

## Research Article

# A Parameter Robust Method for Singularly Perturbed Delay Differential Equations

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Uniform finite difference methods are constructed via nonstandard finite difference methods for the numerical solution of singularly perturbed quasilinear initial value problem for delay differential equations. A numerical method is constructed for this problem which involves the appropriate Bakhvalov meshes on each time subinterval. The method is shown to be uniformly convergent with respect to the perturbation parameter. A numerical example is solved using the presented method, and the computed result is compared with exact solution of the problem.

## 1. Introduction

Delay differential equations are used to model a large variety of practical phenomena in the biosciences, engineering and control theory, and in many other areas of science and technology, in which the time evolution depends not only on present states but also on states at or near a given time in the past (see, e.g., [1–4]). If we restrict the class of delay differential equations to a class in which the highest derivative is multiplied by a small parameter, then it is said to be a singularly perturbed delay differential equation. Such problems arise in the mathematical modeling of various practical phenomena, for example, in population dynamics [4], the study of bistable devices [5], description of the human pupil-light reflex [6], and variational problems in control theory [7]. In the direction of numerical study of singularly perturbed delay differential equation, much can be seen in [8–16].

The numerical analysis of singular perturbation cases has always been far from trivial because of the boundary layer behavior of the solution. Such problems undergo rapid changes within very thin layers near the boundary or inside the problem domain. It is well known that standard numerical methods for solving singular perturbation problems do not give a satisfactory result when the perturbation parameter is sufficiently small. Therefore, it is

important to develop suitable numerical methods for these problems, whose accuracy does not depend on the perturbation parameter, that is, methods that are uniformly convergent with respect to the perturbation parameter [17–20].

In order to construct parameter-uniform numerical methods for singularly perturbed differential equations, two different techniques are applied. Firstly, the fitted operator approach [20] which has coefficients of exponential type adapted to the singular perturbation problems. Secondly, the special mesh approach [19], which constructs meshes adapted to the solution of the problem.

The work contained in this paper falls under the second category. We use the nonstandard finite difference methods originally developed by Bakhvalov for some other problems. One of the simplest ways to derive such methods consists of using a class of special meshes (such as Bakhvalov meshes; see, e.g., [18–24]), which is constructed a priori and depend on the perturbation parameter, the problem data, and the number of corresponding mesh points.

In this paper, we study the following singularly perturbed delay differential problem in the interval  $\bar{I} = [0, T]$ :

$$\varepsilon u'(t) + a(t)u(t) = f(t, u(t-r)), \quad t \in I, \quad (1.1)$$

$$u(t) = \varphi(t), \quad t \in I_0, \quad (1.2)$$

where  $I = (0, T] = \bigcup_{p=1}^m I_p$ ,  $I_p = \{t : r_{p-1} < t \leq r_p\}$ ,  $1 \leq p \leq m$ , and  $r_s = sr$ , for  $0 \leq s \leq m$  and  $I_0 = (-r, 0]$ .  $0 < \varepsilon \leq 1$  is the perturbation parameter, and  $r > 0$  is a constant delay, which is independent of  $\varepsilon$ .  $a(t)$ ,  $\varphi(t)$ , and  $f(t, v)$  are given sufficiently smooth functions satisfying certain regularity conditions in  $\bar{I}$  and  $\bar{I} \times \mathbb{R}$ , respectively moreover

$$a(t) \geq \alpha > 0, \quad \left| \frac{\partial f}{\partial v} \right| \leq M < \infty. \quad (1.3)$$

The solution,  $u(t)$ , displays in general boundary layers on the right side of each point  $t = r_s$  ( $0 \leq s \leq m$ ) for small values of  $\varepsilon$ .

In the present paper we discretize (1.1)–(1.2) using a numerical method which is composed of an implicit finite difference scheme on special Bakhvalov meshes for the numerical solution on each timesubinterval. In Section 2, we state some important properties of the exact solution. In Section 3, we describe the finite difference discretization and introduce Bakhvalov-Shishkin mesh and Bakhvalov mesh. In Section 4, we present the error analysis for the approximate solution. Uniform convergence is proved in the discrete maximum norm. In Section 5, a test example is considered and a comparison of the numerical and exact solutions is presented.

