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Research Article

Meda Inequality for Rearrangements of the Convolution on the Heisenberg Group and Some Applications

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The Meda inequality for rearrangements of the convolution operator on the Heisenberg group $\mathbb{H}_n$ is proved. By using the Meda inequality, an O'Neil-type inequality for the convolution is obtained. As applications of these results, some sufficient and necessary conditions for the boundedness of the fractional maximal operator $M_{\Omega,\alpha}$ and fractional integral operator $I_{\Omega,\alpha}$ with rough kernels in the spaces $L_p(\mathbb{H}_n)$ are found. Finally, we give some comments on the extension of our results to the case of homogeneous groups.

1. Introduction

Let $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$ be the $2n+1$-dimensional Heisenberg group. We define the Lebesgue space on $\mathbb{H}_n$ by

$$L_p(\mathbb{H}_n) = \left\{ f : \|f\|_p \equiv \left(\int_{\mathbb{H}_n} |f(u)|^p \, du \right)^{1/p} < \infty \right\}, \quad 1 \le p < \infty, \tag{1.1}$$

and if $p = \infty$, then $L_\infty(\mathbb{H}_n) = \{ f : \|f\|_{L_\infty} = \operatorname{ess\,sup}_{u \in \mathbb{H}_n} |f(u)| < \infty \}$. The convolution of the functions $f, g \in L_1(\mathbb{H}_n)$ is defined by

$$(f * g)(u) = \int_{\mathbb{H}_n} f(v) g\left(v^{-1} u\right) dv = \int_{\mathbb{H}_n} f\left(u v^{-1}\right) g(v) dv. \tag{1.2}$$

Let $\Omega \in L_s(S_{\mathbb{H}})$, $s \geq 1$, and Ω be homogeneous of degree zero on $\mathbb{H}_n$, and let $0 < \alpha < Q$, where $S_{\mathbb{H}} = S(e,1)$ is the unite sphere centered at the identity of $\mathbb{H}_n$ with radius 1, and $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}_n$. We define the fractional maximal function with rough kernel by

$$M_{\Omega,\alpha} f(u) = \sup_{r>0} \frac{1}{r^{Q-\alpha}} \int_{B(e,r)} |\Omega(v)| \left| f\left(v^{-1}u\right) \right| dv, \tag{1.3}$$

and the fractional integral with rough kernel by

$$I_{\Omega,\alpha} f(u) = \int_{\mathbb{H}_n} \frac{\Omega(v)}{|v|^{Q-\alpha}} f\left(v^{-1}u\right) dv. \tag{1.4}$$

It is clear that, when $\Omega \equiv 1$, $M_{\Omega,\alpha}$ and $I_{\Omega,\alpha}$ are the usual fractional maximal operator M_α and the Riesz potential I_α (see, e.g., [1–5]), respectively.

In this paper, we obtain the Meda inequality for rearrangements of the convolution defined on the Heisenberg group $\mathbb{H}_n$. By using this inequality we get an O'Neil-type inequality for the convolution on $\mathbb{H}_n$. As applications of these inequalities, we find the necessary and sufficient conditions on the parameters for the boundedness of the fractional maximal operator and fractional integral operator with rough kernels from the spaces $L_p(\mathbb{H}_n)$ to $L_q(\mathbb{H}_n)$, $1 < p < q < \infty$, and from the spaces $L_1(\mathbb{H}_n)$ to the weak spaces $WL_q(\mathbb{H}_n)$, $1 < q < \infty$. We also show that the conditions on the parameters ensuring the boundedness cannot be weakened for the fractional maximal operator and fractional integral operator with rough kernels. Finally, we extend our results to the case of homogeneous groups.

2. Preliminaries

Let $\mathbb{H}_n$ be the $2n+1$-dimensional Heisenberg group. That is, $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$, with multiplication

$$(z,t) \cdot (w,s) = (z + w, t + s + 2\mathrm{Im}(z \cdot \overline{w})), \tag{2.1}$$

where $z \cdot \overline{w} = \sum_{j=1}^{n} z_j \overline{w}_j$. The inverse element of $u = (z,t)$ is $u^{-1} = (-z,-t)$, and we write the identity of $\mathbb{H}_n$ as $e = (0,0)$. The Heisenberg group is a connected, simply connected nilpotent Lie group. We define one-parameter dilations on $\mathbb{H}_n$, for $r > 0$, by $\delta_r(z,t) = (rz, r^2 t)$. These dilations are group automorphisms, and the Jacobian determinant is r^Q, where $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}_n$. A homogeneous norm on $\mathbb{H}_n$ is given by

$$|(z,t)| = \left(|z|^2 + |t|\right)^{1/2}. \tag{2.2}$$

With this norm, we define the Heisenberg ball centered at $u = (z,t)$ with radius r by $B(u,r) = \{v \in \mathbb{H}_n : |u^{-1}v| < r\}$ and the Heisenberg sphere by $S(u,r) = \{v \in \mathbb{H}_n : |u^{-1}v| = r\}$. The volume of the ball $B(u,r)$ is $C_Q r^Q$, where C_Q is the volume of the unit ball $B(e,1)$.

