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On a Modified Version of Jensen Inequality

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The right-hand side of the Jensen inequality is multiplied by a constant and the related equation is considered. It is shown that every continuous solution of this equation is of the form $f(x) = cx^d$ for some $c \in \mathbb{R}$, $d \in (-\infty, 0) \cup (1, \infty)$. Further, it is proved that some functions satisfying the inequality considered are bounded below but not above by suitable solutions of the corresponding equation.

Keywords: Jensen's inequality; Functional inequality; A corresponding functional equation

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1 INTRODUCTION

Unless explicitly stated otherwise, we deal with real functions defined on $\mathbb{R}^+ := (0, \infty)$. For these functions we consider some inequalities and equations that are slightly stronger than the classical Jensen inequality. Namely we shall introduce a factor $1 - \delta$ (for some $\delta \in (0, 1)$) on the right-hand side of Jensen's inequality

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x) + f(y)}{2} \tag{1}$$

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getting

$$f\left(\frac{x+y}{2}\right) \leq (1-\delta)\frac{f(x)+f(y)}{2}. \tag{2}$$

In what follows the solutions of (1) will also be termed J-convex. To simplify the notation we will replace $(1-\delta)/2$ by $\gamma \in (0, \frac{1}{2})$ arriving at

$$f\left(\frac{x+y}{2}\right) \leq \gamma[f(x)+f(y)]. \tag{3}$$

It is easily seen that if $f(x) > 0$ for some x then f fails to be a solution of inequality (3) whatever the constant $\gamma \in (0, \frac{1}{2})$. Because of this we shall add some condition which restricts the variability of x and y. It can be done in different ways. One of the posibilities is to be found in a paper of Kolwicz and Płuciennik [1] in connection with some functionals on Orlicz–Bochner spaces and reads as follows:

$$\bigvee_{a\in(0,1)} \bigvee_{\gamma\in(0,\frac{1}{2})} \bigwedge_{x,y;x\leq ay} f\left(\frac{x+y}{2}\right) \leq \gamma[f(x)+f(y)]. \tag{4}$$

Other inequalities of this type are dealt with in a recent work of Hudzik *et al.* [2] but will not be considered in the present paper. Condition (4) still looks stronger than the Jensen inequality but it is not. As it will be seen, there are functions that satisfy (4) and fail to be J-convex. It is obvious that one can find functions fulfilling the Jensen inequality which do not satisfy (4). It will also be shown that there exist discontinuous locally bounded functions that satisfy (4). Summarizing, this condition does not seem to be interesting enough. To make it stronger we shall change it a bit, replacing it by the following requirement:

$$\bigwedge_{a\in(0,1)} \bigvee_{\gamma(a)\in(0,\frac{1}{2})} \bigwedge_{x,y;x\leq ay} f\left(\frac{x+y}{2}\right) \leq \gamma(a)[f(x)+f(y)]. \tag{5}$$

In what follows, we shall assume that $\gamma(a)$ is uniquely determined by a, i.e. γ is a function.

Finally some extreme solutions of inequality (5) can be found by considering the following equation:

$$\bigwedge_{a\in(0,1)} \bigvee_{\gamma(a)\in(0,\frac{1}{2})} f\left(\frac{x+ax}{2}\right) = \gamma(a)[f(x)+f(ax)]. \tag{6}$$

All the facts and notions within the theory of functional equations and inequalities we shall be using in the sequel are to be found in Kuczma's monograph [3], the reader is referred to without explicit indications.

2 RESULTS

Remark 1 A function $f: (0, \infty) \to (0, \infty)$ satisfying (5) or (6) is strictly J-convex.

Example 1 (of a function satisfying condition (4)) Let us consider the function $f(x) = x^2$, $x \in (0, \infty)$ and let $a = \frac{1}{2}$. Take an arbitrary pair x, y fulfilling the inequality $x \leq ay$. We may assume that $y = 2x + \delta$ for some $\delta \geq 0$. Hence we get

$$f\left(\frac{x+y}{2}\right) = \left(\frac{3x+\delta}{2}\right)^2 = \frac{9x^2 + 6x\delta + \delta^2}{4}.$$

On the other hand,

$$f(x) + f(y) = x^2 + 4x^2 + 4x\delta + \delta^2.$$

Our task now is to find a $\gamma \in (0, \frac{1}{2})$ such that

$$9x^2 + 6x\delta + \delta^2 \leq \gamma(20x^2 + 16x\delta + 4\delta^2).$$

We shall show that $\gamma = \frac{9}{20}$ will do this task. To see this we have to show the following inequality:

$$\frac{9x^2 + 6x\delta + \delta^2}{20x^2 + 16x\delta + 4\delta^2} \leq \frac{9}{20},$$

being equivalent to

$$45x^2 + 30x\delta + 5\delta^2 \leq 45x^2 + 36x\delta + 9\delta^2,$$

i.e.

$$3x\delta + 2\delta^2 \geq 0,$$

which obviously is true.