In the works of Amiraliev and Erdogan [9], special meshes (Shishkin mesh) have been used. The method that we propose in this paper uses Bakhvalov-type meshes.

Throughout the paper,  $C$  denotes a generic positive constant independent of  $\varepsilon$  and the mesh parameter. Some specific, fixed constants of this kind are indicated by subscripting  $C$ .

## 2. The Continuous Problem

Before defining the mesh and the finite difference scheme, we show some results about the behavior with respect to the perturbation parameter of the exact solution of problem (1.1)-(1.2) and its derivatives, which we will use in later section for the analysis of an appropriate numerical solution. For any continuous function  $g(t)$ ,  $\|g\|_\infty$  denotes a continuous maximum norm on the corresponding closed interval  $I$ ; in particular we will use  $\|g\|_{\infty,p} = \max_{\bar{I}_p} |g(x)|$ ,  $0 \leq p \leq m$ .

**Lemma 2.1.** *The solution  $u(t)$  of the problem (1.1)-(1.2) satisfies the following estimates:*

$$\|u\|_{\infty,p} \leq C_p, \quad 1 \leq p \leq m, \quad (2.1)$$

where

$$C_p = \|\varphi\|_{\infty,0} (1 + \alpha^{-1}M)^p + \alpha^{-1} \sum_{s=1}^p (1 + \alpha^{-1}M)^{p-s} \|F\|_{\infty,p}, \quad p = 1, 2, \dots, m, \quad (2.2)$$

$$F(t) = f(t, 0),$$

$$|u'| \leq C \left\{ 1 + \frac{(t - r_{p-1})^{p-1}}{\varepsilon^p} \exp\left(-\frac{\alpha(t - r_{p-1})}{\varepsilon}\right) \right\}, \quad t \in I_p, \quad 1 \leq p \leq m, \quad (2.3)$$

provided

$$\left| \frac{\partial f}{\partial t} \right| \leq C, \quad \text{for } t \in \bar{I}, \quad |v| \leq C_0, \quad (2.4)$$

where

$$C_0 = \|\varphi\|_{\infty,0} (1 + \alpha^{-1}M)^m + \alpha^{-1} \|F\|_{\infty,\bar{I}} \left\{ (1 + \alpha^{-1}M)^{m-1} \right\}. \quad (2.5)$$

*Proof.* The quasilinear equation (1.1) can be written in the form

$$\varepsilon u'(t) + a(t)u(t) + b(t)u(t-r) = F(t), \quad t \in I, \quad (2.6)$$

where

$$b(t) = -\frac{\partial f}{\partial v}(t, \tilde{v}), \quad (2.7)$$

$$\tilde{v} = \gamma u(t-r) \quad (0 < \gamma < 1)\text{-intermediate values.}$$

Applying the maximum principle on  $I_p$  gives

$$\begin{aligned} \|u\|_{\infty,p} &\leq |u(r_{p-1})| + \alpha^{-1} \left( \|b\|_{\infty,p} \|u\|_{\infty,p-1} + \|F\|_{\infty,p} \right) \\ &\leq \left( 1 + \alpha^{-1} M \right) \|u\|_{\infty,p-1} + \alpha^{-1} \|F\|_{\infty,p}, \end{aligned} \quad (2.8)$$

which implies the first-order difference inequality

$$w_p \leq \mu w_{p-1} + \psi_p, \quad (2.9)$$

with

$$w_p = \|u\|_{\infty,p}, \quad \mu = 1 + \alpha^{-1} M, \quad \psi_p = \alpha^{-1} \|F\|_{\infty,p}. \quad (2.10)$$

From the last inequality, it follows that

$$w_p \leq w_0 \mu^p + \sum_{s=1}^p \mu^{p-s} \psi_s \quad (2.11)$$

which proves (2.1).