Using coordinates $u = (z,t) = (x+iy,t)$ for points in $\mathbb{H}_n$, the left-invariant vector fields X_j, Y_j, and T on $\mathbb{H}_n$ equal to $\partial/\partial x_j$, $\partial/\partial y_j$, and $\partial/\partial t$ at the origin are given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t}, \tag{2.3}$$

respectively. These $2n+1$ vector fields form a basis for the Lie algebra of $\mathbb{H}_n$ with commutation relations

$$\left[Y_j, X_j\right] = 4T \tag{2.4}$$

for $j = 1,\ldots,n$, and all other commutators equal to 0. It is easy to check that Lebesgue measure $du = dzdt$ is a Haar measure on $\mathbb{H}_n$ (see, e.g., [6]).

Let $f : \mathbb{H}_n \to \mathbb{R}$ be a measurable function. We define rearrangement of f in decreasing order by

$$f^*(t) = \inf\{s > 0 : f_*(s) \le t\}, \quad \forall t \in [0,\infty), \tag{2.5}$$

where $f_*(t)$ denotes the distribution function of f given by

$$f_*(t) = \left|\{u \in \mathbb{H}_n : |f(u)| > t\}\right|, \tag{2.6}$$

and $|A|$ denotes the Haar measure of any measurable subset $A \subset \mathbb{H}_n$.

We note the following properties of rearrangement of functions (see [1, 7, 8]):

(1) if $0 < p < \infty$, then

$$\int_{\mathbb{H}_n} |f(u)|^p du = \int_0^\infty f^*(t)^p \, dt; \tag{2.7}$$

(2) for any $t > 0$,

$$\sup_{|E|=t} \int_E |f(u)| du = \int_0^t f^*(s) \, ds; \tag{2.8}$$

(3)

$$\int_{\mathbb{H}_n} |f(u)g(u)| du \le \int_0^\infty f^*(t)g^*(t)dt. \tag{2.9}$$

The function f^{**} on $(0,\infty)$ is defined by $f^{**}(t) = (1/t)\int_0^t f^*(s)ds$, $t > 0$.

If $1 < p < \infty$, the following inequality is valid:

$$\|f^{**}\|_{L_p(0,\infty)} \le p' \|f^*\|_{L_p(0,\infty)}, \tag{2.10}$$

where $p' = p/(p-1)$.

We denote by $WL_p(\mathbb{H}_n)$ the weak $L_p(\mathbb{H}_n)$ space of all measurable functions f with finite norm

$$\|f\|_{WL_p} = \sup_{t>0} t f_*(t)^{1/p}, \quad 1 \le p < \infty. \tag{2.11}$$

3. Meda Inequality for Rearrangements of the Convolution on the Heisenberg Group $\mathbb{H}_n$

For the convolution

$$(K_\alpha * f)(x) = \int_{\mathbb{R}^n} K_\alpha(x-y) f(y) dy, \tag{3.1}$$

Meda [9] proved the following pointwise rearrangement estimate:

$$(K_\alpha * f)^{**}(t) \le C\left(\delta^\alpha f^{**}(t) + \delta^{\alpha - n/p}\|f\|_p\right), \quad \delta > 0, \tag{3.2}$$

and gave a new proof of the Hardy-Littlewood-Sobolev theorem for $K_\alpha * f$ by using this inequality, where $K_\alpha \in WL_{n/(n-\alpha)}(\mathbb{R}^n)$, $0 < \alpha < n$, and $f \in L_p(\mathbb{R}^n)$, $1 < p < n/\alpha$.

In this section we use the same notation as in the previous sections. We prove the Meda inequality for rearrangements of the convolution on the Heisenberg group $\mathbb{H}_n$.

Theorem 3.1. *Let $g \in WL_r(\mathbb{H}_n)$, $1 < r < \infty$, $f \in L_p(\mathbb{H}_n)$, $1 \le p < r'$. Then for any $\delta > 0$*

$$(f * g)^{**}(t) \le C_1 \delta^{Q/r'} f^{**}(t) + C_2 \delta^{Q/r' - Q/p}\|f\|_p, \tag{3.3}$$

where $C_1 = B_1\|g\|_{WL_r}^r$, $B_1 = 2^r(2^{r-1}-1)^{-1}$, $C_2 = B_2\|g\|_{WL_r}^{r/p'}$, $B_2 = 1$ for $p = 1$, and $B_2 = 2(2^{p'-r}-1)^{-1/p'}$ for $1 < p < r'$.

