Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 981453, 5 pages
doi:10.1155/2009/981453

Research Article

Denseness of Numerical Radius Attaining Holomorphic Functions

Han Ju Lee

Department of Mathematics Education, Dongguk University, Seoul 100-715, South Korea

Correspondence should be addressed to Han Ju Lee, hanjulee@dongguk.edu

Received 21 June 2009; Accepted 6 October 2009

Recommended by Nikolaos Papageorgiou

We study the density of numerical radius attaining holomorphic functions on certain Banach spaces using the Lindenstrauss method. In particular, it is shown that if a complex Banach space X is locally uniformly convex, then the set of all numerical attaining elements of $A(B_X : X)$ is dense in $A(B_X : X)$.

1. Introduction

Let X be a complex Banach space and X^* its dual space. We consider the topological subspace $\Pi(X) = \{(x, x^*) : x^*(x) = 1 = \|x\| = \|x^*\|\}$ of the product space $B_X \times B_{X^*}$, equipped with norm and weak-$*$ topology on the unit ball B_X of X and its dual unit ball B_{X^*}, respectively. It is easy to see that $\Pi(X)$ is a closed subspace of $B_X \times B_{X^*}$.

For two complex Banach spaces X and Y, denote by $C_b(B_X : Y)$ the Banach space of all bounded continuous functions from B_X to Y with sup norm $\|f\| = \sup\{\|f(x)\| : x \in B_X\}$. We are interested in the following two subspaces of $C_b(B_X : Y)$:

$$\begin{aligned} A_b(B_X : Y) &= \{f \in C_b(B_X : Y) : f \text{ is holomorphic on the open unit ball } B_X^\circ\}, \\ A_u(B_X : Y) &= \{f \in A_b(B_X : Y) : f \text{ is uniformly continuous}\}. \end{aligned} \tag{1.1}$$

We denote by $A(B_X : Y)$ either $A_u(B_X : Y)$ or $A_b(B_X : Y)$. When $Y = \mathbb{C}$, we write $A(B_X)$ instead of $A(B_X : \mathbb{C})$. A nonzero function $f \in C_b(B_X : Y)$ is said to be a *strong peak function* at x_0 if whenever there is a sequence $\{x_n\}_{n=1}^\infty$ in B_X with $\lim_n \|f(x_n)\| = \|f\|$, the sequence $\{x_n\}_n$ converges to x_0. The corresponding point x_0 is said to be a *strong peak point* of $A(B_X : Y)$. It is easy to see that x is a strong peak point of $A(B_X : Y)$ if and only if x is a strong peak point of

$A(B_X)$. By the maximum modulus theorem, it is easy to see that if x is a strong peak point of $A(B_X)$ and $X \neq 0$, then x is contained in the unit sphere S_X of X.

Harris [1] introduced the notion of numerical radius $v(f)$ of holomorphic function $f \in A_b(B_X : X)$. More precisely, for each $f \in A_b(B_X : X)$, $v(f) = \sup\{|x^* f(x)| : (x, x^*) \in \Pi(X)\}$. An element $f \in A(B_X : X)$ is said [2] to be *numerical radius attaining* if there is $(x, x^*) \in \Pi(X)$ such that $v(f) = |x^* f(x)|$.

Acosta and Kim [2] showed that if X is a complex Banach space with the Radon-Nikodým property, then the set of all numerical radius attaining elements in $A(B_X : X)$ is dense. In this paper, we show that if X is a locally uniformly convex space or locally uniformly c-convex, order continuous, sequence space, then the set of all numerical radius attaining elements in $A(B_X : X)$ is dense.

We need the notion of numerical boundary. The subset Γ of $\Pi(X)$ is said [3] to be a *numerical boundary* of $A(B_X : X)$ if for every $f \in A(B_X : X)$, $v(f) = \sup\{|x^* f(x)| : (x, x^*) \in \Gamma\}$. For more properties of numerical boundaries, see [3–5].

2. Main Results

The following is an application of the numerical boundary to the density of numerical radius attaining holomorphic functions. Similar application of the norming subset to the density of norm attaining holomorphic functions is given in [4]. We use the Lindenstrauss method [6].