Now we prove (2.3). The proof is verified by induction. For  $p = 1$ , it is known that

$$|u'(t)| \leq C \left\{ 1 + \frac{1}{\varepsilon} \exp\left(-\frac{\alpha t}{\varepsilon}\right) \right\}. \quad (2.12)$$

Now, let (2.3) hold true for  $p = k$ . Differentiating (1.1), we have the relation for  $p = k+1$

$$\varepsilon u''(t) + a(t)u'(t) = g(t), \quad t \in I_{k+1}, \quad (2.13)$$

where

$$g(t) = -u(t) \frac{\partial a}{\partial t} + \frac{\partial f}{\partial t}(t, u(t-r)) + \frac{\partial f}{\partial v}(t, u(t-r))u'(t-r). \quad (2.14)$$

Then, from (2.13) we have the following relation for  $u'(t)$ :

$$u'(t) = u'(r_k) \exp\left(-\frac{1}{\varepsilon} \int_{r_k}^t a(s) ds\right) + \frac{1}{\varepsilon} \int_{r_k}^t g(\tau) \exp\left(-\frac{1}{\varepsilon} \int_{\tau}^t a(s) ds\right) d\tau. \quad (2.15)$$

Using the estimate (2.3) for  $p = k$  and  $t = t_k$ , we have

$$|u'(r_k)| \leq C \left\{ 1 + \frac{r^{k-1}}{\varepsilon^k} \exp\left(-\frac{\alpha r}{\varepsilon}\right) \right\}. \quad (2.16)$$

Hence,

$$|u'(r_k)| \leq C, \quad k \geq 1. \quad (2.17)$$

Furthermore, using now (2.3) for  $p = k$ , we get

$$\begin{aligned} |g(t)| &\leq \left| u(t) \frac{\partial a}{\partial t} \right| + \left| \frac{\partial f}{\partial t}(t, u(t-r)) \right| + \left| \frac{\partial f}{\partial v}(t, u(t-r)) \right| |(u'(t-r))| \\ &\leq C(1 + |u'(t-r)|) \\ &\leq C \left\{ 1 + \frac{(t-r_k)^{k-1}}{\varepsilon^k} \exp\left(-\frac{\alpha(t-r_k)}{\varepsilon}\right) \right\}. \end{aligned} \quad (2.18)$$

Taking into account (2.17) and (2.18) in (2.15), we have

$$\begin{aligned} |u'(t)| &\leq C \exp\left(\frac{-\alpha(t-r_k)}{\varepsilon}\right) \\ &\quad + \frac{1}{\varepsilon} C \int_{r_k}^t \left( 1 + \frac{(\tau-r_k)^{k-1}}{\varepsilon^k} \exp\left(\frac{-\alpha(\tau-r_k)}{\varepsilon}\right) \right) \exp\left(\frac{-\alpha(t-\tau)}{\varepsilon}\right) d\tau \\ &\leq C + C \frac{t-r_k}{\varepsilon} \alpha^{-1} \varepsilon \left( 1 - \exp\left(\frac{-\alpha(t-r_k)}{\varepsilon}\right) \right) \\ &\quad + \frac{1}{\varepsilon} C \exp\left(\frac{-\alpha(t-r_k)}{\varepsilon}\right) \frac{(t-r_k)^k}{k\varepsilon^k} \\ &\leq C \left\{ 1 + \frac{(t-r_k)^k}{\varepsilon^{k+1}} \exp\left(\frac{-\alpha(t-r_k)}{\varepsilon}\right) \right\}, \quad t \in I_{k+1}, \end{aligned} \quad (2.19)$$

which proves (2.3). □

### 3. Discretization and Mesh

Let  $\bar{\omega}_{N_0}$  be any nonuniform mesh on  $\bar{I}$

$$\bar{\omega}_{N_0} = \{0 = t_0 < t_1 < \dots < t_{N_0} = T, \tau_i = t_i - t_{i-1}\} \quad (3.1)$$

which contains by  $N$  mesh point at each subinterval  $I_p (1 \leq p \leq m)$

$$\omega_{N,p} = \{t_i : (p-1)N + 1 \leq i \leq pN\}, \quad 1 \leq p \leq m, \quad (3.2)$$

and consequently,

$$\omega_{N_0} = \bigcup_{p=1}^m \omega_{N,p}. \quad (3.3)$$

To simplify the notation, we set  $g_i = g(t_i)$  for any function  $g(t)$ ; moreover,  $y_i$  denotes an approximation of  $u(t)$  at  $t_i$ . For any mesh function  $\{\omega_i\}$  defined on  $\bar{\omega}_{N_0}$ , we use

$$\omega_{\bar{i},i} = \frac{(\omega_i - \omega_{i-1})}{\tau_i}, \quad (3.4)$$

$$\|\omega\|_{\infty, N, p} = \|\omega\|_{\infty, \omega_{N,p}} := \max_{(p-1)N \leq i \leq pN} |\omega_i|, \quad 1 \leq p \leq m.$$