Proof. Suppose $F_\delta = \{v \in \mathbb{H}_n : |g(v)| \ge \delta^{-Q/r}\}$, then

$$\left|(f * g)(u)\right| \le \left(\int_{F_\delta} + \int_{\mathbb{H}_n \setminus F_\delta}\right)\left|f\left(uv^{-1}\right)\right|\left|g(v)\right| dv = D_1(u) + D_2(u). \tag{3.4}$$

Suppose that $F_\delta = \bigcup_{j=1}^{\infty} F'_{\delta,j}$, where

$$F'_{\delta,j} = \left\{v \in \mathbb{H}_n : 2^{j-1}\delta^{-Q/r} \le \left|g(v)\right| < 2^j \delta^{-Q/r}\right\}. \tag{3.5}$$

Then from equality (2.7) we get

$$\begin{aligned}
\frac{1}{|E|}\int_E D_1(u)du &= \frac{1}{|E|}\int_E \left(\int_{F_\delta} |g(v)|\left|f\left(uv^{-1}\right)\right|dv\right)du \\
&\leq \sum_{j=1}^{\infty} 2^j \delta^{-Q/r} \int_{F'_{\delta,j}} \left(\frac{1}{|E|}\right)\int_E \left|f\left(uv^{-1}\right)\right|du\,dv \\
&\leq \delta^{-Q/r} f^{**}(t) \textstyle\sum_{j=1}^{\infty} 2^j \displaystyle\int_{F'_{\delta,j}} dv \\
&\leq 2^r \delta^{-Q/r} f^{**}(t) \|g\|_{WL_r}^r \sum_{j=1}^{\infty} 2^j \left(2^j \delta^{-Q/r}\right)^{-r} \\
&= C_1 \delta^{Q/r'} f^{**}(t),
\end{aligned} \tag{3.6}$$

where $C_1 = 2^r\left(2^{r-1} - 1\right)^{-1}\|g\|_{WL_r}^r$.

Thus

$$\frac{1}{|E|}\int_E D_1(u)du \leq C_1 \delta^{Q/r'} f^{**}(t). \tag{3.7}$$

Let $p = 1$. We have

$$|D_2(u)| \leq \|f(u\cdot)\|_1 \sup_{\mathbb{H}_n \setminus F_\delta} |g(v)| \leq \|f\|_1 \delta^{-Q/r}. \tag{3.8}$$

Let now $1 < p < r'$. By using Hölder inequality we get

$$\begin{aligned}
|D_2(u)| &\leq \|f(u\cdot)\|_p \left(\int_{\mathbb{H}_n \setminus F_\delta} |g(v)|^{p'} dv\right)^{1/p'} \\
&\leq \|f\|_p \left(\int_{\mathbb{H}_n \setminus F_\delta} |g(v)|^{p'} dv\right)^{1/p'}.
\end{aligned} \tag{3.9}$$

Since $g \in WL_r(\mathbb{H}_n)$ and $\mathbb{H}_n \setminus F_\delta = \bigcup_{j=1}^{\infty} B_{\delta,j}$, where

$$B_{\delta,j} = \left\{v \in \mathbb{H}_n : 2^{-j}\delta^{-Q/r} \leq |g(v)| < 2^{-j+1}\delta^{-Q/r}\right\}, \tag{3.10}$$

then

$$\int_{\mathbb{H}_n\setminus F_\delta} |g(v)|^{p'} dv = \sum_{j=1}^{\infty} \int_{B_{\delta,j}} |g(v)|^{p'} dv$$
$$\leq \sum_{j=1}^{\infty} (2^{-j+1}\delta^{-Q/r})^{p'} \int_{\{v\in\mathbb{H}_n:|g(v)|\geq 2^{-j}\delta^{-Q/r}\}} dv$$
$$\leq 2^{p'}\delta^{-Qp'/r}\|g\|_{WL_r}^{r} \sum_{j=1}^{\infty} 2^{-jp'}(2^{-j}\delta^{-Q/r})^{-r} \tag{3.11}$$
$$= 2^{p'}\delta^{Q-Qp'/r}\|g\|_{WL_r}^{r} \sum_{j=1}^{\infty} 2^{-j(p'-r)}$$
$$= C_2^{p'}\delta^{Q-Q/rp'},$$

where $C_2 = 2(2^{p'-r}-1)^{-1/p'} \mathrm{P}g\mathrm{P}_{WL_r}^{r/p'}$. Hence

$$|D_2(u)| \leq C_2\|f\|_p \delta^{Q/r'-Q/p}. \tag{3.12}$$

Thus

$$\frac{1}{|E|}\int_E |(f*g)(u)| du \leq C_1\delta^{Q/r'} f^{**}(t) + C_2\delta^{Q/r'-Q/p}\|f\|_p. \tag{3.13}$$