Theorem 2.1. *Suppose that X is a Banach space and there is a numerical boundary $\Gamma \subset \Pi(X)$ of $A(B_X : X)$ such that for every $(x, x^*) \in \Gamma$, x is a strong peak point of $A(B_X)$. Then the set of numerical radius attaining elements in $A(B_X : X)$ is dense.*

Proof. We may assume that $\Gamma = \{(x_\alpha, x_\alpha^*)\}_\alpha$ and $\varphi_\alpha(x_\alpha) = 1$ for each α, where each φ_α is a strong peak function in $A(B_X)$. Notice that if $f \in A(B_X : X)$ and $v(f) = 0$, then $v(f) = 0 = |x^* f(x)|$ for any $(x, x^*) \in \Pi(X)$ and f attains its numerical radius. Hence we have only to show that if $f \in A(B_X : X)$, $v(f) = 1$ and $\epsilon > 0$, then there is $\widehat{f} \in A(B_X : X)$ such that $\widehat{f}$ attains its numerical radius and $\|f - \widehat{f}\| < \epsilon$.

Let $f \in A$ with $v(f) = 1$ and ϵ with $0 < \epsilon < 1/3$ be given. We choose a monotonically decreasing sequence $\{\epsilon_k\}$ of positive numbers so that

$$2\sum_{i=1}^{\infty}\epsilon_i < \epsilon, \quad 2\sum_{i=k+1}^{\infty}\epsilon_i < \epsilon_k^2, \quad \epsilon_k < \frac{1}{10k}, \quad k = 1, 2, \ldots. \tag{2.1}$$

We next choose inductively sequences $\{f_k\}_{k=1}^{\infty}$, $\{(x_{\alpha_k}, x_{\alpha_k}^*)\}_{k=1}^{\infty}$ in Γ satisfying

$$f_1 = f, \tag{2.2}$$

$$\left|x_{\alpha_k}^* f_k(x_{\alpha_k})\right| \geq v(f_k) - \epsilon_k^2, \tag{2.3}$$

$$f_{k+1}(x) = f_k(x) + \lambda_k \epsilon_k \widetilde{\varphi}_{\alpha_k}(x) \cdot x_{\alpha_k}, \tag{2.4}$$

$$\left|\widetilde{\varphi}_{\alpha_k}(x)\right| > 1 - \frac{1}{k} \quad \text{implies } \|x - x_{\alpha_k}\| < \frac{1}{k}, \tag{2.5}$$

where each λ_j is chosen in $S_{\mathbb{C}}$ to satisfy $|x^*_{\alpha_j} f_j(x_{\alpha j}) + \lambda_j \epsilon_j| = |x^*_{\alpha_j} f_j(x_{\alpha_j})| + \epsilon_k$ and each $\tilde{\varphi}_{\alpha_j}$ is $\varphi^{n_j}_{\alpha_j}$ for some positive integer n_j. Having chosen these sequences, we verify that the following hold:

$$\|f_j - f_k\| \le 2\sum_{i=j}^{k-1}\epsilon_i, \quad \|f_k\| \le \|f\| + \frac{1}{3}, \quad j < k,\ k = 2,3,\dots, \tag{2.6}$$

$$v(f_{k+1}) \ge v(f_k) + \epsilon_k - \epsilon_k^2, \quad k = 1,2,\dots, \tag{2.7}$$

$$v(f_k) \ge v(f_j), \quad j < k,\ k = 2,3,\dots, \tag{2.8}$$

$$\left|\tilde{\varphi}_{\alpha_j}(x_{\alpha_k})\right| > 1 - \frac{1}{j}, \quad j < k,\ k = 2,3,\dots. \tag{2.9}$$

Assertion (2.6) is easy by using induction on k. By (2.3) and (2.4),

$$\begin{aligned} v(f_{k+1}) &\ge \left|x^*_{\alpha_k} f_{k+1}(x_{\alpha_k})\right| = \left|x^*_{\alpha_k} f_k(x_{\alpha_k}) + \lambda_k\epsilon_k\tilde{\varphi}_{\alpha_k}(x_{\alpha_k})\right| \\ &= \left|x^*_{\alpha_k} f_k(x_{\alpha_k})\right| + \epsilon_k \ge v(f_k) - \epsilon_k^2 + \epsilon_k, \end{aligned} \tag{2.10}$$

so the relation (2.7) is proved. Therefore (2.8) is an immediate consequence (2.1) and (2.7). For $j < k$, by the triangle inequality, (2.3) and (2.6), we have

$$\begin{aligned} \left|x^*_{\alpha_k} f_{j+1}(x_{\alpha_k})\right| &\ge \left|x^*_{\alpha_k} f_k(x_{\alpha_k})\right| - \|f_k - f_{j+1}\| \\ &\ge v(f_k) - \epsilon_k^2 - 2\sum_{i=j+1}^{k-1}\epsilon_i \ge v(f_{j+1}) - 2\epsilon_j^2. \end{aligned} \tag{2.11}$$