Example 2 (of a function satisfying condition (4) that is neither J-convex nor continuous) Let

$$f(x) = \begin{cases} x^2, & x \neq 10, \\ 100 + \varepsilon, & x = 10. \end{cases}$$

We are going to show that there exists an $\varepsilon > 0$ such that f satisfies (4) for $a = \frac{1}{2}$ and $\gamma = \frac{19}{40}$. Let us distinguish three cases:

I. $(x+y)/2, x, y \neq 10$; it sufficies to apply Example 1.

II. $x = 10$ or $y = 10$. In this case $f(x) \geq x^2$, $f(y) \geq y^2$, $f((x+y)/2) = ((x+y)/2)^2$. Again by Example 1 we get the desired inequality for the function $g(x) = x^2$. Hence by condition (4) f is satisfied.

III. $(x+y)/2 = 10$. There exists an $\alpha \in \mathbb{R}^+$ such that $x = 10 - \alpha$, $y = 10 + \alpha$. Recall that $x/y \leq \frac{1}{2}$, which forces α to be greater than $\frac{10}{3}$. Let us remind that we want to show the existence of an $\varepsilon > 0$ such that

$$100 + \varepsilon \leq \frac{19}{40}[(10 - \alpha)^2 + (10 + \alpha)^2],$$

i.e.

$$20\varepsilon \leq 19\alpha^2 - 100.$$

But we have already observed that $\alpha > \frac{10}{3}$. It implies that $\alpha^2 > 10$ and

$$19\alpha^2 - 100 > 90,$$

i.e. we may take $\varepsilon := \frac{9}{2}$.

Remark 2 Let function $f\colon (0, \infty) \to (0, \infty)$ be a solution of Eq. (6). If the function γ occurring in this equation is increasing then f satisfies inequalities (4) and (5).

Remark 3 Function $f_\alpha(x) = cx^\alpha$, $x \in (0, \infty)$, $(\alpha \in \mathbb{R})$ yields a solution to the equation

$$f\left(\frac{x + ax}{2}\right) = \gamma(a)[f(x) + f(ax)]$$

postulated for every $x \in \mathbb{R}^+$ and $a \in (0,1)$ provided that $\gamma = \gamma_\alpha$ where $\gamma_\alpha : (0,1) \to \mathbb{R}$ is given by the following formula:

$$\gamma_\alpha(a) = \frac{(1+a)^\alpha}{2^\alpha(1+a^\alpha)}, \quad a \in (0,1).$$

Proof We have

$$\begin{aligned} f_\alpha\left(\frac{x+ax}{2}\right) &= c\frac{(x+ax)^\alpha}{2^\alpha} = c\frac{x^\alpha(1+a)^\alpha}{2^\alpha} \\ &= \frac{(1+a)^\alpha}{2^\alpha(1+a^\alpha)} c(1+a^\alpha)x^\alpha \\ &= \frac{(1+a)^\alpha}{2^\alpha(1+a^\alpha)} (cx^\alpha + ca^\alpha x^\alpha) \\ &= \gamma_\alpha(a)[f_\alpha(x) + f_\alpha(ax)]. \end{aligned}$$

LEMMA 1 *Let $f : (0,\infty) \to \mathbb{R}$ be a solution of Eq. (6) for some strictly increasing function γ. Then there exists a function $\alpha : (0,\infty) \to \mathbb{R}$ such that*

$$f(bx) = \alpha(b)f(x)$$

for every $b, x \in \mathbb{R}^+$.

Proof Without loss of generality we may assume that $f \neq 0$. For the function $f = 0$ the claim holds true with any function α.

At first we shall prove the lemma for the arguments $b \in (1,\infty)$.