Hence from equality (2.8) we get (3.3), and therefore Theorem 3.1 is proved. □

Corollary 3.2. *Let Ω be homogeneous of degree zero on $\mathbb{H}_n$, $\Omega \in L_{Q/(Q-\alpha)}(S_{\mathbb{H}})$, $0 < \alpha < Q$, and let $f \in L_p(\mathbb{H}_n)$, $1 \leq p < Q/\alpha$. Then for any $\delta > 0$ the following inequality holds:*

$$(I_{\Omega,\alpha} f)^{**}(t) \leq C_3\delta^{\alpha} f^{**}(t) + C_4\delta^{\alpha-Q/p}\|f\|_p, \tag{3.14}$$

where $C_3 = B_3(A/Q)$, $C_4 = B_4(A/Q)^{1/p'}$, $B_3 = 2^{Q/(Q-\alpha)}(2^{\alpha/(Q-\alpha)}-1)^{-1}$, $B_4 = 2(2^{p'-Q/(Q-\alpha)}-1)^{-1/p'}$ for $1 < p < Q/\alpha$, and $B_4 = 1$ for $p = 1$ and

$$A = \|\Omega\|_{L_{Q/(Q-\alpha)}}^{Q/(Q-\alpha)}, \qquad \|\Omega\|_{L_{Q/(Q-\alpha)}} = \left(\int_{S_{\mathbb{H}}} |\Omega(u)|^{Q/(Q-\alpha)} du\right)^{(Q-\alpha)/Q}. \tag{3.15}$$

Proof. If we take $g(u) = \Omega(u)/|u|^{Q-\alpha}$, $0 < \alpha < Q$, and $r = Q/(Q-\alpha)$ in Theorem 3.1, then the proof of Corollary 3.2 is straightforward, where Ω is homogeneous of degree zero on $\mathbb{H}_n$ and $\Omega \in L_{Q/(Q-\alpha)}(S_{\mathbb{H}})$. In this case

$$g_*(t) = (\mathrm{A}/\mathrm{Q})t^{-Q/(Q-\alpha)}, \qquad g^*(t) = (\mathrm{A}/\mathrm{Q})t^{1-\alpha/Q}. \tag{3.16}$$

Therefore$g \in WL_{Q/(Q-\alpha)}(\mathbb{H}_n)$ and $\|g\|_{WL_{Q/(Q-\alpha)}} = (A/Q)^{1-\alpha/Q}$. □

Corollary 3.3. *Let $f \in L_p(\mathbb{H}_n)$, $1 \le p < Q/\alpha$, $0 < \alpha < Q$, then for the Riesz potential*

$$I_\alpha f(u) = \int_{\mathbb{H}_n} \left|v^{-1}u\right|^{\alpha-Q} f(v)dv \tag{3.17}$$

for all $0 < t < \infty$ the following inequality holds:

$$\left(I_\alpha f\right)^{**}(t) \le C_5 \delta^\alpha f^{**}(t) + C_6 \delta^{\alpha-Q/p} \|f\|_p, \tag{3.18}$$

where $C_5 = C_Q B_3$ and $C_6 = C_Q^{1/p'} B_4$ for $1 \le p < Q/\alpha$.

Proof. By the same argument in Corollary 3.2 if we take

$$g(u) = |u|^{\alpha-Q} \in WL_{Q/(Q-\alpha)}(\mathbb{H}_n), \quad 0 < \alpha < Q, \tag{3.19}$$

in Theorem 3.1, we easily get the proof of the Corollary. In this case

$$\begin{gathered} g_*(t) = C_Q t^{-Q/(Q-\alpha)}, \qquad g^*(t) = \left(C_Q t^{-1}\right)^{1-\alpha/Q}, \\ \|g\|_{WL_{Q/(Q-\alpha)}} = C_Q^{1-\alpha/Q}, \end{gathered} \tag{3.20}$$

where C_Q is the volume of the unit ball B_1. □

Note that, the following estimate

$$M_{\Omega,\alpha} f(u) \le I_{|\Omega|,\alpha}\left(|f|\right)(u) \tag{3.21}$$

is valid. Indeed, for all $r > 0$ we have

$$\begin{aligned} I_{|\Omega|,\alpha}\left(|f|\right)(u) &\ge \int_{B(e,r)} \frac{|\Omega(v)|}{|y|^{Q-\alpha}} \left|f\left(v^{-1}u\right)\right| dv \\ &\ge \frac{1}{r^{Q-\alpha}} \int_{B(e,r)} |\Omega(v)| \left|f\left(v^{-1}u\right)\right| dv. \end{aligned} \tag{3.22}$$

Taking supremum over all $r > 0$, we get (3.21).

From Corollary 3.2 and inequality (3.21) we get the following

Corollary 3.4. *Suppose that Ω is homogeneous of degree zero on $\mathbb{H}_n$, $\Omega \in L_{Q/(Q-\alpha)}(S_{\mathbb{H}})$, $0 < \alpha < Q$ and let $f \in L_p(\mathbb{H}_n)$, $1 \le p < Q/\alpha$. Then for all $\delta > 0$ the following inequality holds:*

$$\left(M_{\Omega,\alpha} f\right)^{**}(t) \le C_3 \delta^\alpha f^{**}(t) + C_4 \|f\|_p \delta^{\alpha-Q/p}. \tag{3.23}$$

4. O'Neil-Type Inequality for the Convolution on $\mathbb{H}_n$ and Some Applications

In this section we prove an O'Neil-type inequality (see [10]) for the convolution on the Heisenberg group $\mathbb{H}_n$. With the help of this inequality, we obtain some sufficient and necessary conditions for the boundedness of the fractional maximal operator $M_{\Omega,\alpha}$ and fractional integral operator $I_{\Omega,\alpha}$ with rough kernels in the spaces $L_p(\mathbb{H}_n)$.