Hence by (2.4) and (2.7),

$$\begin{aligned} \epsilon_j\left|\tilde{\varphi}_{\alpha_j}(x_{\alpha_k})\right| + v(f_j) &\ge \left|x^*_{\alpha_k} f_{j+1}(x_{\alpha_k})\right| \ge v(f_{j+1}) - 2\epsilon_j^2 \\ &\ge v(f_j) + \epsilon_j - 4\epsilon_j^2, \end{aligned} \tag{2.12}$$

so that

$$\left|\tilde{\varphi}_{\alpha_j}(x_{\alpha_k})\right| \ge 1 - 4\epsilon_j > 1 - \frac{1}{j}, \tag{2.13}$$

and this proves (2.9). Let $\hat{f} \in A$ be the limit of $\{f_k\}$ in the norm topology. By (2.1) and (2.6), $\|\hat{f} - f\| = \lim_n \|f_n - f_1\| \le 2\sum_{i=1}^{\infty}\epsilon_i \le \epsilon$ holds. The relations (2.5) and (2.9) mean that the sequence $\{x_{\alpha_k}\}$ converges to a point $\tilde{x}$, say and by (2.3), we have $v(\hat{f}) = \lim_n v(f_n) = \lim_n |x^*_{\alpha_n} f_n(x_{\alpha_n})| = |\tilde{x}^*\hat{f}(\tilde{x})|$, where $\tilde{x}^*$ is a weak-$*$ limit point of $\{x^*_{\alpha_k}\}_k$ in B_{X^*}. Then it is easy to see that $|\tilde{x}^*(\tilde{x})| = 1$. Hence $\hat{f}$ attains its numerical radius. This concludes the proof. □

Recall that a Banach space X is said to be *locally uniformly convex* if $x \in S_X$ and there is a sequence $\{x_n\}$ in B_X satisfying $\lim_n \|x_n + x\| = 2$, then $\lim_n \|x_n - x\| = 0$.

Corollary 2.2. *Let X be a locally uniformly convex Banach space. Then the set of numerical radius attaining elements in $A(B_X : X)$ is dense.*

Proof. Let $\Gamma = \Pi(X)$ and notice that every element in S_X is a strong peak point for $A_u(B_X)$. Indeed, if $x \in S_X$, choose $x^* \in S_{X^*}$ so that $x^*(x) = 1$. Set $f(y) = (x^*(y) + 1)/2$ for $y \in B_X$. Then $f \in A(B_X)$ and $f(x) = 1$. If $\lim_n |f(x_n)| = 1$ for some sequence $\{x_n\}$ in B_X, then $\lim_n x^*(x_n) = 1$. Since $|x^*(x_n) + x^*(x)| \le \|x_n + x\| \le 2$ for every n, $\|x_n + x\| \to 2$ and $\|x_n - x\| \to 0$ as $n \to \infty$. By Theorem 2.1, we get the desired result. □

It was shown in [7] that if a Banach sequence space X is locally uniformly c-convex and order continuous, then the set of all strong peak points for $A(B_X)$ is dense in S_X. Therefore, the set of all strong peak points for $A(B_X)$ is dense in S_X. For the definition of a Banach sequence space and order continuity, see [8, 9]. For the characterization of local uniform c-convexity in function spaces, see [7, 10].

Corollary 2.3. *Suppose that X is a locally uniformly c-convex order continuous Banach sequence space. Then the set of numerical radius attaining elements in $A_u(B_X : X)$ is dense.*

Proof. Let $\Gamma = \{(x, x^*) \in \Pi(X) : x$ be a strong peak point of $A_u(B_X)\}$. Then by [11, Theorem 2.5], and the remark above the Corollary 2.3, Γ is a numerical boundary of $A_u(B_X : X)$. Hence the proof is complete by Theorem 2.1. □

Acknowledgment

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2009-0069100).

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