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*Review Article*

# Nonlinear $\mathcal{L}$-Random Stability of an ACQ Functional Equation

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We prove the generalized Hyers-Ulam stability of the following additive-cubic-quartic functional equation: $11f(x+2y)+11f(x-2y) = 44f(x+y)+44f(x-y)+12f(3y)-48f(2y)+60f(y)-66f(x)$ in complete latticetic random normed spaces.

## 1. Introduction

Random theory is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. It has also very useful applications in various fields, for example, population dynamics, chaos control, computer programming, nonlinear dynamical systems, nonlinear operators, statistical convergence, and so forth. The random topology proves to be a very useful tool to deal with such situations where the use of classical theories breaks down. The usual uncertainty principle of Werner Heisenberg leads to a generalized uncertainty principle, which has been motivated by string theory and noncommutative geometry. In strong quantum gravity regime space-time points are determined in a random manner. Thus impossibility of determining the position of particles gives the space-time a random structure. Because of this random structure, position space representation of quantum mechanics breaks down, and therefore a generalized normed space of quasiposition eigenfunction is required. Hence, one needs to discuss on a new family of random norms. There are many situations where the norm of a vector is not possible to be found and the concept of random norm seems to be more suitable in such cases, that is, we can deal with such situations by modeling the inexactness through the random norm [1, 2].

The stability problem of functional equations originated from a question of Ulam [3] concerning the stability of group homomorphisms. Hyers [4] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [5] for additive mappings and by Th. M. Rassias [6] for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias [6] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* or as *Hyers-Ulam-Rassias stability* of functional equations

$$\begin{aligned} 11f(x+2y)+11f(x-2y) &= 44f(x+y)+44f(x-y)+12f(3y) \\ &\quad -48f(2y)+60f(y)-66f(x). \end{aligned} \tag{1.1}$$

A generalization of the Th. M. Rassias theorem was obtained by Găvruţa [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias approach.

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [6, 8–24]).

In [25], Jun and Kim considered the following cubic functional equation:

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{1.2}$$

It is easy to show that the function $f(x)=x^3$ satisfies the functional equation (1.2), which is called a *cubic functional equation*, and every solution of the cubic functional equation is said to be a *cubic mapping*.

In [26], Lee et al. considered the following quartic functional equation:

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \tag{1.3}$$

It is easy to show that the function $f(x)=x^4$ satisfies the functional equation (1.3), which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*.

The study of stability of functional equations is important problem in nonlinear sciences and application in solving integral equation via VIM [27–29] PDE and ODE [30–34]. Let $X$ be a set A function $d : X \times X \to [0,\infty]$ is called a *generalized metric* on $X$ if $d$ satisfies

(1) $d(x,y)=0$ if and only if $x=y$;

(2) $d(x,y)=d(y,x)$ for all $x,y\in X$;

(3) $d(x,z)\le d(x,y)+d(y,z)$ for all $x,y,z\in X$.

We recall a fundamental result in fixed point theory.

**Theorem 1.1** (see [35, 36]). *Let $(X,d)$ be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then for each given element $x\in X$, either*

$$d\left(J^n x, J^{n+1}x\right)=\infty \tag{1.4}$$

*for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that*

(1) $d(J^n x, J^{n+1}x) < \infty$, *for all* $n \geq n_0$;

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* $y^*$ *of* $J$;

(3) $y^*$ *is the unique fixed point of* $J$ *in the set* $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;

(4) $d(y, y^*) \leq (1/(1-L))d(y, Jy)$ *for all* $y \in Y$.

In 1996, Isac and Th. M. Rassias [37] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [38–43]).

## 2. Preliminaries

The theory of random normed spaces (RN-spaces) is important as a generalization of deterministic result of linear normed spaces and also in the study of random operator equations. The RN-spaces may also provide us the appropriate tools to study the geometry of nuclear physics and have important application in quantum particle physics. The generalized Hyers-Ulam stability of different functional equations in random normed spaces, RN-spaces and fuzzy normed spaces has been recently studied by Alsina [44], Mirmostafaee and Moslehian [45] and Mirzavaziri and Moslehian [40], Miheţ and Radu [46], Miheţ et al. [47, 48], Baktash et al. [49], and Saadati et al. [50].

Let $\mathcal{L} = (L, \geq_L)$ be a complete lattice, that is, a partially ordered set in which every nonempty subset admits supremum and infimum, and $0_{\mathcal{L}} = \inf L$, $1_{\mathcal{L}} = \sup L$. The space of latticetic random distribution functions, denoted by $\Delta_L^+$, is defined as the set of all mappings $F : \mathbb{R} \cup \{-\infty, +\infty\} \to L$ such that $F$ is left continuous and nondecreasing on $\mathbb{R}$, $F(0) = 0_{\mathcal{L}}$, $F(+\infty) = 1_{\mathcal{L}}$.

$D_L^+ \subseteq \Delta_L^+$ is defined as $D_L^+ = \{F \in \Delta_L^+ : l^- F(+\infty) = 1_{\mathcal{L}}\}$, where $l^- f(x)$ denotes the left limit of the function $f$ at the point $x$. The space $\Delta_L^+$ is partially ordered by the usual point-wise ordering of functions, that is, $F \geq G$ if and only if $F(t) \geq_L G(t)$ for all $t$ in $\mathbb{R}$. The maximal element for $\Delta_L^+$ in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0_{\mathcal{L}}, & \text{if } t \leq 0, \\ 1_{\mathcal{L}}, & \text{if } t > 0. \end{cases} \tag{2.1}$$

*Definition 2.1* (see [51]). A *triangular norm* ($t$-norm) on $L$ is a mapping $\mathcal{T} : (L)^2 \to L$ satisfying the following conditions:

(a) $(\forall x \in L)$ $(\mathcal{T}(x, 1_{\mathcal{L}}) = x)$ (boundary condition);

(b) $(\forall (x, y) \in (L)^2)$ $(\mathcal{T}(x, y) = \mathcal{T}(y, x))$ (commutativity);

(c) $(\forall (x, y, z) \in (L)^3)$ $(\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z))$ (associativity);

(d) $(\forall (x, x', y, y') \in (L)^4)$ $(x \leq_L x'$ and $y \leq_L y' \Rightarrow \mathcal{T}(x, y) \leq_L \mathcal{T}(x', y'))$ (monotonicity).

Let $\{x_n\}$ be a sequence in $L$ which converges to $x \in L$ (equipped order topology). The $t$-norm $\mathcal{T}$ is said to be a *continuous t-norm* if

$$\lim_{n\to\infty} \mathcal{T}(x_n, y) = \mathcal{T}(x, y), \tag{2.2}$$

for all $y \in L$.

A $t$-norm $\mathcal{T}$ can be extended (by associativity) in a unique way to an $n$-array operation taking for $(x_1, \ldots, x_n) \in L^n$ the value $\mathcal{T}(x_1, \ldots, x_n)$ defined by

$$\mathcal{T}_{i=1}^{0} x_i = 1, \quad \mathcal{T}_{i=1}^{n} x_i = \mathcal{T}\left(\mathcal{T}_{i=1}^{n-1} x_i, x_n\right) = \mathcal{T}(x_1, \ldots, x_n). \tag{2.3}$$

$\mathcal{T}$ can also be extended to a countable operation taking for any sequence $(x_n)_{n\in N}$ in $L$ the value

$$\mathcal{T}_{i=1}^{\infty} x_i = \lim_{n\to\infty} \mathcal{T}_{i=1}^{n} x_i. \tag{2.4}$$

The limit on the right side of (2.4) exists since the sequence $(\mathcal{T}_{i=1}^{n} x_i)_{n\in\mathbb{N}}$ is nonincreasing and bounded from below.