So, let $x_0 \in \mathbb{R}^+$ and $b \in (1,\infty)$ be arbitrarily fixed. Put $y_0 := f(x_0)$ and $y_b := f(bx_0)$. Let $\Gamma_1 := \gamma(1/b)$. Then we obtain:

$$f\left(\frac{bx_0 + x_0}{2}\right) = \Gamma_1[f(x_0) + f(bx_0)] = \Gamma_1(y_0 + y_b).$$

Denote $\Gamma_2 := \gamma(1/(2b-1))$ to get

$$y_b = f(bx_0) = f\left(\frac{(2b-1)x_0 + x_0}{2}\right) = \Gamma_2[f((2b-1)x_0) + f(x_0)],$$

whence

$$f((2b-1)x_0) = \frac{1}{\Gamma_2} y_b - y_0.$$

Now put $\Gamma_3 := \gamma(b/(2b-1))$. Then

$$\begin{aligned} f\left(\frac{3b-1}{2}x_0\right) &= f\left(\frac{(2b-1)x_0 + bx_0}{2}\right) = \Gamma_3[f((2b-1)x_0) + f(bx_0)] \\ &= \Gamma_3\left(\frac{1}{\Gamma_2} y_b - y_0 + y_b\right) = \frac{\Gamma_3(1+\Gamma_2)}{\Gamma_2} y_b - \Gamma_3 y_0. \end{aligned}$$

Finally, let $\Gamma_4 := \gamma((b+1)/(3b-1))$. Then we have:

$$\begin{aligned} y_b = f(bx_0) &= f\left(\frac{[(3b-1)/2]x_0 + [(b+1)/2]x_0}{2}\right) \\ &= \Gamma_4\left[f\left(\frac{3b-1}{2}x_0\right) + f\left(\frac{b+1}{2}x_0\right)\right] \\ &= \Gamma_4\left[\frac{\Gamma_3(1+\Gamma_2)}{\Gamma_2} y_b - \Gamma_3 y_0 + \Gamma_1(y_0 + y_b)\right] \\ &= \Gamma_4\left[\frac{\Gamma_3(1+\Gamma_2) + \Gamma_1\Gamma_2}{\Gamma_2} y_b + (\Gamma_1 - \Gamma_3)y_0\right] \\ &= \frac{\Gamma_4[\Gamma_3(1+\Gamma_2) + \Gamma_1\Gamma_2]}{\Gamma_2} y_b + \Gamma_4(\Gamma_1 - \Gamma_3)y_0. \end{aligned}$$

Now we get

$$y_b[\Gamma_2 - \Gamma_4(\Gamma_3(1+\Gamma_2) + \Gamma_1\Gamma_2)] = y_0\Gamma_2\Gamma_4(\Gamma_1 - \Gamma_3).$$

Let us define the following two functions: $g(b) := [\Gamma_2 - \Gamma_4(\Gamma_3(1+\Gamma_2) + \Gamma_1\Gamma_2)]$ and $h(b) = \Gamma_2\Gamma_4(\Gamma_1 - \Gamma_3)$, $b \in (1, \infty)$ then

$$f(bx_0)g(b) = f(x_0)h(b).$$

Since the function γ is strictly increasing, we have $h(b) \neq 0$. It means that $f(x_0) = (g(b)/h(b))f(bx_0)$. Recall that the point x_0 was arbitrarily fixed.

It follows that the equality

$$f(x) = \frac{g(b)}{h(b)} f(bx)$$

holds for every $x \in \mathbb{R}^+$ and $b \in (1, \infty)$. Since $f \neq 0$, we infer that $g(b) \neq 0$ for every $b \in (1, \infty)$. Thus

$$f(bx) = \frac{\Gamma_2\Gamma_4(\Gamma_1 - \Gamma_3)}{\Gamma_2 - \Gamma_3\Gamma_4(1 + \Gamma_2) - \Gamma_1\Gamma_2\Gamma_4} f(x) \tag{7}$$

and it remains to put

$$\alpha(b) := \frac{\Gamma_2\Gamma_4(\Gamma_1 - \Gamma_3)}{\Gamma_2 - \Gamma_3\Gamma_4(1 + \Gamma_2) - \Gamma_1\Gamma_2\Gamma_4}$$

where $\Gamma_1, \dots, \Gamma_4$ are defined as above.

Fix now arbitrarily $a, b \in (0, 1)$. Then from the first part of the proof we obtain that

$$f(x_0) = f\left(\frac{1}{b} bx_0\right) = \alpha\left(\frac{1}{b}\right) f(bx_0).$$

Hence

$$f(bx_0) = \frac{1}{\alpha(1/b)} f(x_0).$$

We see that for $b \in (0, 1)$ the required result is achieved by setting $\alpha(b) := 1/(\alpha(1/\mathrm{b}))$. For $b = 1$ we put $\alpha(b) := 1$.

Now we are going to reverse Remark 3 partially.

PROPOSITION *If a function f is a solution of Eq.* (6) *with* $\gamma(a) = (1 + a)^2/[4(1 + a^2)]$, $a \in (0, 1)$, *then* $f(x) = cx^2$, $x \in (0, \infty)$ *for some* $c \in \mathbb{R}$.

