_n (1-\beta_n) d(x_n, z_n)^2 \le d(x_n, p)^2 - d(x_{n+1}, p)^2. \tag{5.8}$$

This implies

$$\sum_{n=0}^{\infty} a^2(1-b) d(x_n, z_n)^2 \le d(x_1, p)^2 < \infty. \tag{5.9}$$

Thus, $\lim_{n\to\infty} d(x_n, z_n)^2 = 0$ and hence $\lim_{n\to\infty} d(x_n, z_n) = 0$. Since $z_n \in Tx_n$,

$$\operatorname{dist}(x_n, Tx_n) \le d(x_n, z_n). \tag{5.10}$$

Therefore, $\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0$. Furthermore Condition I implies

$$\lim_{n\to\infty} \operatorname{dist}(x_n, F(T)) = 0. \tag{5.11}$$

The proof of remaining part closely follows the proof of [2, Theorem 3.8], simply replacing $\|\cdot\|$ with $d(\cdot,\cdot)$. □

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