Note that we put $\mathcal{T} = T$ whenever $L = [0,1]$. If $T$ is a $t$-norm then $x_T^{(n)}$ is defined for all $x \in [0,1]$ and $n \in N \cup \{0\}$ by 1, if $n = 0$ and $T(x_T^{(n-1)}, x)$, if $n \geq 1$. A $t$-norm $T$ is said to be *of Hadžić-type* (we denote by $T \in \mathcal{H}$) if the family $(x_T^{(n)})_{n\in N}$ is equicontinuous at $x = 1$ (cf. [52]).

*Definition 2.2* (see [51]). A continuous $t$-norm $\mathcal{T}$ on $L = [0,1]^2$ is said to be *continuous t-representable* if there exist a continuous $t$-norm $*$ and a continuous $t$-conorm $\diamond$ on $[0,1]$ such that, for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in L$,

$$\mathcal{T}(x, y) = (x_1 * y_1, x_2 \diamond y_2). \tag{2.5}$$

For example,

$$\begin{aligned} \mathcal{T}(a, b) &= (a_1 b_1, \min\{a_2 + b_2, 1\}), \\ \mathbf{M}(a, b) &= (\min\{a_1, b_1\}, \max\{a_2, b_2\}) \end{aligned} \tag{2.6}$$

for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in [0,1]^2$ are continuous $t$-representable.

Define the mapping $\mathcal{T}_\wedge$ from $L^2$ to $L$ by

$$\mathcal{T}_\wedge(x, y) = \begin{cases} x, & \text{if } y \geq_L x, \\ y, & \text{if } x \geq_L y. \end{cases} \tag{2.7}$$

Recall (see [52, 53]) that if $\{x_n\}$ is a given sequence in $L$, $(\mathcal{T}_\wedge)_{i=1}^{n} x_i$ is defined recurrently by $(\mathcal{T}_\wedge)_{i=1}^{1} x_i = x_1$ and $(\mathcal{T}_\wedge)_{i=1}^{n} x_i = \mathcal{T}_\wedge((\mathcal{T}_\wedge)_{i=1}^{n-1} x_i, x_n)$ for $n \geq 2$.

A negation on $\mathcal{L}$ is any decreasing mapping $\mathcal{N} : L \to L$ satisfying $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L$, then $\mathcal{N}$ is called an involutive negation. In the following, $\mathcal{L}$ is endowed with a (fixed) negation $\mathcal{N}$.

*Definition 2.3.* A *latticetic random normed space* is a triple $(X, \mu, \mathcal{T}_\wedge)$, where $X$ is a vector space and $\mu$ is a mapping from $X$ into $D_L^+$ such that the following conditions hold:

(LRN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;

(LRN2) $\mu_{\alpha x}(t) = \mu_x(t/|\alpha|)$ for all $x$ in $X$, $\alpha \neq 0$ and $t \geq 0$;

(LRN3) $\mu_{x+y}(t+s) \geq_L \mathcal{T}_\wedge(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

We note that from (LPN2) it follows that $\mu_{-x}(t) = \mu_x(t)$ $(x \in X, t \geq 0)$.

*Example 2.4.* Let $L = [0,1] \times [0,1]$ and operation $\leq_L$ be defined by

$$\begin{aligned} L &= \{(a_1, a_2) : (a_1, a_2) \in [0,1] \times [0,1],\ a_1 + a_2 \leq 1\}, \\ (a_1, a_2) \leq_L (b_1, b_2) &\Longleftrightarrow a_1 \leq b_1, \quad a_2 \geq b_2, \quad \forall a = (a_1, a_2),\ b = (b_1, b_2) \in L. \end{aligned} \tag{2.8}$$

Then $(L, \leq_L)$ is a complete lattice (see [51]). In this complete lattice, we denote its units by $0_L = (0,1)$ and $1_L = (1,0)$. Let $(X, \|\cdot\|)$ be a normed space. Let $\mathcal{T}(a,b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$ for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in [0,1] \times [0,1]$ and $\mu$ be a mapping defined by

$$\mu_x(t) = \left(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|}\right), \quad \forall t \in \mathbb{R}^+. \tag{2.9}$$

Then $(X, \mu, \mathcal{T})$ is a latticetic random normed space.

If $(X, \mu, \mathcal{T}_\wedge)$ is a latticetic random normed space, then

$$\mathcal{V} = \{V(\varepsilon, \lambda) : \varepsilon >_L 0_{\mathcal{L}},\ \lambda \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}\}, \qquad V(\varepsilon, \lambda) = \{x \in X : F_x(\varepsilon) >_L \mathcal{N}(\lambda)\} \tag{2.10}$$

is a complete system of neighborhoods of null vector for a linear topology on $X$ generated by the norm $F$.

*Definition 2.5.* Let $(X, \mu, \mathcal{T}_\wedge)$ be a latticetic random normed space.

(1) A sequence $\{x_n\}$ in $X$ is said to be *convergent* to $x$ in $X$ if, for every $t > 0$ and $\varepsilon \in L \setminus \{0_{\mathcal{L}}\}$, there exists a positive integer $N$ such that $\mu_{x_n - x}(t) >_L \mathcal{N}(\varepsilon)$ whenever $n \geq N$.

(2) A sequence $\{x_n\}$ in $X$ is called *Cauchy sequence* if, for every $t > 0$ and $\varepsilon \in L \setminus \{0_{\mathcal{L}}\}$, there exists a positive integer $N$ such that $\mu_{x_n - x_m}(t) >_L \mathcal{N}(\varepsilon)$ whenever $n \geq m \geq N$.

(3) A latticetic random normed spaces $(X, \mu, \mathcal{T}_\wedge)$ is said to be *complete* if and only if every Cauchy sequence in $X$ is convergent to a point in $X$.

**Theorem 2.6.** *If $(X, \mu, \mathcal{T}_\wedge)$ is a latticetic random normed space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$.*

*Proof.* The proof is the same as classical random normed spaces, see [54]. □

**Lemma 2.7.** *Let* $(X, \mu, \mathcal{T}_\wedge)$ *be a latticetic random normed space and* $x \in X$*. If*

$$\mu_x(t) = C, \quad \forall t > 0, \tag{2.11}$$

*then* $C = 1_{\mathcal{L}}$ *and* $x = 0$.