Proof We shall start with evaluating coefficients $\Gamma_1, \dots, \Gamma_4$ occurring in Lemma 1. Observe that

$$\Gamma_1 = \gamma\left(\frac{1}{b}\right) = \frac{1/b^2 + 2/b + 1}{4(1/b^2 + 1)} = \frac{1 + 2b + b^2}{4 + 4b^2} = \frac{(1 + b)^2}{4(1 + b^2)} = \gamma(b),$$

i.e. $\gamma(1/b)=\gamma(b)$. This property will prove to be useful in our further calculations which read as follows:

$$\Gamma_2=\gamma\left(\frac{1}{2b-1}\right)=\gamma(2b-1)=\frac{(2b-1+1)^2}{4[(2b-1)^2+1]}=\frac{b^2}{4b^2-4b+2},$$

$$\Gamma_3=\gamma\left(\frac{b}{2b-1}\right)=\frac{(1+b/(2b-1))^2}{4\left(b^2/(2b-1)^2+1\right)}=\frac{(2b-1)^2+2b(2b-1)+b^2}{4b^2+4(2b-1)^2}$$

$$=\frac{4b^2-4b+1+4b^2-2b+b^2}{4b^2+16b^2-16b+4}=\frac{(3b-1)^2}{4(5b^2-4b+1)},$$

$$\Gamma_4=\gamma\left(\frac{b+1}{3b-1}\right)=\frac{((b+1)/(3b-1)+1)^2}{4\left(((b+1)/(3b-1))^2+1\right)}$$

$$=\frac{(4b)^2}{4(b^2+2b+1+9b^2-6b+1)}=\frac{2b^2}{5b^2-2b+1}.$$

Now we are going to evaluate $\alpha(b)$. To this end we shall use the formula (7). Start with the numerator:

$$N=\frac{b^2}{2(2b^2-2b+1)}\frac{2b^2}{5b^2-2b+1}\left(\frac{(1+b)^2}{4(1+b^2)}-\frac{(3b-1)^2}{4(5b^2-4b+1)}\right)$$

$$=\frac{b^4(1+b)^2(5b^2-4b+1)-b^4(3b-1)^2(1+b^2)}{4(2b^2-2b+1)(5b^2-2b+1)(1+b^2)(5b^2-4b+1)}$$

$$=b^4(5b^4-9b^4+10b^3-4b^3+6b^3+b^2+5b^2-8b^2-9b^2-b^2$$

$$+2b-4b+6b+1-1)/(4(2b^2-2b+1)(5b^2-2b+1)$$

$$\times(1+b^2)(5b^2-4b+1))$$

$$=\frac{b^5(-b^3+3b^2-3b+1)}{(2b^2-2b+1)(5b^2-2b+1)(1+b^2)(5b^2-4b+1)}.$$

Now calculate the denominator:

$$
\begin{aligned}
D = & \frac{b^2}{2(2b^2-2b+1)} - \frac{(3b-1)^2 2b^2}{4(5b^2-4b+1)(5b^2-2b+1)} \\
& \times \left(1 + \frac{b^2}{2(2b^2-2b+1)}\right) \\
& - \frac{(1+b)^2 b^2 2b^2}{4(1+b^2)2(2b^2-2b+1)(5b^2-2b+1)} \\
= & \frac{b^2}{2(2b^2-2b+1)} - \frac{(3b-1)^2 2b^2}{4(5b^2-4b+1)(5b^2-2b+1)} \left(\frac{5b^2-4b+2}{2(2b^2-2b+1)}\right) \\
& - \frac{(1+b)^2 b^2 2b^2}{4(1+b^2)2(2b^2-2b+1)(5b^2-2b+1)} \\
= & b^2 \left(\frac{(1+b^2)2(5b^2-2b+1)(5b^2-4b+1)}{4(2b^2-2b+1)(5b^2-4b+1)(5b^2-2b+1)(b^2+1)} \right. \\
& \left. - \frac{(1+b^2)(3b-1)^2(5b^2-4b+2) + (1+b)^2 b^2 (5b^2-4b+1)}{4(2b^2-2b+1)(5b^2-4b+1)(5b^2-2b+1)(b^2+1)} \right).
\end{aligned}
$$