Theorem 4.1. (1) *Let* $g \in WL_r(\mathbb{H}_n)$, $1 < r < \infty$, $f \in L_p(\mathbb{H}_n)$, $1 < p < r'$, *and* $1/p - 1/q = 1/r'$. *Then* $f * g \in L_q(\mathbb{H}_n)$ *and*

$$\|f * g\|_q \le 2B_1^{1-p/r'} B_2^{p/r'} (p')^{p/q} \|g\|_{WL_r} \|f\|_p. \tag{4.1}$$

(2) *Let* $f \in L_1(\mathbb{H}_n)$, $g \in WL_q(\mathbb{H}_n)$, *and* $1 < q < \infty$. *Then* $f * g \in WL_q(\mathbb{H}_n)$ *and*

$$\|f * g\|_{WL_q} \le 2B_1^{1/q} \|g\|_{WL_q} \|f\|_1. \tag{4.2}$$

Proof. Step 1. $g \in WL_r(\mathbb{H}_n)$, $1 < r < \infty$, $f \in L_p(\mathbb{H}_n)$, $1 < p < r'$, and $1/p - 1/q = 1/r'$. If we take

$$\delta = \left(\frac{C_1 f^{**}(t)}{C_2\|f\|_p}\right)^{-p/Q} \tag{4.3}$$

in (3.3), then we get

$$\begin{aligned}(f * g)^{**}(t) &\le 2(C_1 f^{**}(t))^{1-p/r'} (C_2\|f\|_p)^{p/r'} \\ &= 2C_1^{1-p/r'} C_2^{p/r'} f^{**}(t)^{p/q} \|f\|_p^{1-p/q} \\ &= 2B_1^{1-p/r'} B_2^{p/r'} \|g\|_{WL_r} \|f\|_p^{1-p/q} f^{**}(t)^{p/q}.\end{aligned} \tag{4.4}$$

Thus

$$\begin{aligned}\|(f * g)\|_q &\le \|(f * g)^{**}(t)\|_{L_q(0,\infty)} \\ &\le 2B_1^{1-p/r'} B_2^{p/r'} \|g\|_{WL_r} \|f\|_p^{1-p/q} \left\|(f^{**})^{p/q}\right\|_{L_q(0,\infty)} \\ &= 2B_1^{1-p/r'} B_2^{p/r'} \|g\|_{WL_r} \|f\|_p^{1-p/q} \|f^{**}\|_{L_p(0,\infty)}^{p/q} \\ &\le 2B_1^{1-p/r'} B_2^{p/r'} (p')^{p/q} \|g\|_{WL_r} \|f\|_p.\end{aligned} \tag{4.5}$$

Step 2. Let $p = 1$, $1 < q < \infty$, $f \in L_1(\mathbb{H}_n)$ and $g \in WL_q(\mathbb{H}_n)$.

We take

$$\delta = \left(\frac{C_1 f^{**}(t)}{C_2 \|f\|_p} \right)^{-1/Q} \tag{4.6}$$

in (3.3), then we get

$$(f * g)^{**}(t) \le 2B_1^{1/q} \, \|g\|_{WL_q} \|f\|_1^{1/q'} f^{**}(t)^{1/q}. \tag{4.7}$$

Thus

$$\begin{aligned} \|(f * g)\|_{WL_q} &= \sup_{t>0} t^{1/q} (f * g)^*(t) \\ &\le 2B_1^{1/q} \|g\|_{WL_q} \|f\|_1^{1/q'} \sup_{t>0} t^{1/q} f^{**}(t)^{1/q} \\ &= 2B_1^{1/q} \|g\|_{WL_q} \|f\|_1^{1/q'} \sup_{t>0} \left(\int_0^t f^*(s) ds \right)^{1/q} \\ &\le 2B_1^{1/q} \|g\|_{WL_q} \|f\|_1^{1/q'} \|f^*\|_{L_1(0,\infty)}^{1/q} \\ &= 2B_1^{1/q} \|g\|_{WL_q} \|f\|_1, \end{aligned} \tag{4.8}$$

and therefore the proof of Theorem 4.1 is completed. □

Corollary 4.2. *Let Ω be homogeneous of degree zero on $\mathbb{H}_n$ and $\Omega \in L_{Q/(Q-\alpha)}(S_{\mathbb{H}})$, $0 < \alpha < Q$.*

(1) *If $1 < p < Q/\alpha$, $f \in L_p(\mathbb{H}_n)$ and $1/p - 1/q = \alpha/Q$, then $M_{\Omega,\alpha} f$, $I_{\Omega,\alpha} f \in L_q(\mathbb{H}_n)$ and*

$$\|M_{\Omega,\alpha} f\|_q \le \|I_{\Omega,\alpha} f\|_q \le 2(A/Q)^{1-\alpha/Q} \, B_3^{1-\alpha p/Q} B_4^{\alpha p/Q} (p')^{p/q} \|f\|_p. \tag{4.9}$$