*Proof.* Let $\mu_x(t) = C$ for all $t > 0$. Since $\mathrm{Ran}(\mu) \subseteq D_L^+$, we have $C = 1_{\mathcal{L}}$, and by (LRN1) we conclude that $x = 0$. □

## 3. Generalized Hyers-Ulam Stability of the Functional Equation (1.1): An Odd Case

One can easily show that an even mapping $f : X \to Y$ satisfies (1.1) if and only if the even mapping $f : X \to Y$ is a quartic mapping, that is,

$$f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y), \tag{3.1}$$

and that an odd mapping $f : X \to Y$ satisfies (1.1) if and only if the odd mapping $f : X \to Y$ is an additive-cubic mapping, that is,

$$f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x). \tag{3.2}$$

It was shown in Lemma 2.2 of [55] that $g(x) := f(2x) - 2f(x)$ and $h(x) := f(2x) - 8f(x)$ are cubic and additive, respectively, and that $f(x) = (1/6)g(x) - (1/6)h(x)$.

For a given mapping $f : X \to Y$, we define

$$\begin{aligned} Df(x, y) : = {} & 11f(x + 2y) + 11f(x - 2y) - 44f(x + y) - 44f(x - y) \\ & - 12f(3y) + 48f(2y) - 60f(y) + 66f(x) \end{aligned} \tag{3.3}$$

for all $x, y \in X$.

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in complete LRN-spaces: an odd case.

**Theorem 3.1.** *Let* $X$ *be a linear space,* $(Y, \mu, \mathcal{T}_\wedge)$ *a complete LRN -space and* $\Phi$ *a mapping from* $X^2$ *to* $D_L^+$ $(\Phi(x, y)$ *is denoted by* $\Phi_{x,y})$ *such that, for some* $0 < \alpha < 1/8$,

$$\Phi_{2x,2y}(t) \leq_L \Phi_{x,y}(\alpha t) \quad (x, y \in X, \ t > 0). \tag{3.4}$$

*Let* $f : X \to Y$ *be an odd mapping satisfying*

$$\mu_{Df(x,y)}(t) \geq_L \Phi_{x,y}(t) \tag{3.5}$$

*for all $x, y \in X$ and all $t > 0$. Then*

$$C(x) := \lim_{n\to\infty} 8^n \left( f\left(\frac{x}{2^{n-1}}\right) - 2f\left(\frac{x}{2^n}\right)\right) \tag{3.6}$$

*exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that*

$$\mu_{f(2x)-2f(x)-C(x)}(t) \geq \mathcal{T}_\wedge\left(\Phi_{0,x}\left(\frac{(33-264\alpha)}{17\alpha}t\right), \Phi_{2x,x}\left(\frac{(33-264\alpha)}{17\alpha}t\right)\right) \tag{3.7}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* Letting $x = 0$ in (3.5), we get

$$\mu_{12f(3y)-48f(2y)+60f(y)}(t) \geq_L \Phi_{0,y}(t) \tag{3.8}$$

for all $y \in X$ and all $t > 0$.

Replacing $x$ by $2y$ in (3.5), we get

$$\mu_{11f(4y)-56f(3y)+114f(2y)-104f(y)}(t) \geq_L \Phi_{2y,y}(t) \tag{3.9}$$

for all $y \in X$ and all $t > 0$.

By (3.8) and (3.9),

$$\begin{aligned}
&\mu_{f(4y)-10f(2y)+16f(y)}\left(\frac{14}{33}t + \frac{1}{11}t\right) \\
&\quad \geq_L \mathcal{T}_\wedge\left(\mu_{(14/33)(12f(3y)-48f(2y)+60f(y))}\left(\frac{14}{33}t\right), \mu_{(1/11)(11f(4y)-56f(3y)+114f(2y)-104f(y))}\left(\frac{1}{11}t\right)\right) \\
&\quad \geq_L \mathcal{T}_\wedge(\Phi_{0,y}(t), \Phi_{2y,y}(t))
\end{aligned} \tag{3.10}$$

for all $y \in X$ and all $t > 0$. Letting $y := x/2$ and $g(x) := f(2x) - 2f(x)$ for all $x \in X$, we get

$$\mu_{g(x)-8g(x/2)}\left(\frac{17}{33}t\right) \geq_L \mathcal{T}_\wedge(\Phi_{0,x/2}(t), \Phi_{x,x/2}(t)) \tag{3.11}$$

for all $x \in X$ and all $t > 0$.

Consider the set

$$S := \{g : X \longrightarrow Y\}, \tag{3.12}$$

and introduce the generalized metric on $S$:

$$d(g,h) = \inf\{u \in \mathbb{R}^+ : \mu_{g(x)-h(x)}(ut) \geq_L \mathcal{T}_\wedge(\Phi_{0,x}(t), \Phi_{2x,x}(t)),\ \forall x \in X,\ \forall t > 0\}, \tag{3.13}$$

where, as usual, $\inf \emptyset = +\infty$. It is easy to show that $(S, d)$ is complete. (See the proof of Lemma 2.1 of [46].)

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 8g\left(\frac{x}{2}\right) \tag{3.14}$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \geq_L \mathcal{T}_\wedge(\Phi_{0,x}(t), \Phi_{2x,x}(t)) \tag{3.15}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}
\mu_{Jg(x)-Jh(x)}(8\alpha\varepsilon t) &= \mu_{8g(x/2)-8h(x/2)}(8\alpha\varepsilon t) \\
&= \mu_{g(x/2)-h(x/2)}(\alpha\varepsilon t) \\
&\geq_L \mathcal{T}_\wedge(\Phi_{0,x/2}(\alpha t), \Phi_{x,x/2}(\alpha t)) \\
&\geq_L \mathcal{T}_\wedge(\Phi_{0,x}(t), \Phi_{2x,x}(t))
\end{aligned} \tag{3.16}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that

$$d(Jg, Jh) \leq 8\alpha\varepsilon. \tag{3.17}$$

This means that

$$d(Jg, Jh) \leq 8\alpha d(g, h) \tag{3.18}$$

for all $g, h \in S$.

It follows from (3.11) that

$$\mu_{g(x)-8g(x/2)}\left(\frac{17}{33}\alpha t\right) \geq_L \mathcal{T}_\wedge(\Phi_{0,x}(t), \Phi_{2x,x}(t)) \tag{3.19}$$

for all $x \in X$ and all $t > 0$. So

$$d(g, Jg) \leq \frac{17}{33}\alpha. \tag{3.20}$$

By Theorem 1.1, there exists a mapping $C : X \to Y$ satisfying the following:

(1) $C$ is a fixed point of $J$, that is,

$$C\left(\frac{x}{2}\right) = \frac{1}{8}C(x) \tag{3.21}$$

for all $x \in X$. Since $g : X \to Y$ is odd, $C : X \to Y$ is an odd mapping. The mapping $C$ is a unique fixed point of $J$ in the set

$$M = \{g \in S : d(f,g) < \infty\}. \tag{3.22}$$

This implies that $C$ is a unique mapping satisfying (3.21) such that there exists a $u \in (0,\infty)$ satisfying

$$\mu_{g(x)-C(x)}(ut) \geq_L \mathcal{T}_\wedge(\Phi_{0,x}(t), \Phi_{2x,x}(t)) \tag{3.23}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n g, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 8^n g\left(\frac{x}{2^n}\right) = C(x) \tag{3.24}$$

for all $x \in X$;

(3) $d(g,C) \leq (1/(1-8\alpha))d(g,Jg)$, which implies the inequality

$$d(g,C) \leq \frac{17\alpha}{33-264\alpha}. \tag{3.25}$$

This implies that inequality (3.7) holds.