For the difference approximation to (1.1), we integrate (1.1) over  $(t_{i-1}, t_i)$

$$\varepsilon u_{\bar{i},i} + \tau^{-1} \int_{t_{i-1}}^{t_i} a(t)u(t)dt = \tau^{-1} \int_{t_{i-1}}^{t_i} f(t, u(t-r))dt, \quad (3.5)$$

which yields the relation

$$\varepsilon u_{\bar{i},i} + a_i u_i + R_i = f(t_i, u_{i-N}), \quad 1 \leq i \leq N_0, \quad (3.6)$$

with the local truncation error

$$R_i = -\tau_i^{-1} \int_{t_{i-1}}^{t_i} \left\{ (t - t_{i-1}) \frac{d}{dt} (a(t)u(t)) \right\} dt$$

$$- \tau_i^{-1} \int_{t_{i-1}}^{t_i} \left\{ (t_{i-1} - t) \frac{d}{dt} f(t, u(t-r)) \right\} dt. \quad (3.7)$$

As a consequence of (3.6), we propose the following difference scheme for approximation to (1.1)-(1.2):

$$\varepsilon y_{\bar{i},i} + a_i y_i = f(t_i, u_{i-N}), \quad 1 \leq i \leq N_0, \quad (3.8)$$

$$y_i = \varphi_i, \quad -N \leq i \leq 0.$$

We consider two special discretization meshes, both dense in the boundary layer. We illustrate that the essential idea of Bakhvalov [21] by constructing special nonuniform meshes and has been combined with various difference schemes in numerous papers [22, 23].

### 3.1. Bakhvalov-Shishkin Mesh

Let us introduce a non-uniform mesh  $\omega_{N,p}$  which will be generated as follows. For the even number  $N$ , the non-uniform mesh  $\omega_{N,p}$  divides each of the interval  $[r_{p-1}, \sigma_p]$  and  $[\sigma_p, r_p]$  into  $N/2$  subintervals, where the transition point  $\sigma_p$ , which separates the fine and coarse portions of the mesh is defined by

$$\sigma_p = r_{p-1} + \alpha^{-1} \theta_p \varepsilon \ln N, \quad 1 \leq p \leq m, \quad (3.9)$$

where  $\theta_1 \geq 1$  and  $\theta_p > 1$  ( $2 \leq p \leq m$ ) are some constants. We will assume throughout the paper that  $\varepsilon \leq N^{-1}$ , as is generally the case in practice.

Hence, if  $\tau_p$  denote the step sizes in  $[\sigma_p, r_p]$ , we have

$$\tau_p = 2(r_p - \sigma_p)N^{-1}, \quad 1 \leq p \leq m. \quad (3.10)$$

The corresponding mesh points are

$$t_i = \begin{cases} r_{p-1} - \alpha^{-1} \theta_p \varepsilon \ln \left[ 1 - \frac{(1 - N^{-1})2i}{N} \right], & i = (p-1)N, \dots, \left(p - \frac{1}{2}\right)N, \\ \sigma_p + \left(i - \frac{N}{2}\right)\tau_p, & i = \left(p - \frac{1}{2}\right)N + 1, \dots, pN, \quad 1 \leq p \leq m. \end{cases} \quad (3.11)$$

### 3.2. Bakhvalov Mesh

In order the difference scheme (3.8), to be  $\varepsilon$ -uniform convergent, we will use the fitted form of  $\omega_{N,p}$ . This is a special non-uniform mesh which is condensed in the boundary layer. The fitted special non-uniform mesh  $\omega_{N,p}$  on the interval  $[r_{p-1}, r_p]$  is formed by dividing the interval into two subintervals  $[r_{p-1}, \sigma_p]$  and  $[\sigma_p, r_p]$ , where

$$\sigma_p = r_{p-1} - \alpha^{-1} \theta_p \varepsilon \ln \varepsilon, \quad 1 \leq p \leq m. \quad (3.12)$$