Expand the numerator of the latter fraction:

$$
\begin{aligned}
N(D) = & b^2\{(1+b^2)[50b^4 - 45b^4 - 20b^3 - 40b^3 + 36b^3 + 30b^3 + 10b^2 \\
& \qquad + 10b^2 + 16b^2 - 18b^2 - 5b^2 - 24b^2 - 8b - 4b \\
& \qquad + 4b + 12b + 2 - 2] - (1+b)^2(5b^4 - 4b^3 + b^2)\} \\
= & b^2[(1+b^2)b(5b^3 + 6b^2 - 11b + 4) - (1+b)^2(5b^4 - 4b^3 + b^2)] \\
= & b^3(5b^3 + 6b^2 - 11b + 4 + 5b^2 + 6b^4 - 11b^3 + 4b^2 - 5b^3 \\
& \qquad + 4b^2 - b - 5b^5 + 4b^4 - b^3 - 10b^4 + 8b^3 - 2b^2) \\
= & b^3(-4b^3 + 12b^2 - 12b + 4).
\end{aligned}
$$

Thus the denominator looks as follows:

$$
D = \frac{b^3(-b^3 + 3b^2 - 3b + 1)}{(2b^2-2b+1)(5b^2-2b+1)(1+b^2)(5b^2-4b+1)},
$$

and consequently

$$\alpha(b) = \frac{N}{D} = b^2$$

for every $b > 1$. On the other hand, from the proof of Lemma 1, we obtain $\alpha(b) = 1/(\alpha(1/b)) = b^2$ for $b \in (0, 1)$. It means $\alpha(b) = b^2$ for all $b \in \mathbb{R}^+$. Finally

$$f(x) = f(1x) = x^2 f(1)$$

for every $x \in \mathbb{R}^+$, as claimed.

LEMMA 2 *Let $f\colon (0, \infty) \to \mathbb{R}$ be a continuous solution of Eq.* (6). *Then condition $f(x) + f(ax) \equiv 0$ for some $a \in (0,1)$ implies that f equals zero everywhere.*

Proof Fix an arbitrary point $x \in (0, \infty)$. We shall show that $f(x) = 0$. By assumption we get

$$f\left(\frac{x}{a}\right) = -f(x) = f(ax)$$

for some $a \in (0, 1)$. We know that $ax < x < x/a$. Suppose that $f(x) \neq 0$. Then there are two possibilities: $f(x) > 0 > f(ax) = f(x/a)$ or $f(x) < 0 < f(ax) = f(x/a)$. In both cases there exist points $b \in (ax, x)$ and $c \in (x, x/a)$ such that $f(b) = f(c) = 0$. By Eq. (6) we obtain

$$f\left(\frac{b+c}{2}\right) = f\left(\frac{b + (c/b)b}{2}\right) = \gamma\left(\frac{c}{b}\right)[f(b) + f(c)] = 0.$$

Similarly, one can prove that

$$f\left(\frac{m}{2^n} b + \frac{2^n - m}{2^n} c\right) = 0$$

for every $n \in \mathbb{N}$ and $m \in \mathbb{N}$, $m < 2^n$. Numbers of this form form a dense subset of the interval (b, c) and it follows that $f(x) = 0$, a contradiction.

THEOREM 1 *If f is a continous solution of Eq.* (6) *with a strictly increasing function γ, then $f(x) = cx^d$, $x \in (0, \infty)$ for some $c \in \mathbb{R}$ and $d \in (-\infty, 0) \cup (1, \infty)$.*

Proof Let $f: (0, \infty) \to \mathbb{R}$ be a continuous solution of Eq. (6) and let $\gamma: (0,1) \to (0, \frac{1}{2})$ be a strictly increasing function. At first we are going to show that γ is continuous as well. To this aim take an arbitrary point $a \in (0,1)$ and two sequences (a_n^1) and (a_n^2) both convergent to a and such that $\gamma(a_n^1) \to a_1$, $\gamma(a_n^2) \to a_2$. We get $a_1[f(x)+f(ax)] = \lim \gamma(a_n^1)[f(x) + f(a_n^1 x)] = \lim f((x + a_n^1 x)/2) = f((x + ax)/2) = \lim f((x + a_n^2 x)/2) = \lim \gamma(a_n^2)[f(x) + f(a_n^2 x)] = a_2[f(x) = f(ax)]$ for every $x \in (0, \infty)$. Hence

$$a_1[f(x) + f(ax)] = a_2[f(x) + f(ax)],$$

which jointly with Lemma 2 leads to the continuity of γ.