From $Dg(x,y) = Df(2x,2y) - 2Df(x,y)$, by (3.5), we deduce that

$$\mu_{Df(2x,2y)}(t) \geq_L \Phi_{2x,2y}(t), \qquad \mu_{-2Df(x,y)}(t) = \mu_{Df(x,y)}\left(\frac{t}{2}\right) \geq_L \Phi_{x,y}\left(\frac{t}{2}\right), \tag{3.26}$$

and so, by (LRN3) and (3.4), we obtain

$$\mu_{Dg(x,y)}(3t) \geq_L \mathcal{T}_\wedge\left(\mu_{Df(2x,2y)}(t), \mu_{-2Df(x,y)}(2t)\right) \geq_L \mathcal{T}_\wedge\left(\Phi_{2x,2y}(t), \Phi_{x,y}(t)\right) \geq_L \Phi_{2x,2y}(t). \tag{3.27}$$

It follows that

$$\mu_{8^n Dg(x/2^n, y/2^n)}(3t) = \mu_{Dg(x/2^n, y/2^n)}\left(3\frac{t}{8^n}\right) \geq_L \Phi_{x/2^{n-1}, y/2^{n-1}}\left(\frac{t}{8^n}\right) \geq_L \cdots \geq_L \Phi_{x,y}\left(\frac{1}{8}\frac{t}{(8\alpha)^{n-1}}\right) \tag{3.28}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $0 < 8\alpha < 1$,

$$\lim_{n\to\infty} \Phi_{x,y}\left(\frac{t}{(8\alpha)^n}\right) = 1_{\mathcal{L}} \tag{3.29}$$

for all $x, y \in X$ and all $t > 0$. Then

$$\mu_{DC(x,y)}(t) = 1_{\mathcal{L}} \tag{3.30}$$

for all $x, y \in X$ and all $t > 0$. Thus the mapping $C : X \to Y$ is cubic, as desired. $\square$

**Corollary 3.2.** *Let $\theta \geq 0$ and let $p$ be a real number with $p > 3$. Let $X$ be a normed vector space with norm $\|\cdot\|$ and let $(X, \mu, \mathcal{T}_\wedge)$ be an LRN -space in which $L = [0,1]$ and $\mathcal{T}_\wedge = \min$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\mu_{Df(x,y)}(t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{3.31}$$

*for all $x, y \in X$ and all $t > 0$. Then*

$$C(x) := \lim_{n\to\infty} 8^n \left( f\left(\frac{x}{2^{n-1}}\right) - 2f\left(\frac{x}{2^n}\right)\right) \tag{3.32}$$

*exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that*

$$\mu_{f(2x)-2f(x)-C(x)}(t) \geq \frac{33(2^p-8)t}{33(2^p-8)t + 17(1+2^p)\theta\|x\|^p} \tag{3.33}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* The proof follows from Theorem 3.1 by taking

$$\Phi_{x,y}(t) := \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{3.34}$$

for all $x, y \in X$. Then we can choose $\alpha = 2^{-p}$ and we get the desired result. $\square$

**Theorem 3.3.** *Let $X$ be a linear space, $(Y, \mu, \mathcal{T}_\wedge)$ a complete LRN -space and $\Phi$ a mapping from $X^2$ to $D_L^+$ ($\Phi(x,y)$ is denoted by $\Phi_{x,y}$) such that, for some $0 < \alpha < 8$,*

$$\Phi_{x,y}(\alpha t) \geq_L \Phi_{x/2,y/2}(t) \quad (x, y \in X,\ t > 0). \tag{3.35}$$

*Let $f : X \to Y$ be an odd mapping satisfying (1.1). Then*

$$C(x) := \lim_{n\to\infty} \frac{1}{8^n}\left( f\left(2^{n+1}x\right) - 2f(2^n x)\right) \tag{3.36}$$

*exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that*

$$\mu_{f(2x)-2f(x)-C(x)}(t) \geq \mathcal{T}_\wedge\left(\Phi_{0,x}\left(\frac{(264-33\alpha)}{17}t\right), \Phi_{2x,x}\left(\frac{(264-33\alpha)}{17}t\right)\right) \tag{3.37}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 3.1.

Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{8}g(2x) \tag{3.38}$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \geq_L \mathcal{T}_\wedge(\Phi_{0,x}(t), \Phi_{2x,x}(t)) \tag{3.39}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}
\mu_{Jg(x)-Jh(x)}\left(\frac{\alpha}{8}\varepsilon t\right) &= \mu_{(1/8)g(2x)-(1/8)h(2x)}\left(\frac{\alpha}{8}\varepsilon t\right) \\
&= \mu_{g(2x)-h(2x)}(\alpha\varepsilon t) \\
&\geq_L \mathcal{T}_\wedge(\Phi_{0,2x}(\alpha t), \Phi_{4x,2x}(\alpha t)) \\
&\geq_L \mathcal{T}_\wedge(\Phi_{0,x}(t), \Phi_{2x,x}(t))
\end{aligned} \tag{3.40}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that

$$d(Jg, Jh) \leq \frac{\alpha}{8}\varepsilon. \tag{3.41}$$

This means that

$$d(Jg, Jh) \leq \frac{\alpha}{8}d(g, h) \tag{3.42}$$

for all $g, h \in S$.

It follows from (3.11) that

$$\mu_{g(x)-(1/8)g(2x)}\left(\frac{17}{264}t\right) \geq_L \mathcal{T}_\wedge(\Phi_{0,x}(t), \Phi_{2x,x}(t)) \tag{3.43}$$

for all $x \in X$ and all $t > 0$. So $d(g, Jg) \leq 17/264$.

By Theorem 1.1, there exists a mapping $C : X \to Y$ satisfying the following:

(1) $C$ is a fixed point of $J$, that is,

$$C(2x) = 8C(x) \tag{3.44}$$

for all $x \in X$. Since $g : X \to Y$ is odd, $C : X \to Y$ is an odd mapping. The mapping $C$ is a unique fixed point of $J$ in the set

$$M = \{g \in S : d(f, g) < \infty\}. \tag{3.45}$$

This implies that $C$ is a unique mapping satisfying (3.44) such that there exists a $u \in (0, \infty)$ satisfying

$$\mu_{g(x)-C(x)}(ut) \geq_L \mathcal{T}_\wedge(\Phi_{0,x}(t), \Phi_{2x,x}(t)) \tag{3.46}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n g, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{8^n} g(2^n x) = C(x) \tag{3.47}$$

for all $x \in X$;

(3) $d(g, C) \leq (1/(1-\alpha/8)) d(g, Jg)$, which implies the inequality

$$d(g, C) \leq \frac{17}{264 - 33\alpha}. \tag{3.48}$$

This implies that inequality (3.37) holds.