In practice one usually has  $\sigma_p \leq r_p$ . So, the mesh is fine on  $[r_{p-1}, \sigma_p]$  and coarse on  $[\sigma_p, r_p]$ . The corresponding mesh points are

$$t_i = \begin{cases} r_{p-1} - \alpha^{-1} \theta_p \varepsilon \ln \left[ 1 - \frac{(1 - \varepsilon)2i}{N} \right], & i = (p-1)N, \dots, \left(p - \frac{1}{2}\right)N, \\ \sigma_p + \left(i - \frac{N}{2}\right)\tau_p, & i = \left(p - \frac{1}{2}\right)N + 1, \dots, pN, \quad 1 \leq p \leq m. \end{cases} \quad (3.13)$$

#### 4. Stability and Convergence Analysis

To investigate the convergence of the method, note that the error function  $z_i = y_i - u_i$ ,  $0 \leq i \leq N_0$ , is the solution of the discrete problem

$$\begin{aligned} \varepsilon z_{i,i} + a_i z_i + R_i &= f(t_i, y_{i-N}) - f(t_i, u_{i-N}), \quad 1 \leq i \leq N_0, \\ z_i &= \varphi_i, \quad -N \leq i \leq 0, \end{aligned} \quad (4.1)$$

where the truncation error  $R_i$  is given by (3.7).

**Lemma 4.1.** *Let  $y_i$  be an approximate solution of (1.1)-(1.2). Then, the following estimate holds*

$$\|y\|_{\infty, \omega_{N,p}} \leq \|\varphi\|_{\infty, \omega_{N,0}} (1 + \alpha^{-1}M)^p + \alpha^{-1} \sum_{k=1}^p \|f\|_{\infty, \omega_{N,k}} (1 + \alpha^{-1}M)^{p-1}, \quad 1 \leq p \leq m. \quad (4.2)$$

*Proof.* The proof follows easily by induction in  $p$ , by analogy with differential case.  $\square$

**Lemma 4.2.** *Let  $z_i$  be the solution of (4.1). Then, the following estimate holds:*

$$\|z\|_{\infty, N, p} \leq C \sum_{k=1}^p \|R\|_{\infty, \omega_{N,k}}, \quad 1 \leq p \leq m. \quad (4.3)$$

*Proof.* It evidently follows from (4.2) by taking  $\varphi \equiv 0$  and  $f \equiv R$ .  $\square$

**Lemma 4.3.** *Under the above assumptions of Section 1 and Lemma 2.1, for the error function  $R_i$ , the following estimate holds:*

$$\|R\|_{\infty, \omega_{N,p}} \leq CN^{-1}, \quad 1 \leq p \leq m. \quad (4.4)$$

*Proof.* From explicit expression (3.7) for  $R_i$ , on an arbitrary mesh, we have

$$|R_i| \leq \tau_i^{-1} \int_{t_{i-1}}^{t_i} (t - t_{i-1}) \left| \frac{d}{dt} (a(t)u(t) - f(t, u(t-r))) \right| dt, \quad 1 \leq i \leq N_0. \quad (4.5)$$

This inequality together with (2.1) enables us to write

$$|R_i| \leq C \left\{ \tau_i + \int_{t_{i-1}}^{t_i} (|u'(t)| + |u'(t-r)|) dt \right\}, \quad 1 \leq i \leq N_0. \quad (4.6)$$



From here, in view of (2.3), it follows that

$$|R_i| \leq C \left\{ \tau_i + \frac{1}{\varepsilon} \int_{t_{i-1}}^{t_i} e^{-\alpha t/\varepsilon} dt \right\}, \quad \text{for } 1 \leq i \leq N, \tag{4.7}$$

$$|R_i| \leq C \left\{ \tau_i + \int_{t_{i-1}}^{t_i} \frac{(t - r_{p-1})^{p-1}}{\varepsilon^p} e^{-\alpha(t-r_{p-1})/\varepsilon} dt + \int_{t_{i-1}}^{t_i} \frac{(t - r_{p-1})^{p-2}}{\varepsilon^{p-1}} e^{-\alpha(t-r_{p-1})/\varepsilon} dt \right\}, \tag{4.8}$$

for  $t_i \in I_p (p > 1)$ .