(2) *If $f \in L_1(\mathbb{H}_n)$ and $1 - 1/q = \alpha/Q$, then $M_{\Omega,\alpha} f$, $I_{\Omega,\alpha} f \in WL_q(\mathbb{H}_n)$ and*

$$\|M_{\Omega,\alpha} f\|_{WL_q} \le \|I_{\Omega,\alpha} f\|_{WL_q} \le 2(A/Q)^{1-\alpha/Q} B_3^{1/q} \, \|f\|_1. \tag{4.10}$$

Corollary 4.3 ([1]). *Let $0 < \alpha < Q$.*

(1) *If $1 < p < Q/\alpha$, $f \in L_p(\mathbb{H}_n)$ and $1/p - 1/q = \alpha/Q$, then $M_\alpha f, I_\alpha f \in L_q(\mathbb{H}_n)$ and*

$$\|M_\alpha f\|_q \le \|I_\alpha f\|_q \le 2C_Q^{1-\alpha/Q} B_3^{1-\alpha p/Q} B_4^{\alpha p/Q} (p')^{p/q} \|f\|_p. \tag{4.11}$$

(2) *If $f \in L_1(\mathbb{H}_n)$ and $1 - 1/q = \alpha/Q$, then $M_\alpha f, I_\alpha f \in WL_q(\mathbb{H}_n)$ and*

$$\|M_\alpha f\|_{WL_q} \le \|I_\alpha f\|_{WL_q} \le C_Q^{1-\alpha/Q} \, B_3^{1/Q} \, \|f\|_1. \tag{4.12}$$

In the following we give necessary and sufficient conditions for the (L_p, L_q)-boundedness of the fractional integral operator $I_{\Omega,\alpha}$ with rough kernel on $\mathbb{H}_n$.

Theorem 4.4. *Let Ω be homogeneous of degree zero on $\mathbb{H}_n$ and $\Omega \in L_{Q/(Q-\alpha)}(S_{\mathbb{H}})$, $0 < \alpha < Q$.*

(1) *If $1 < p < Q/\alpha$, then the condition $1/p - 1/q = \alpha/Q$ is necessary and sufficient for the boundedness of $I_{\Omega,\alpha}$ from $L_p(\mathbb{H}_n)$ to $L_q(\mathbb{H}_n)$.*

(2) *If $p = 1$, then the condition $1 - 1/q = \alpha/Q$ is necessary and sufficient for the boundedness of $I_{\Omega,\alpha}$ from $L_1(\mathbb{H}_n)$ to $WL_q(\mathbb{H}_n)$.*

Proof. Sufficiency of Theorem 4.4 follows from Corollary 4.2.

Necessity

(1) Suppose that the operator $I_{\Omega,\alpha}$ is bounded from $L_p(\mathbb{H}_n)$ to $L_q(\mathbb{H}_n)$ and $1 < p < Q/\alpha$.
Define $f_t(u) =: f(\delta_t u)$ for $t > 0$. Then it can be easily shown that

$$\left\|f_t\right\|_p = t^{-Q/p}\left\|f\right\|_p, \qquad \left(I_{\Omega,\alpha}f_t\right)(u) = t^{-\alpha}I_{\Omega,\alpha}f(\delta_t u),$$
$$\left\|I_{\Omega,\alpha}f_t\right\|_q = t^{-\alpha-Q/q}\left\|I_{\Omega,\alpha}f\right\|_q. \tag{4.13}$$

Since the operator $I_{\Omega,\alpha}$ is bounded from $L_p(\mathbb{H}_n)$ to $L_q(\mathbb{H}_n)$, we have

$$\left\|I_{\Omega,\alpha}f\right\|_q \le C\left\|f\right\|_p, \tag{4.14}$$

where C is independent of f. Then we get

$$\left\|I_{\Omega,\alpha}f\right\|_q = t^{\alpha+Q/q}\left\|I_{\Omega,\alpha}f_t\right\|_q \le Ct^{\alpha+Q/q}\left\|f_t\right\|_p = Ct^{\alpha+Q/q-Q/p}\left\|f\right\|_p. \tag{4.15}$$

If $1/p < 1/q + \alpha/Q$, then for all $f \in L_p(\mathbb{H}_n)$ we have $\|I_{\Omega,\alpha}f\|_{L_q} = 0$ as $t \to 0$.
As well as if $1/p > 1/q + \alpha/Q$, then for all $f \in L_p(\mathbb{H}_n)$ we have $\|I_{\Omega,\alpha}f\|_q = 0$ as $t \to \infty$.
Therefore we get $1/p = 1/q + \alpha/Q$.