The rest of the proof is similar to the proof of Theorem 3.1. □

**Corollary 3.4.** *Let $\theta \geq 0$, and let $p$ be a real number with $0 < p < 3$. Let $X$ be a normed vector space with norm $\|\cdot\|$, and let $(X, \mu, \mathcal{T}_\wedge)$ be an LRN-space in which $L = [0,1]$ and $\mathcal{T}_\wedge = \min$. Let $f : X \to Y$ be an odd mapping satisfying* (3.31). *Then*

$$C(x) := \lim_{n\to\infty} \frac{1}{8^n} \left( f\left(2^{n+1} x\right) - 2 f(2^n x) \right) \tag{3.49}$$

*exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that*

$$\mu_{f(2x)-2f(x)-C(x)}(t) \geq \frac{33(8-2^p)t}{33(8-2^p)t + 17(1+2^p)\theta\|x\|^p} \tag{3.50}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* The proof follows from Theorem 3.3 by taking

$$\Phi_{x,y}(t) := \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{3.51}$$

for all $x, y \in X$. Then we can choose $\alpha = 2^p$, and we get the desired result. □

**Theorem 3.5.** *Let $X$ be a linear space, $(X, \mu, \mathcal{T}_\wedge)$ an LRN-space and let $\Phi$ be a mapping from $X^2$ to $D_L^+$ ($\Phi(x, y)$ is denoted by $\Phi_{x,y}$) such that, for some $0 < \alpha < 1/2$,*

$$\Phi_{x,y}(\alpha t) \geq_L \Phi_{2x,2y}(t) \quad (x, y \in X,\ t > 0). \tag{3.52}$$

*Let $f : X \to Y$ be an odd mapping satisfying* (3.5). *Then*

$$A(x) := \lim_{n\to\infty} 2^n \left( f\left(\frac{x}{2^{n-1}}\right) - 8f\left(\frac{x}{2^n}\right)\right) \tag{3.53}$$

*exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$\mu_{f(2x)-8f(x)-A(x)}(t) \geq_L \mathcal{T}_\wedge\left(\Phi_{0,x}\left(\frac{(33-66\alpha)}{17\alpha}t\right), \Phi_{2x,x}\left(\frac{(33-66\alpha)}{17\alpha}t\right)\right) \tag{3.54}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 3.1.

Letting $y := x/2$ and $h(x) := f(2x) - 8f(x)$ for all $x \in X$ in (3.10), we get

$$\mu_{h(x)-2h(x/2)}\left(\frac{17}{33}t\right) \geq_L \mathcal{T}_\wedge(\Phi_{0,x/2}(t), \Phi_{x,x/2}(t)) \tag{3.55}$$

for all $x \in X$ and all $t > 0$.

Now we consider the linear mapping $J : S \to S$ such that

$$Jh(x) := 2h\left(\frac{x}{2}\right) \tag{3.56}$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \geq_L \mathcal{T}_\wedge(\Phi_{0,x}(t), \Phi_{2x,x}(t)) \tag{3.57}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}
\mu_{Jg(x)-Jh(x)}(2\alpha\varepsilon t) &= \mu_{2g(x/2)-2h(x/2)}(2\alpha\varepsilon t) \\
&= \mu_{g(x/2)-h(x/2)}(\alpha\varepsilon t) \\
&\geq_L \mathcal{T}_\wedge(\Phi_{0,x/2}(\alpha t), \Phi_{x,x/2}(\alpha t)) \\
&\geq_L \mathcal{T}_\wedge(\Phi_{0,x}(t), \Phi_{2x,x}(t))
\end{aligned} \tag{3.58}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq 2\alpha\varepsilon$. This means that

$$d(Jg, Jh) \leq 2\alpha d(g, h) \tag{3.59}$$

for all $g, h \in S$.

It follows from (3.55) that

$$\mu_{h(x)-2h(x/2)}\left(\frac{17}{33}\alpha t\right) \geq_L \mathcal{T}_\wedge(\Phi_{0,x}(t), \Phi_{2x,x}(t)) \tag{3.60}$$

for all $x \in X$ and all $t > 0$. So $d(h, Jh) \leq 17\alpha/33$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) $A$ is a fixed point of $J$, that is,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{3.61}$$

for all $x \in X$. Since $h : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping $A$ is a unique fixed point of $J$ in the set

$$M = \{g \in S : d(f, g) < \infty\}. \tag{3.62}$$

This implies that $A$ is a unique mapping satisfying (3.61) such that there exists a $u \in (0, \infty)$ satisfying

$$\mu_{h(x)-A(x)}(ut) \geq_L \mathcal{T}_\wedge(\Phi_{0,x}(t), \Phi_{2x,x}(t)) \tag{3.63}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n h, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n h\left(\frac{x}{2^n}\right) = A(x) \tag{3.64}$$

for all $x \in X$;

(3) $d(h, A) \leq (1/(1-2\alpha))d(h, Jh)$, which implies the inequality

$$d(h, A) \leq \frac{17\alpha}{33 - 66\alpha}. \tag{3.65}$$

This implies that inequality (3.54) holds.

The rest of the proof is similar to the proof of Theorem 3.1. □

**Corollary 3.6.** *Let $\theta \geq 0$, and let $p$ be a real number with $p > 1$. Let $X$ be a normed vector space with norm $\|\cdot\|$, and let $(X, \mu, \mathcal{T}_\wedge)$ be an LRN-space in which $L = [0,1]$ and $\mathcal{T}_\wedge = \min$. Let $f : X \to Y$ be an odd mapping satisfying (3.31). Then*

$$A(x) := \lim_{n\to\infty} 2^n \left(f\left(\frac{x}{2^{n-1}}\right) - 8f\left(\frac{x}{2^n}\right)\right) \tag{3.66}$$

*exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$\mu_{f(2x)-8f(x)-A(x)}(t) \geq \frac{33(2^p - 2)t}{33(2^p - 2)t + 17(1 + 2^p)\theta\|x\|^p} \tag{3.67}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* The proof follows from Theorem 3.5 by taking

$$\Phi_{x,y}(t) := \frac{t}{t + \theta\left(\|x\|^p + \|y\|^p\right)} \tag{3.68}$$

for all $x, y \in X$. Then we can choose $\alpha = 2^{-p}$ and we get the desired result. $\square$

**Theorem 3.7.** *Let $X$ be a linear space, $(X, \mu, \mathcal{T}_\wedge)$ an LRN-space and let $\Phi$ be a mapping from $X^2$ to $D_L^+$ ($\Phi(x,y)$ is denoted by $\Phi_{x,y}$) such that, for some $0 < \alpha < 2$,*

$$\Phi_{x,y}(\alpha t) \geq_L \Phi_{x/2,y/2}(t) \quad \left(x, y \in X, t > 0\right). \tag{3.69}$$

*Let $f : X \to Y$ be an odd mapping satisfying* (3.5). *Then*

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n}\left(f\left(2^{n+1}x\right) - 8f(2^n x)\right) \tag{3.70}$$

*exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$\mu_{f(2x)-8f(x)-A(x)}(t) \geq_L \mathcal{T}_\wedge\left(\Phi_{0,x}\left(\frac{(66-33\alpha)}{17}t\right), \Phi_{2x,x}\left(\frac{(66-33\alpha)}{17}t\right)\right) \tag{3.71}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 3.1.