Applying the inequality  $x^k e^{-x} \leq C e^{-\gamma x}$ ,  $0 < \gamma < 1$ ,  $x \in [0, \infty)$  to (4.7), we deduce

$$|R_i| \leq C \left\{ \tau_i + \frac{1}{\varepsilon} \int_{t_{i-1}}^{t_i} e^{-\alpha(t-r_{p-1})/\theta_p \varepsilon} dt \right\}, \quad \text{for } t_i \in I_p, \theta_p > 1, p > 1. \tag{4.9}$$

Combining (4.7) and (4.9), we can write

$$|R_i| \leq C \left\{ \tau_i + \frac{1}{\varepsilon} \int_{t_{i-1}}^{t_i} e^{-\alpha(t-r_{p-1})/\theta_p \varepsilon} dt \right\}, \quad \text{for } t_i \in I_p, p = 1, 2, \dots, m, \theta_1 \geq 1, \theta_p > 1 (p \geq 2) \tag{4.10}$$

where

$$\tau_i = \tau_p, \quad \left( p - \frac{1}{2} \right) N + 1 \leq i \leq pN. \tag{4.11}$$

At each submesh  $\omega_{N,p}$ , we estimate the truncation error  $R_i$  for Bakhvalov-Shishkin mesh as follows. We estimate  $R_i$  on  $[r_{p-1}, \sigma_p]$  and  $[\sigma_p, r_p]$  separately. We consider that  $t_i \in [\sigma_p, r_p]$ . We obtain from (4.10) that

$$\begin{aligned} |R_i| &\leq C \left\{ \tau_p + \alpha^{-1} \theta_p \left( e^{-\alpha(t_{i-1}-r_{p-1})/\theta_p \varepsilon} - e^{-\alpha(t_i-r_{p-1})/\theta_p \varepsilon} \right) \right\} \\ &= C \left\{ \tau_p + \alpha^{-1} \theta_p N^{-1} e^{-\alpha(i-1-(p-1/2)N)\tau_p/\theta_p \varepsilon} \left( 1 - e^{-\alpha\tau_p/\theta_p \varepsilon} \right) \right\}. \end{aligned} \tag{4.12}$$

This implies that

$$|R_i| \leq CN^{-1}. \tag{4.13}$$

On the other hand, in the layer region  $[r_{p-1}, \sigma_p]$ , (4.10) becomes

$$|R_i| \leq C \left\{ \tau_i + \alpha^{-1} \theta_p \left( e^{-\alpha(t_{i-1}-r_{p-1})/\theta_p \varepsilon} - e^{-\alpha(t_i-r_{p-1})/\theta_p \varepsilon} \right) \right\}. \tag{4.14}$$

Hereby, since

$$\begin{aligned} \tau_i &= t_i - t_{i-1} \\ &= \alpha^{-1} \theta_p \varepsilon \left\{ -\ln \left[ 1 - \frac{(1-N^{-1})2i}{N} \right] + \ln \left[ 1 - \frac{(1-N^{-1})2(i-1)}{N} \right] \right\} \end{aligned} \quad (4.15)$$

$$\leq 2\alpha^{-1} \theta_p \varepsilon (1 - N^{-1}) \leq CN^{-1},$$

$$e^{-\alpha t_{i-1}/\varepsilon} - e^{-\alpha t_i/\varepsilon} = 2(1 - N^{-1})N^{-1} \quad (4.16)$$

then

$$|R_i| \leq 4\alpha^{-1} \theta_p CN^{-1}, \quad (p-1)N \leq i \leq \left(p - \frac{1}{2}\right)N, \quad 1 \leq p \leq m. \quad (4.17)$$

We estimate the truncation error  $R_i$  for Bakhvalov mesh as follows. We consider first  $t_i \in [\sigma_p, r_p]$ . In  $[\sigma_p, r_p]$ ; that is, outside the layer  $|u'(t)| \leq C$  and  $|u'(t-r)| \leq C(\varepsilon^{-p} e^{-at/\varepsilon} \leq 1)$  by (2.1) and (4.7). Hereby, we get from (4.7) and (4.10) that

$$|R_i| \leq C\tau_i, \quad (p-1)N \leq i \leq \left(p - \frac{1}{2}\right)N. \quad (4.18)$$

Hence,

$$|R_i| \leq 2CrN^{-1}, \quad (p-1)N \leq i \leq \left(p - \frac{1}{2}\right)N. \quad (4.19)$$

Next, we estimate  $R_i$  for  $[r_{p-1}, \sigma_p]$ .