(2) Suppose that the operator $I_{\Omega,\alpha}$ is bounded from $L_1(\mathbb{H}_n)$ to $WL_q(\mathbb{H}_n)$. It is easy to show that

$$\left\|f_t\right\|_1 = t^{-Q}\left\|f\right\|_1, \qquad \left(I_{\Omega,\alpha}f_t\right)(u) = t^{-\alpha}\left(I_{\Omega,\alpha}f\right)(\delta_t u),$$
$$\left\|I_{\Omega,\alpha}f_t\right\|_{WL_q} = t^{-\alpha-Q/q}\left\|I_{\Omega,\alpha}f\right\|_{WL_q}. \tag{4.16}$$

By the boundedness of $I_{\Omega,\alpha}$ from $L_1(\mathbb{H}_n)$ to $WL_q(\mathbb{H}_n)$, we have

$$\left\|I_{\Omega,\alpha}f\right\|_{WL_q} \le C\left\|f\right\|_1, \tag{4.17}$$

where C is independent of f. Then we have

$$
\begin{aligned}
(I_{\Omega,\alpha} f_t)_*(\tau) &= t^{-Q} (I_{\Omega,\alpha} f)_*(t^\alpha \tau),\\
\|I_{\Omega,\alpha} f_t\|_{WL_q} &= t^{-\alpha - Q/q} \|I_{\Omega,\alpha} f\|_{WL_q},\\
\|I_{\Omega,\alpha} f\|_{WL_q} = t^{\alpha+Q/q} \|I_{\Omega,\alpha} f_t\|_{WL_q} &\le C t^{\alpha+Q/q} \|f_t\|_1 = C t^{\alpha+Q/q-Q} \|f\|_1.
\end{aligned}
\tag{4.18}
$$

If $1 < 1/q + \alpha/Q$, then for all $f \in L_1(\mathbb{H}_n)$ we have $\|I_{\Omega,\alpha} f\|_{WL_q} = 0$ as $t \to 0$.

If $1 > 1/q + \alpha/Q$, then for all $f \in L_1(\mathbb{H}_n)$ we have $\|I_{\Omega,\alpha} f\|_{WL_q} = 0$ as $t \to \infty$.

Therefore we get the equality $1 = 1/q + \alpha/Q$. □

Corollary 4.5. *Let $0 < \alpha < Q$, Ω be homogeneous of degree zero on $\mathbb{H}_n$ and $\Omega \in L_{Q/(Q-\alpha)}(S_{\mathbb{H}})$.*

(1) *If $1 < p < Q/\alpha$, then the condition $1/p - 1/q = \alpha/Q$ is necessary and sufficient for the boundedness of $M_{\Omega,\alpha}$ from $L_p(\mathbb{H}_n)$ to $L_q(\mathbb{H}_n)$.*

(2) *If $p = 1$, then the condition $1 - 1/q = \alpha/Q$ is necessary and sufficient for the boundedness of $M_{\Omega,\alpha}$ from $L_1(\mathbb{H}_n)$ to $WL_q(\mathbb{H}_n)$.*

5. Conclusions

In this section we make some comments on the extension of above results to the case of homogeneous groups. Homogeneous groups can be exemplified by an n-dimensional Euclidean space, Heisenberg groups, and so on. We begin by reviewing some definitions (see [1–3, 5]).

Let $\mathbb{G}$ be a homogeneous group, that is, $\mathbb{G}$ is a connected and simply connected nilpotent Lie group which is endowed with a family of dilations $\{\delta_r\}_{r>0}$. We recall that a family of dilations on the Lie algebra $\mathbf{g}$ of $\mathbb{G}$ is a one parameter family of automorphisms of $\mathbf{g}$ of the form $\{\exp(A \log r) : r > 0\}$, where A is diagonalizable linear operator on $\mathbf{g}$ with positive eigenvalues $1 = d_1 \le d_2 \le \cdots \le d_n$, $Q = \dim(\mathbb{G})$. Then the exponential map from $\mathbf{g}$ to $\mathbb{G}$ defines the corresponding family of dilations $\{\delta_r\}_{r>0}$ on $\mathbb{G}$. We will often use the abbreviated notation $\delta_r x = rx$ for $x \in \mathbb{G}$ and $r > 0$.

We fix a homogeneous norm on $\mathbb{G}$, that is, a continuous map $|\cdot| : \mathbb{G} \to [0,\infty)$ that is C^∞ on $\mathbb{G} \setminus \{e\}$ and satisfies

$$
\begin{gathered}
\left|x^{-1}\right| = |x| \quad \forall x \in \mathbb{G},\\
|\delta_r x| = r|x| \quad \forall x \in \mathbb{G}, r > 0,\\
|x| = 0 \Longleftrightarrow x = e.
\end{gathered}
\tag{5.1}
$$

The ball $B(x,r)$ of radius $r > 0$ and centered $x \in \mathbb{G}$ is defined as

$$
B(x,r) = \left\{ y \in \mathbb{G} : \left|x^{-1} y\right| < r \right\},
\tag{5.2}
$$

Let $Q = d_1 + d_2 + \cdots + d_n$ be the homogeneous dimension of $\mathbb{G}$, and let γ be the minimal constant such that

$$|xy| \le \gamma(|x| + |y|) \quad \forall x, y \in \mathbb{G}. \tag{5.3}$$

Onto the group $\mathbb{G}$ we fixed the normed Haar measure in such a manner that the measure of the unit ball $B(e,1)$ is equal to 1. The Haar measure of any measurable set $E \subset \mathbb{G}$ will be denoted by $|E|$, and the integral on E with respect to this measure by $\int_E f(x)dx$.