Consider the linear mapping $J : S \to S$ such that

$$Jh(x) := \frac{1}{2}h(2x) \tag{3.72}$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \geq_L \mathcal{T}_\wedge(\Phi_{0,x}(t), \Phi_{2x,x}(t)) \tag{3.73}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}
\mu_{Jg(x)-Jh(x)}(L\varepsilon t) &= \mu_{(1/2)g(2x)-(1/2)h(2x)}\left(\frac{\alpha}{2}\varepsilon t\right) \\
&= \mu_{g(2x)-h(2x)}(\alpha\varepsilon t) \\
&\geq_L \mathcal{T}_\wedge(\Phi_{0,2x}(\alpha t), \Phi_{4x,2x}(\alpha t)) \\
&\geq_L \mathcal{T}_\wedge(\Phi_{0,x}(t), \Phi_{2x,x}(t))
\end{aligned} \tag{3.74}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that

$$d(Jg, Jh) \leq \frac{\alpha}{2}\varepsilon. \tag{3.75}$$

This means that

$$d(Jg, Jh) \leq \frac{\alpha}{2} d(g,h) \tag{3.76}$$

for all $g, h \in S$.

It follows from (3.55) that

$$\mu_{h(x)-(1/2)h(2x)}\left(\frac{17}{66}t\right) \geq_L \mathcal{T}_\wedge(\Phi_{0,x}(t), \Phi_{2x,x}(t)) \tag{3.77}$$

for all $x \in X$ and all $t > 0$. So $d(h, Jh) \leq 17/66$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) $A$ is a fixed point of $J$, that is,

$$A(2x) = 2A(x) \tag{3.78}$$

for all $x \in X$. Since $h : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping $A$ is a unique fixed point of $J$ in the set

$$M = \{g \in S : d(f,g) < \infty\}. \tag{3.79}$$

This implies that $A$ is a unique mapping satisfying (3.78) such that there exists a $u \in (0,\infty)$ satisfying

$$\mu_{h(x)-A(x)}(ut) \geq_L \mathcal{T}_\wedge(\Phi_{0,x}(t), \Phi_{2x,x}(t)) \tag{3.80}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n h, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{2^n} h(2^n x) = A(x) \tag{3.81}$$

for all $x \in X$;

(3) $d(h, A) \leq (1/(1-\alpha/2))d(h, Jh)$, which implies the inequality

$$d(h, A) \leq \frac{17}{66 - 33\alpha}. \tag{3.82}$$

This implies that inequality (3.71) holds.

The rest of the proof is similar to the proof of Theorem 3.1. □

**Corollary 3.8.** *Let $\theta \geq 0$, and let $p$ be a real number with $0 < p < 1$. Let $X$ be a normed vector space with norm $\|\cdot\|$, and let $(X, \mu, \mathcal{T}_\wedge)$ be an LRN-space in which $L = [0,1]$ and $\mathcal{T}_\wedge = \min$. Let $f : X \to Y$ be an odd mapping satisfying* (3.31). *Then*

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n}\left(f\left(2^{n+1}x\right) - 8f(2^n x)\right) \tag{3.83}$$

*exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$\mu_{f(2x)-8f(x)-A(x)}(t) \geq \frac{33(2-2^p)t}{33(2-2^p)t + 17(1+2^p)\theta\|x\|^p} \tag{3.84}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* The proof follows from Theorem 3.7 by taking

$$\Phi_{x,y}(t) := \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{3.85}$$

for all $x, y \in X$. Then we can choose $\alpha = 2^p$ and we get the desired result. □

## 4. Generalized Hyers-Ulam Stability of the Functional Equation (1.1): An Even Case

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $Df(x,y) = 0$ in complete RN-spaces: an even case.

**Theorem 4.1.** *Let $X$ be a linear space, $(X, \mu, \mathcal{T}_\wedge)$ an LRN-space and let $\Phi$ be a mapping from $X^2$ to $D_L^+$ ($\Phi(x,y)$ is denoted by $\Phi_{x,y}$) such that, for some $0 < \alpha < 1/16$,*

$$\Phi_{x,y}(\alpha t) \geq_L \Phi_{2x,2y}(t) \quad (x, y \in X,\ t > 0). \tag{4.1}$$

*Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and* (3.5). *Then*

$$Q(x) := \lim_{n\to\infty} 16^n f\left(\frac{x}{2^n}\right) \tag{4.2}$$

*exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that*

$$\mu_{f(x)-Q(x)}(t) \geq_L \mathcal{T}_\wedge\left(\Phi_{0,x}\left(\frac{(22-352\alpha)}{13\alpha}t\right), \Phi_{x,x}\left(\frac{(22-352\alpha)}{13\alpha}t\right)\right) \tag{4.3}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* Letting $x = 0$ in (3.5), we get

$$\mu_{12f(3y)-70f(2y)+148f(y)}(t) \geq_L \Phi_{0,y}(t) \tag{4.4}$$

for all $y \in X$ and all $t > 0$.

Letting $x = y$ in (3.5), we get

$$\mu_{f(3y)-4f(2y)-17f(y)}(t) \geq_L \Phi_{y,y}(t) \tag{4.5}$$

for all $y \in X$ and all $t > 0$.

By (4.4) and (4.5),

$$\begin{aligned}
&\mu_{f(2y)-16f(y)}\left(\frac{1}{22}t + \frac{12}{22}t\right) \\
&\quad \geq_L \mathcal{T}_\wedge\left(\mu_{(1/22)(12f(3y)-70f(2y)+148f(y))}\left(\frac{1}{22}t\right), \mu_{(12/22)(f(3y)-4f(2y)-17f(y))}\left(\frac{12}{22}t\right)\right) \\
&\quad \geq_L \mathcal{T}_\wedge(\Phi_{0,y}(t), \Phi_{y,y}(t))
\end{aligned} \tag{4.6}$$

for all $y \in X$ and all $t > 0$.

Consider the set

$$S := \{g : X \longrightarrow Y\}, \tag{4.7}$$

and introduce the generalized metric on $S$

$$d(g,h) = \inf\{u \in \mathbb{R}_+ : N(g(x) - h(x), ut) \geq_L \mathcal{T}_\wedge(\Phi_{0,x}(t), \Phi_{x,x}(t)),\ \forall x \in X,\ \forall t > 0\}, \tag{4.8}$$

where, as usual, $\inf \emptyset = +\infty$. It is easy to show that $(S,d)$ is complete. (See the proof of Lemma 2.1 of [46].)

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 16g\left(\frac{x}{2}\right) \tag{4.9}$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \geq_L \mathcal{T}_\wedge(\Phi_{0,x}(t), \Phi_{x,x}(t)) \tag{4.10}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}
\mu_{Jg(x)-Jh(x)}(16\alpha\varepsilon t) &= \mu_{16g(x/2)-16h(x/2)}(16\alpha\varepsilon t) \\
&= \mu_{g(x/2)-h(x/2)}(\alpha\varepsilon t) \\
&\geq_L \mathcal{T}_\wedge(\Phi_{0,x/2}(\alpha t), \Phi_{x/2,x/2}(\alpha t)) \\
&\geq_L \mathcal{T}_\wedge(\Phi_{0,x}(t), \Phi_{x,x}(t))
\end{aligned} \tag{4.11}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that

$$d(Jg, Jh) \leq 16\alpha\varepsilon. \tag{4.12}$$

This means that

$$d(Jg, Jh) \leq 16\alpha d(g,h) \tag{4.13}$$

for all $g, h \in S$.