Since

$$\begin{aligned} \tau_i &= t_i - t_{i-1} \\ &= \alpha^{-1} \theta_p \varepsilon \left\{ -\ln \left[ 1 - \frac{(1-\varepsilon)2i}{N} \right] + \ln \left[ 1 - \frac{(1-\varepsilon)2(i-1)}{N} \right] \right\} \end{aligned} \quad (4.20)$$

$$\leq 2\alpha^{-1} \theta_p (1 - \varepsilon)N^{-1},$$

$$e^{-\alpha t_{i-1}/\varepsilon} - e^{-\alpha t_i/\varepsilon} = 2(1 - \varepsilon)N^{-1}, \quad (4.21)$$

recalling that  $\varepsilon \leq N^{-1}$ , it then follows from (4.12) that

$$|R_i| \leq 4\alpha^{-1} \theta_p CN^{-1}. \quad (4.22)$$

Thus, the proof is completed.  $\square$

Combining the previous lemmas gives us the following convergence result.

**Table 1:** Maximum Errors and Rates of Convergence for the Bakhvalov-Shishkin Mesh on  $\omega_{N,1}$ .

$\varepsilon$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
$2^{-2}$	0.00978429 0.987	0.00493577 0.993	0.00247899 0.996	0.00124229 0.998	0.00062184
$2^{-4}$	0.016348 0.975	0.00831665 0.987	0.0041954 0.993	0.00210714 0.996	0.00105595
$2^{-6}$	0.0230541 0.963	0.0118195 0.980	0.00598914 0.990	0.00301454 0.995	0.00151234
$2^{-8}$	0.0298948 0.952	0.0154465 0.975	0.00785801 0.987	0.00396404 0.993	0.00199094
$2^{-10}$	0.0366571 0.942	0.0190685 0.967	0.0097511 0.984	0.00492979 0.991	0.00247866
$2^{-12}$	0.0432959 0.931	0.022705 0.963	0.0116405 0.980	0.00589844 0.990	0.00296889
$2^{-14}$	0.0493475 0.911	0.0262615 0.958	0.0135164 0.977	0.00686448 0.988	0.00345923
$2^{-16}$	0.0560001 0.911	0.0297789 0.52	0.0153866 0.975	0.00782756 0.987	0.00394867

**Theorem 4.4.** Let  $u$  be the solution of (1.1)-(1.2), and let  $y$  be the solution of (3.8). Then, for both meshes the following estimate holds:

$$\|y - u\|_{\infty, \bar{\omega}_{N,p}} \leq CN^{-1}, \quad 1 \leq p \leq m, \quad (4.23)$$

where  $C$  is a constant independent of  $N$  and  $\varepsilon$ .

## 5. Numerical Results

We begin with an example from Driver [2] for which we possess the exact solution

$$\begin{aligned} \varepsilon u'(t) + u(t) &= u(t-1), \quad t \in [0, T], \\ u(t) &= 1+t, \quad -1 \leq t \leq 0. \end{aligned} \quad (5.1)$$

The exact solution for  $0 \leq t \leq 2$  is given by

$$u(t) = \begin{cases} -\varepsilon + t + (1 + \varepsilon)e^{-t/\varepsilon}, & t \in [0, 1], \\ -1 - 2\varepsilon + t + (1 + \varepsilon)e^{-t/\varepsilon} + \left[ \varepsilon - \frac{1}{\varepsilon} + \left(1 + \frac{1}{\varepsilon}\right)t \right] e^{(1-t)/\varepsilon}, & t \in (1, 2]. \end{cases} \quad (5.2)$$