It readily follows that

$$|\delta_t E| = t^Q |E|, \qquad d(\delta_t x) = t^Q dx. \tag{5.4}$$

In particular, for arbitrary $x \in G$ and $r > 0$ we have $|B(e,r)| = r^Q$.

The spaces $L_p(\mathbb{G})$ and $WL_p(\mathbb{G})$ are defined the same as in the case $\mathbb{H}_n$. For example, $L_p(\mathbb{G})$ is defined as the set of all measurable functions f on $\mathbb{G}$ with finite norm

$$\|f\|_{L_p(\mathbb{G})} = \left(\int_{\mathbb{G}} |f(x)|^p dx\right)^{1/p}, \quad 1 \le p < \infty. \tag{5.5}$$

The convolution of the functions $f, g \in L_1(\mathbb{G})$ is defined by

$$(f * g)(x) = \int_{\mathbb{G}} f(y) g\left(y^{-1}x\right) dy = \int_{\mathbb{G}} f\left(xy^{-1}\right) g(y) dy. \tag{5.6}$$

By using the methods of Theorem 3.1 it is not hard to show the Meda inequality for rearrangements of the convolution on the homogeneous group $\mathbb{G}$.

For a homogeneous group $\mathbb{G}$, it is well known that (see [1, page 14]) there exists a unique countably additive measure σ defined on $S_{\mathbb{G}}$ such that the equality

$$\int_{\mathbb{G}} f(x) dx = \int_0^\infty t^{Q-1} \left(\int_{S_{\mathbb{G}}} f(\delta_t \xi) d\sigma(\xi)\right) dt \tag{5.7}$$

holds for any $f \in L_1(\mathbb{G})$, where $S_{\mathbb{G}} = \{x \in \mathbb{G} : |x| = 1\}$. The equality (5.7) is an important tool for proving the following two theorems.

The fractional integral $R_{\Omega,\alpha}$ with rough kernel on $\mathbb{G}$ is defined by

$$R_{\Omega,\alpha} f(u) = \int_{\mathbb{G}} \frac{\Omega(y)}{|y|^{Q-\alpha}} f\left(y^{-1}x\right) dy, \quad 0 < \alpha < Q. \tag{5.8}$$

In Theorem 5.1, with the help of Meda type inequality for rearrangements we show that an O'Neil-type inequality for the convolution (see [10]) on the homogeneous group $\mathbb{G}$ is valid.

Theorem 5.1. (1) *Let* $g \in WL_r(\mathbb{G})$, $1 < r < \infty$, $f \in L_p(\mathbb{G})$, $1 < p < r'$, *and* $1/p - 1/q = 1/r'$. *Then* $f * g \in L_q(\mathbb{G})$ *and*

$$\|f * g\|_{L_q(\mathbb{G})} \le 2B_1^{1-p/r'} B_2^{p/r'} (p')^{p/q} \|g\|_{WL_r(\mathbb{G})} \|f\|_{L_p(\mathbb{G})}, \tag{5.9}$$

where B_1 *and* B_2 *are defined in Theorem 3.1.*

(2) *Let* $f \in L_1(\mathbb{G})$, $g \in WL_q(\mathbb{G})$, *and* $1 < q < \infty$. *Then* $f * g \in WL_q(\mathbb{G})$ *and*

$$\|f * g\|_{WL_q(\mathbb{G})} \le 2B_1^{1/q} \ \|g\|_{WL_q(\mathbb{G})} \ \|f\|_{L_1(\mathbb{G})}. \tag{5.10}$$

In Theorem 5.2 we give necessary and sufficient conditions for the (L_p, L_q)-boundedness of the fractional integral operator $R_{\Omega,\alpha}$ with rough kernel on $\mathbb{G}$.

Theorem 5.2. *Let* Ω *be homogeneous of degree zero on* $\mathbb{G}$ *and* $\Omega \in L_{Q/(Q-\alpha)}(S_{\mathbb{G}})$, $0 < \alpha < Q$.

(1) *If* $1 < p < Q/\alpha$, *then the condition* $1/p - 1/q = \alpha/Q$ *is necessary and sufficient for the boundedness of* $R_{\Omega,\alpha}$ *from* $L_p(\mathbb{G})$ *to* $L_q(\mathbb{G})$.

(2) *If* $p = 1$, *then the condition* $1 - 1/q = \alpha/Q$ *is necessary and sufficient for the boundedness of* $R_{\Omega,\alpha}$ *from* $L_1(\mathbb{G})$ *to* $WL_q(\mathbb{G})$.

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