A careful inspection of the construction of the function α occurring in Lemma 1 shows that α is continuous on the set $\mathbb{R}^+ \backslash \{1\}$. On the other hand, we know that $f(bx) = \alpha(b)f(x)$. Consequently $\alpha(ab)f(x) = f(abx) = \alpha(a)f(bx) = \alpha(a)\alpha(b)f(x)$. Hence α is a solution of the equation

$$\phi(ab) = \phi(a)\phi(b),$$

that is continuous everywhere but at one point. Therefore we know that $\alpha(b) = b^d$ for every $b > 0$ and certain $d \in \mathbb{R}$. Finally

$$f(x) = f(1x) = \alpha(x)f(1) = f(1)x^d = cx^d.$$

Note that $d \in (-\infty, 0) \cup (1, \infty)$. Indeed, if we had $d = 1$, then the function γ_α occurring in Eq. (6) would be equal to $\frac{1}{2}$ whereas if $d \in (0, 1)$, then the same function

$$a \to \gamma_\alpha(a) = \frac{(1 + a)^\alpha}{2^\alpha(1 + a^\alpha)}$$

would not have values in $(0, \frac{1}{2})$ because $\lim_{a \to 0} \gamma_\alpha(x) = \frac{1}{2^\alpha} > \frac{1}{2}$, the function γ_α is continuous so there would exist a point $a_0 \in (0, 1)$ such that $\gamma_\alpha(a_0) > \frac{1}{2}$. Finally, if d were zero, then γ_α would be equal to $\frac{1}{2}$.

Summarizing, we have shown that $d \in (-\infty, 0) \cup (1, \infty)$.

COROLLARY *If $f:(0,\infty)\to(0,\infty)$ is a continuous solution of Eq.* (6), *with a strictly increasing function γ, such that* $\lim_{x\to 0} f(x)=0$, *then $f(x)=cx^d$ for some $c>0$ and $d>1$.*

Proof From Theorem 1 we obtain $f(x)=cx^d$, $x\in(0,\infty)$ for some $c\in\mathbb{R}$ and $d\in(-\infty,0)\cup(1,\infty)$. The constant c is positive because f is positive, d is also positive because for the negative values of d we would have $\lim_{x\to 0} f(x)=\infty$.

THEOREM 2 *Let $f:(0,\infty)\to(0,\infty)$ be an arbitrary continuous function satisfying inequality* (5) *such that*

$$\lim_{x\to 0} f(x)=0.$$

Then

$$\bigwedge_{b>0}\ \bigvee_{c>0,d>1}\ \bigwedge_{x\in(b,\infty)} cx^d\le f(x).$$

Proof At first we shall assume that $b=1$. Put

$$g_\alpha(x):=f(1)x^\alpha,\quad \alpha\in\mathbb{R}.$$

Then $g_\alpha(1)=f(1)$ for every α. Denoting $c:=f(1)$, we shall distinguish two cases.

(1) $\bigvee_{\alpha>1}\bigvee_{x_0}\bigwedge_{x>x_0} f(x)\ge g_\alpha(x)=cx^\alpha$. Consider the function given by the formula $G_\alpha(x):=f(x)/g_\alpha(x)$. Clearly $G_\alpha\colon[1,x_0]\to\mathbb{R}^+$ is continuous on compact interval. Hence function G_α is bounded and

$$p:=\inf_{x\in[1,x_0]} G_\alpha(x)$$

is finite. Obviously

$$G_\alpha(x)\ge p\quad\text{for all } x\in[1,x_0],$$

whence

$$f(x)\ge pcx^\alpha\quad\text{for } x\in[1,x_0].$$

Put $g(x) := \min\{p, 1\}cx^{\alpha}$.This function satisfies the desired inequality on the whole interval $[1, \infty)$ because it is bounded by functions pg_{α} and g_{α}, which satisfy our inequality on $[1, x_0]$ and (x_0, ∞), respectively.

(2) $\bigwedge_{\alpha>1} \bigwedge_{x_0} \bigvee_{x>x_0} f(x) < g_{\alpha}(x)$. We are going to show that the above assumption leads to a contradiction.

To see this let us extend f on $[0, \infty)$ by putting $f(0) = 0$. Let us still denote this function by f. Continuity and strict convexity of such an extension is transparent. We shall prove that $f(x) > cx$ for every $x > 1$. Suppose the contrary:

$$f(x_0) \leq cx_0 \quad \text{for some } x_0 > 1.$$

Since $f(0) = 0 = c0$ we get

$$f(x) \leq cx \quad \text{for all } x \in (0, x_0).$$

In particular $f(1) < c$, a contradiction. Therefore:

$$\bigvee_{\alpha_0>1} \bigwedge_{1<\alpha<\alpha_0} c2^{\alpha} < f(2).$$

Fix an α satisfying the above condition. By assumption, one can find an $x > 2$ such that $f(x) < cx^{\alpha}$. From the continuity of the function f we infer that there exists a point $x_0 > 2$ such that $f(x_0) = cx^{\alpha}$. For the same reason we may find a neighbourhood (a, b) of the point 2 such that for every $x \in (a, b)$ we have $cx^{\alpha} < f(x)$. By choosing the maximal neighbourhood satisfying the above condition we obtain points

$$a_{\alpha} < 2 < b_{\alpha} \tag{8}$$

such that

$$\bigwedge_{x \in (a_{\alpha}, b_{\alpha})} f(x) > cx^{\alpha}$$

and $f(a_{\alpha}) = c(a_{\alpha})^{\alpha}$, $f(b_{\alpha}) = c(b_{\alpha})^{\alpha}$. Hence

$$f\left(\frac{a_{\alpha} + b_{\alpha}}{2}\right) > c\left(\frac{a_{\alpha} + b_{\alpha}}{2}\right)^{\alpha}.$$

On the other hand, by Remark 3,

$$c\left(\frac{a_\alpha + b_\alpha}{2}\right)^\alpha = \gamma_\alpha\left(\frac{a_\alpha}{b_\alpha}\right)[c(a_\alpha)^\alpha + c(b_\alpha)^\alpha],$$

and consequently,

$$f\left(\frac{a_\alpha + b_\alpha}{2}\right) > \gamma_\alpha\left(\frac{a_\alpha}{b_\alpha}\right)[f(a_\alpha) + f(b_\alpha)].$$

Recall that the function γ_α is defined by the formula

$$\gamma_\alpha(a) = \frac{(1+a)^\alpha}{2^\alpha(1+a^\alpha)}, \quad a \in (0,1).$$

Thus we have

$$f\left(\frac{a_\alpha + b_\alpha}{2}\right) > \frac{(1 + a_\alpha/b_\alpha)^\alpha}{2^\alpha(1 + (a_\alpha/b_\alpha)^\alpha)}[f(a_\alpha) + f(b_\alpha)],$$

and setting $h(a) := (1+a)^\alpha/(1+a^\alpha)$, $a \in (0,\infty)$, we may write

$$f\left(\frac{a_\alpha + b_\alpha}{2}\right) > \frac{1}{2^\alpha}h\left(\frac{a_\alpha}{b_\alpha}\right)[f(a_\alpha) + f(b_\alpha)]. \tag{9}$$

Observe that $h(0) = 1$, $h(1) = 2^{\alpha-1}$ and, moreover, $\lim_{a\to\infty} h(a) = 1$. The derivative

$$h'(a) = \frac{\alpha(1+a)^{\alpha-1}(1+a^\alpha) - \alpha a^{\alpha-1}(1+a)^\alpha}{(1+a^\alpha)^2}$$

vanishes at a point $a \in (0,\infty)$ if and only if

$$(1+a)^{\alpha-1}(1+a^\alpha) - a^{\alpha-1}(1+a)^\alpha = 0,$$

which states that

$$a^{\alpha-1} = 1.$$

Consequently, the only zero of this derivative is just $a = 1$. Jointly with the previous observations it implies that $a = 1$ is a maximum of function h which happens to be increasing from 0 to 1 and decreasing from 1 to ∞. Hence by (9) we have got

$$f\left(\frac{a_\alpha + b_\alpha}{2}\right) > \frac{1}{2^\alpha}[f(a_\alpha) + f(b_\alpha)]. \tag{10}$$

Now setting $\alpha_n := 1 + 1/n$ we get the sequence $(a_{\alpha_n}, b_{\alpha_n})$ satisfying inequality (10). This sequence satisfies also $a_{\alpha_n}/b_{\alpha_n} \not\to 1$. Suppose the contrary. Then in view of (8), $a_{\alpha_n} \to 2 \leftarrow b_{\alpha_n}$. From the continuity of f we get $f(2) = c2$. But $f(2) > c2^\alpha > 2c$, a contradiction. Since $a_{\alpha_n}/b_{\alpha_n} \not\to 1$, one can choose a subsequence $(a_{\alpha_{n_k}}, b_{\alpha_{n_k}})$ such that $a_{\alpha_{n_k}}/b_{\alpha_{n_k}} \to a < 1$. Denote this sequence by (c_n, d_n). For all but finitely many $n \in \mathbb{N}$ we have $c_n/d_n < (1+a)/2$. Hence, by (5), we obtain

$$f\left(\frac{c_n + d_n}{2}\right) \leq \gamma\left(\frac{1+a}{2}\right)[f(c_n) + f(d_n)],$$

for almost all $n \in \mathbb{N}$, which contradicts inequality (10).