**Table 2:** Maximum Errors and Rates of Convergence for the Bakhvalov-Shishkin Mesh on  $\omega_{N,2}$ .

$\varepsilon$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
$2^{-2}$	0.0120441 0.983	0.00609088 0.991	0.00306261 0.995	0.00153567 0.997	0.00076893
$2^{-4}$	0.0204344 0.939	0.0106567 0.973	0.00542574 0.985	0.0027399 0.992	0.00137664
$2^{-6}$	0.0206243 0.741	0.0123374 0.894	0.00663473 0.936	0.00346693 0.960	0.00178218
$2^{-8}$	0.0251094 0.951	0.0129806 0.975	0.00660313 0.929	0.00346667 0.951	0.00192158
$2^{-10}$	0.0308922 0.945	0.0160402 0.968	0.00819434 0.992	0.00414173 0.996	0.00208219
$2^{-12}$	0.0358208 0.909	0.0190729 0.963	0.00978373 0.981	0.00495569 0.990	0.00249403
$2^{-14}$	0.0418982 0.924	0.0220722 0.957	0.0113657 0.978	0.00576776 0.988	0.00290598
$2^{-16}$	0.0471824 0.915	0.0250121 0.951	0.0129303 0.975	0.00657754 0.987	0.0033174

**Table 3:** Maximum Errors and Rates of Convergence for the Bakhvalov Mesh on  $\omega_{N,1}$ .

$\varepsilon$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
$2^{-2}$	0.0140074 0.987	0.00709303 0.993	0.00356936 0.996	0.00179046 0.998	0.000896684
$2^{-4}$	0.0241181 0.975	0.0143603 0.987	0.00831665 0.993	0.00471467 0.996	0.00263101
$2^{-6}$	0.0230541 0.963	0.0137267 0.980	0.00794974 0.990	0.00450667 0.995	0.00251493
$2^{-8}$	0.0227881 0.952	0.0135684 0.975	0.00785801 0.987	0.00445467 0.993	0.00248591
$2^{-10}$	0.0227216 0.942	0.0135288 0.967	0.00783508 0.984	0.00444167 0.991	0.00247866
$2^{-12}$	0.0227050 0.931	0.0135189 0.963	0.00782935 0.980	0.00443842 0.990	0.00247684
$2^{-14}$	0.0227008 0.911	0.0135164 0.958	0.0782791 0.977	0.00443761 0.988	0.00247639
$2^{-16}$	0.0226998 0.911	0.0135158 0.52	0.00782756 0.975	0.0044374 0.987	0.00247628

We define the computed parameter-uniform maximum error  $e_\varepsilon^{N,p}$  as follows:

$$e_\varepsilon^{N,p} = \|y - u\|_{\infty, \omega_{N,p}}, \quad p = 1, 2, \quad (5.3)$$

**Table 4:** Maximum Errors and Rates of Convergence for the Bakhvalov Mesh on  $\omega_{N,2}$ .

$\varepsilon$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
$2^{-2}$	0.0121386 0.983	0.00613925 0.991	0.00308755 0.995	0.00154829 0.997	0.000775281
$2^{-4}$	0.0202600 0.939	0.0120754 0.973	0.00698853 0.985	0.00396095 0.992	0.00221017
$2^{-6}$	0.0206243 0.741	0.0115426 0.894	0.00668021 0.936	0.0037862 0.960	0.00211266
$2^{-8}$	0.0191427 0.951	0.0114094 0.975	0.00660313 0.929	0.00374251 0.951	0.00208828
$2^{-10}$	0.0190868 0.945	0.0113761 0.968	0.00658386 0.992	0.00373159 0.996	0.00208219
$2^{-12}$	0.0190729 0.909	0.0113678 0.963	0.00657904 0.981	0.00372886 0.990	0.00208066
$2^{-14}$	0.0190694 0.924	0.0113657 0.957	0.0657784 0.978	0.00372817 0.988	0.00208028
$2^{-16}$	0.0190685 0.915	0.0113652 0.951	0.00657754 0.975	0.003728 0.987	0.00208019

where  $y$  is the numerical approximation to  $u$  for various values of  $N, \varepsilon$ . We also define the computed parameter-uniform rate of convergence to be

$$r^{N,p} = \frac{\ln(e^{N,p}/e^{2N,p})}{\ln 2}, \quad p = 1, 2. \quad (5.4)$$

The values of  $\varepsilon$  for which we solve the test problem are  $\varepsilon = 2^{-i}, i = 2, 4, \dots, 16$ . Tables 1, 2, 3, and 4 verify the  $\varepsilon$ -uniform convergence of the numerical solution on both subintervals, and computed rates are essentially in agreement with our theoretical analysis.

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