It follows from (4.6) that

$$\mu_{f(x)-16f(x/2)}\left(\frac{13}{22}\alpha t\right) \geq_L \mathcal{T}_\wedge(\Phi_{0,x}(t), \Phi_{x,x}(t)) \tag{4.14}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq 13\alpha/22$.

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) $Q$ is a fixed point of $J$, that is,

$$Q\left(\frac{x}{2}\right) = \frac{1}{16}Q(x) \tag{4.15}$$

for all $x \in X$. Since $f : X \to Y$ is even, $Q : X \to Y$ is an even mapping. The mapping $Q$ is a unique fixed point of $J$ in the set

$$M = \{g \in S : d(f,g) < \infty\}. \tag{4.16}$$

This implies that $Q$ is a unique mapping satisfying (4.15) such that there exists a $u \in (0,\infty)$ satisfying

$$\mu_{f(x)-Q(x)}(ut) \geq_L \mathcal{T}_\wedge(\Phi_{0,x}(t), \Phi_{x,x}(t)) \tag{4.17}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 16^n f\left(\frac{x}{2^n}\right) = Q(x) \tag{4.18}$$

for all $x \in X$;

(3) $d(f,Q) \leq (1/(1-16\alpha))d(f,Jf)$, which implies the inequality

$$d(f,Q) \leq \frac{13\alpha}{22-352\alpha}. \tag{4.19}$$

This implies that inequality (4.3) holds.

The rest of the proof is similar to the proof of Theorem 3.1. □

**Corollary 4.2.** *Let $\theta \geq 0$, and let $p$ be a real number with $p > 4$. Let $X$ be a normed vector space with norm $\|\cdot\|$, and let $(X, \mu, \mathcal{T}_\wedge)$ be an LRN-space in which $L = [0,1]$ and $\mathcal{T}_\wedge = \min$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.31). Then*

$$Q(x) := \lim_{n\to\infty} 16^n f\left(\frac{x}{2^n}\right) \tag{4.20}$$

*exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that*

$$\mu_{f(x)-Q(x)}(t) \geq \frac{11(2^p - 16)t}{11(2^p - 16)t + 13\theta\|x\|^p} \tag{4.21}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* The proof follows from Theorem 4.1 by taking

$$\Phi_{x,y}(t) := \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{4.22}$$

for all $x, y \in X$. Then we can choose $\alpha = 2^{-p}$, and we get the desired result. □

Similarly, we can obtain the following. We will omit the proof.

**Theorem 4.3.** *Let $X$ be a linear space, $(X, \mu, \mathcal{T}_\wedge)$ an LRN -space and let $\Phi$ be a mapping from $X^2$ to $D_L^+$ ($\Phi(x,y)$ is denoted by $\Phi_{x,y}$) such that, for some $0 < \alpha < 16$,*

$$\Phi_{x,y}(\alpha t) \geq_L \Phi_{x/2,y/2}(t) \quad (x, y \in X,\ t > 0). \tag{4.23}$$

*Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.5). Then*

$$Q(x) := \lim_{n\to\infty} \frac{1}{16^n} f(2^n x) \tag{4.24}$$

*exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that*

$$\mu_{f(x)-Q(x)}(t) \geq_L \mathcal{T}_\wedge\left(\Phi_{0,x}\left(\frac{(352 - 22\alpha)}{13}t\right), \Phi_{x,x}\left(\frac{(352 - 22\alpha)}{13}t\right)\right) \tag{4.25}$$

*for all $x \in X$ and all $t > 0$.*

**Corollary 4.4.** *Let $\theta \geq 0$, and let $p$ be a real number with $0 < p < 4$. Let $X$ be a normed vector space with norm $\|\cdot\|$, and let $(X, \mu, \mathcal{T}_\wedge)$ be an LRN-space in which $L = [0,1]$ and $\mathcal{T}_\wedge = \min$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.31). Then*

$$Q(x) := \lim_{n\to\infty} \frac{1}{16^n} f(2^n x) \tag{4.26}$$

*exists for each* $x \in X$ *and defines a quartic mapping* $Q : X \to Y$ *such that*

$$\mu_{f(x)-Q(x)}(t) \geq \frac{11(16-2^p)t}{11(16-2^p)t + 13\theta\|x\|^p} \tag{4.27}$$

*for all* $x \in X$ *and all* $t > 0$.

*Proof.* The proof follows from Theorem 4.3 by taking

$$\Phi_{x,y}(t) := \frac{t}{t + \theta\left(\|x\|^p + \|y\|^p\right)} \tag{4.28}$$

for all $x, y \in X$. Then we can choose $\alpha = 2^p$, and we get the desired result. □

## References

[1] M. S. El Naschie, "On a fuzzy Kähler-like manifold which is consistent with the two slit experiment," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 6, no. 2, pp. 95–98, 2005.

[2] L. D. G. Sigalotti and A. Mejias, "On El Naschie's conjugate complex time, fractal $E^{(\infty)}$ space-time and faster-than-light particles," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 7, no. 4, pp. 467–472, 2006.

[3] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Publishers, New York, NY, USA, 1960.

[4] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[5] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.

[6] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[7] P. Găvruţa, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.

[8] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, Mass, USA, 1998.

[9] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Fla, USA, 2001.

[10] C. Park, "Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras," *Bulletin des Sciences Mathématiques*, vol. 132, no. 2, pp. 87–96, 2008.

[11] C. Park, "Hyers-Ulam-Rassias stability of a generalized Apollonius-Jensen type additive mapping and isomorphisms between $C^*$-algebras," *Mathematische Nachrichten*, vol. 281, no. 3, pp. 402–411, 2008.

[12] C. Park and J. Cui, "Generalized stability of $C^*$-ternary quadratic mappings," *Abstract and Applied Analysis*, vol. 2007, Article ID 23282, 6 pages, 2007.

[13] C. Park and A. Najati, "Homomorphisms and derivations in $C^*$-algebras," *Abstract and Applied Analysis*, vol. 2007, Article ID 80630, 12 pages, 2007.

[14] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Bulletin des Sciences Mathématiques*, vol. 108, no. 4, pp. 445–446, 1984.

[15] J. M. Rassias, "Refined Hyers-Ulam approximation of approximately Jensen type mappings," *Bulletin des Sciences Mathématiques*, vol. 131, no. 1, pp. 89–98, 2007.

[16] J. M. Rassias and M. J. Rassias, "Asymptotic behavior of alternative Jensen and Jensen type functional equations," *Bulletin des Sciences Mathématiques*, vol. 129, no. 7, pp. 545–558, 2005.

[17] Th. M. Rassias, "Problem 16; 2, Report of the 27th International Symposium on Functional Equations," *Aequationes Mathematicae*, vol. 39, pp. 292–293, 1990.

[18] Th. M. Rassias, "On the stability of the quadratic functional equation and its applications," *Universitatis Babeş-Bolyai. Studia. Mathematica*, vol. 43, no. 3, pp. 89–124, 1998.

[19] Th. M. Rassias, "The problem of S. M. Ulam for approximately multiplicative mappings," *Journal of Mathematical Analysis and Applications*, vol. 246, no. 2, pp. 352–378, 2000.