The proof is now complete in the case $b = 1$. To finish the proof take an arbitrary $b \in \mathbb{R}^+$. We have shown that for every $x > 1$ one has $f(x) > cx^\alpha$. If $b > 1$ then we have the inequality trivially satisfied for $x > b$. Now let $b \in (0,1)$ and let us consider function $G_\alpha(x) := f(x)/cx^\alpha$. Set $p := \min_{x\in[b,1]}(f(x)/cx^\alpha)$. Thus $f(x) \geq pcx^\alpha$ for every $x \in [b,1]$ and $f(x) > \min\{p, 1\}cx^\alpha$ for every $x > b$.

Remark 4 All functions of the form $(0,\infty) \ni x \to \sum_{n=1}^{k} c_n x^{d_n}$, $c_i \in \mathbb{R}^+; d_i \in (1,\infty)$, $i \in \{1,\ldots,k\}$ satisfy inequality (5).

Proof Put $f_i(x) := c_i x^{d_i}$ for $i = 1,\ldots,n$. Then

$$\begin{aligned}
\sum_{n=1}^{k} f_n\left(\frac{x+y}{2}\right) &= \sum_{n=1}^{k} \gamma_{d_n}\left(\frac{y}{x}\right)\frac{f_n(x)+f_n(y)}{2} \\
&\leq \sum_{n=1}^{k} \max_{i=1,\ldots,k} \gamma_{d_i}\left(\frac{y}{x}\right)\frac{f_n(x)+f_n(y)}{2} \\
&= \max_{i=1,\ldots,k} \gamma_{d_i}\left(\frac{y}{x}\right)\sum_{n=1}^{k}(f_n(x)+f_n(y)) \\
&= \gamma\left(\frac{y}{x}\right)\sum_{n=1}^{k}(f_n(x)+f_n(y)) \\
&\leq \gamma(a)\sum_{n=1}^{k}(f_n(x)+f_n(y))
\end{aligned}$$

for every pair (x, y) satisfying the inequality $y/x \leq a$ (γ_{d_i} are increasing).

The following example will show that there exist functions satisfying inequality (5) which are not bounded above by any solution of Eq. (6).

Example 3 The function $f(x) = x^2 e^x$ satisfies inequality (5).

To see this, at first we shall show that

$$e^{bx}x^2 + e^x(bx)^2 \le e^{bx}(bx)^2 + e^x x^2 \quad \text{for all } b \in (0,1). \tag{11}$$

This inequality may equivalently be written as

$$1 + b^2 e^{x-bx} \le b^2 + e^{x-bx}$$

and since $e^{x-bx} > 1$ we are able to write $e^{x-bx} = 1 + \delta$ for some $\delta > 0$. The inequality considered looks now as follows:

$$1 + b^2(1+\delta) \le b^2 + 1 + \delta,$$

i.e. $b^2\delta < \delta$, which is just obvious because $b < 1$. Now,

$$\begin{aligned}
f\left(\frac{x+bx}{2}\right)^2 &= e^{(x+bx)/2}\left(\frac{x+bx}{2}\right)^2 \le \frac{e^x + e^{bx}}{2}\left(\frac{x+bx}{2}\right)^2 \\
&= \frac{e^x + e^{bx}}{2}\frac{(1+b)^2}{4(1+b^2)}(x^2 + (bx)^2) \\
&= \gamma_2(b)\left[\frac{e^x x^2}{2} + \frac{e^{bx}(bx)^2}{2} + \frac{e^x(bx)^2}{2} + \frac{e^{bx}x^2}{2}\right] \\
&= \gamma_2(b)\left[\frac{f(x)}{2} + \frac{f(bx)}{2} + \frac{e^x(bx)^2}{2} + \frac{e^{bx}x^2}{2}\right],
\end{aligned}$$

where, as previously, we have put $\gamma_2(b) := (1+b)^2/[4(1+b^2)]$, $b \in (0,1)$. Clearly γ_2 is strictly increasing.

Finally inequality (11) enables us to write

$$\begin{aligned}
&\gamma_2(b)\left[\frac{f(x)}{2} + \frac{f(bx)}{2} + \frac{e^x(bx)^2}{2} + \frac{e^{bx}x^2}{2}\right] \le \gamma_2(b)\left[\frac{2f(x)}{2} + \frac{2f(bx)}{2}\right] \\
&= \gamma_2(b)[f(x) + f(bx)]
\end{aligned}$$

and the proof has been completed.

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