[20] Th. M. Rassias, "On the stability of functional equations in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 251, no. 1, pp. 264–284, 2000.
[21] Th. M. Rassias, "On the stability of functional equations and a problem of Ulam," *Acta Applicandae Mathematicae*, vol. 62, no. 1, pp. 23–130, 2000.
[22] Th. M. Rassias and P. Šemrl, "On the behavior of mappings which do not satisfy Hyers-Ulam stability," *Proceedings of the American Mathematical Society*, vol. 114, no. 4, pp. 989–993, 1992.
[23] Th. M. Rassias and P. Šemrl, "On the Hyers-Ulam stability of linear mappings," *Journal of Mathematical Analysis and Applications*, vol. 173, no. 2, pp. 325–338, 1993.
[24] Th. M. Rassias and K. Shibata, "Variational problem of some quadratic functionals in complex analysis," *Journal of Mathematical Analysis and Applications*, vol. 228, no. 1, pp. 234–253, 1998.
[25] K.-W. Jun and H.-M. Kim, "The generalized Hyers-Ulam-Rassias stability of a cubic functional equation," *Journal of Mathematical Analysis and Applications*, vol. 274, no. 2, pp. 267–278, 2002.
[26] S. H. Lee, S. M. Im, and I. S. Hwang, "Quartic functional equations," *Journal of Mathematical Analysis and Applications*, vol. 307, no. 2, pp. 387–394, 2005.
[27] R. Saadati, S. M. Vaezpour, and B. E. Rhoades, "$T$-stability approach to variational iteration method for solving integral equations," *Fixed Point Theory and Applications*, vol. 2009, Article ID 393245, 9 pages, 2009.
[28] J. H. He, "Variational iteration method—a kind of nonlinear analytical technique: some examples," *International Journal of Non-Linear Mechanics*, vol. 34, pp. 699–708, 1999.
[29] J.-H. He, "A review on some new recently developed nonlinear analytical techniques," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 1, no. 1, pp. 51–70, 2000.
[30] J.-H. He and X.-H. Wu, "Variational iteration method: new development and applications," *Computers & Mathematics with Applications*, vol. 54, no. 7-8, pp. 881–894, 2007.
[31] J.-H. He, "Variational iteration method—some recent results and new interpretations," *Journal of Computational and Applied Mathematics*, vol. 207, no. 1, pp. 3–17, 2007.
[32] H. Ozer, "Application of the variational iteration method to the boundary value problems with jump discontinuities arising in solid mechanics," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 8, no. 4, pp. 513–518, 2007.
[33] J. Biazar and H. Ghazvini, "He's variational iteration method for solving hyperbolic differential equations," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 8, pp. 311–314, 2007.
[34] Z. M. Odibat and S. Momani, "Application of variational iteration method to nonlinear differential equations of fractional order," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 7, no. 1, pp. 27–34, 2006.
[35] L. Cădariu and V. Radu, "Fixed points and the stability of Jensen's functional equation," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 1, article 4, p. 7, 2003.
[36] J. B. Diaz and B. Margolis, "A fixed point theorem of the alternative, for contractions on a generalized complete metric space," *Bulletin of the American Mathematical Society*, vol. 74, pp. 305–309, 1968.
[37] G. Isac and Th. M. Rassias, "Stability of $\psi$-additive mappings: applications to nonlinear analysis," *International Journal of Mathematics and Mathematical Sciences*, vol. 19, no. 2, pp. 219–228, 1996.
[38] L. Cădariu and V. Radu, "On the stability of the Cauchy functional equation: a fixed point approach," in *Iteration Theory*, vol. 346, pp. 43–52, Karl-Franzens-Universität Graz, Graz, Austria, 2004.
[39] L. Cădariu and V. Radu, "Fixed point methods for the generalized stability of functional equations in a single variable," *Fixed Point Theory and Applications*, vol. 2008, Article ID 749392, 15 pages, 2008.
[40] M. Mirzavaziri and M. S. Moslehian, "A fixed point approach to stability of a quadratic equation," *Bulletin of the Brazilian Mathematical Society*, vol. 37, no. 3, pp. 361–376, 2006.
[41] C. Park, "Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras," *Fixed Point Theory and Applications*, vol. 2007, Article ID 50175, 15 pages, 2007.
[42] C. Park, "Generalized Hyers-Ulam stability of quadratic functional equations: a fixed point approach," *Fixed Point Theory and Applications*, vol. 2008, Article ID 493751, 9 pages, 2008.
[43] V. Radu, "The fixed point alternative and the stability of functional equations," *Fixed Point Theory*, vol. 4, no. 1, pp. 91–96, 2003.
[44] C. Alsina, "On the stability of a functional equation arising in probabilistic normed spaces," in *General Inequalities*, vol. 80, pp. 263–271, Birkhäuser, Basel, Switzerland, 1987.
[45] A. K. Mirmostafaee and M. S. Moslehian, "Fuzzy approximately cubic mappings," *Information Sciences*, vol. 178, no. 19, pp. 3791–3798, 2008.
[46] D. Miheţ and V. Radu, "On the stability of the additive Cauchy functional equation in random normed spaces," *Journal of Mathematical Analysis and Applications*, vol. 343, no. 1, pp. 567–572, 2008.

[47] D. Miheţ, R. Saadati, and S. M. Vaezpour, “The stability of the quartic functional equation in random normed spaces,” *Acta Applicandae Mathematicae*, vol. 110, no. 2, pp. 797–803, 2011.

[48] D. Miheţ, R. Saadati, and S. M. Vaezpour, “The stability of anadditive functional equation in Menger probabilistic $\varphi$-normed spaces,” *Mathematica Slovaca*. In press.

[49] E. Baktash, Y. J. Cho, M. Jalili, R. Saadati, and S. M. Vaezpour, “On the stability of cubic mappings and quadratic mappings in random normed spaces,” *Journal of Inequalities and Applications*, vol. 2008, Article ID 902187, 11 pages, 2008.

[50] R. Saadati, S. M. Vaezpour, and Y. J. Cho, “Erratumml: A note to paper “On the stability of cubic mappings and quartic mappings in random normed spaces”,” *Journal of Inequalities and Applications*, vol. 2009, Article ID 214530, 6 pages, 2009.

[51] G. Deschrijver and E. E. Kerre, “On the relationship between some extensions of fuzzy set theory,” *Fuzzy Sets and Systems*, vol. 133, no. 2, pp. 227–235, 2003.

[52] O. Hadžić and E. Pap, *Fixed Point Theory in Probabilistic Metric Spaces*, vol. 536 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.

[53] O. Hadžić, E. Pap, and M. Budinčević, “Countable extension of triangular norms and their applications to the fixed point theory in probabilistic metric spaces,” *Kybernetika*, vol. 38, no. 3, pp. 363–382, 2002.

[54] B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, North-Holland Series in Probability and Applied Mathematics, North-Holland Publishing, New York, NY, USA, 1983.

[55] M. Eshaghi-Gordji, S. Kaboli-Gharetapeh, C. Park, and S. Zolfaghri, “Stability of an additive-cubic-quartic functional equation,” *Advances in Difference Equations*, vol. 2009, Article ID 395693, p. 20, 2009.