Estimates for Polynomials Orthogonal with Respect to Some Gegenbauer–Sobolev Type Inner Product

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In this paper we obtain some estimates in $[-1, 1]$ for orthogonal polynomials with respect to an inner product of Sobolev-type

$$\langle f, g \rangle = \int_{-1}^{1} fg \, d\mu_0 + \int_{-1}^{1} f'g' \, d\mu_1$$

where

$$d\mu_0 = \frac{\Gamma(2\alpha + 2)}{2^{(2\alpha+1)}\Gamma^2(\alpha + 1)} (1 - x^2)^\alpha \, dx + M[\delta(x + 1) + \delta(x - 1)]$$

$$d\mu_1 = N[\delta(x + 1) + \delta(x - 1)], \quad M, N \geq 0 \text{ and } \alpha > -1$$

Finally, the asymptotic behavior of such polynomials in $[-1, 1]$ is analyzed.

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1. INTRODUCTION

In [3], Bavinck and Meijer introduced a nonstandard inner product involving derivatives

$$\langle f, g \rangle = \int_{-1}^{1} fg \, d\mu_\alpha + M[\, f(1)g(1) + f(-1)g(-1)\,]$$
$$+ N[\, f'(1)g'(1) + f'(-1)g'(-1)\,] \tag{1.1}$$

where $M \geq 0$, $N \geq 0$ and

$$d\mu_\alpha(x) = \frac{\Gamma(2\alpha + 2)}{2^{(2\alpha+1)}\Gamma^2(\alpha + 1)}(1 - x^2)^\alpha \, dx, \quad \alpha > -1$$

is the probability measure corresponding to the Gegenbauer (or ultraspherical) polynomials.

Such inner products are called Sobolev-type inner products. Notice that (1.1) represents a particular case of the symmetric case analyzed in [2].

In [3], the authors studied the representation of polynomials orthogonal with respect to (1.1) in terms of the ultraspherical polynomials, their expression as hypergeometric functions $_4F_3$ as well as they obtained a second order linear differential equation that such polynomials satisfy.

In a second paper [4], they proved that the zeros are real and simple. For $N > 0$ and the degree of the polynomial large enough, there exists exactly one pair of real zeros $\pm\rho_n$ outside $(-1, 1)$. Furthermore, a five-term recurrence relation for these polynomials is obtained. Notice that in the case $M = N = 0$, the zeros are real and simple, they are located in $(-1, 1)$ and the polynomials satisfy a three-term recurrence relation.

The aim of our contribution is to obtain estimates for such nonstandard orthogonal polynomials through their pointwise behavior as well as for the uniform norm. Thus, we generalize some previous work of us relative to the Legendre–Sobolev type inner product ($\alpha = 0$), [6]. In Section 2, we summarize some results about ultraspherical polynomials that we will need later on, as well as some basic properties for polynomials orthogonal with respect to (1.1).

In Section 3, we obtain the estimate of the norm of the orthonormalized polynomials $\{\hat{B}_n^{(\alpha)}\}_{n\geq 0}$ with respect to (1.1). In Section 4, we find estimates for such polynomials in $(-1, 1)$, an estimate for their uniform norm as well as their behavior at the ends of the interval. We also find an estimate for the uniform behavior in compact subsets of the domain $\mathbb{C}\backslash[-1, 1]$.

2. GEGENBAUER POLYNOMIALS: BASIC PROPERTIES

If the nth ultraspherical polynomial is given by the Rodrigues formula

$$R_n^{(\alpha)}(x) = \frac{(-1)^n \Gamma(\alpha + 1)}{2^n \Gamma(n + \alpha + 1)} (1 - x^2)^{-\alpha} D^n\left((1 - x^2)^{n+\alpha}\right)$$

$\alpha > -1$, then it is very well known that

$$\int_{-1}^{1} R_n^{(\alpha)}(x) R_m^{(\alpha)}(x)(1 - x^2)^\alpha \, \mathrm{d}x = 0 \quad \text{for } m \neq n.$$

Furthermore $R_n^{(\alpha)}(1) = 1$.
On the other hand, for $k = 0, 1, \ldots,$

$$D^k R_n^{(\alpha)}(x) = \frac{(-1)^k (n + 2\alpha + 1)_k (-n)_k}{2^k (\alpha + 1)_k} R_{n-k}^{(\alpha+k)}(x). \qquad (2.1)$$

Here $(a)_n$ is the shifted factorial defined by

$$(a)_n := a(a + 1) \cdots (a + n - 1) = \frac{\Gamma(a + n)}{\Gamma(a)} \quad \text{for } n \geq 1.$$

Taking into account the representation of the Gegenbauer polynomials

$$R_n^{(\alpha)}(x) = {}_2F_1\left(-n, n + 2\alpha + 1; \alpha + 1; \frac{1 - x}{2}\right)$$

as an hypergeometric function, we deduce that the leading coefficient $k_n(\alpha)$ of $R_n^{(\alpha)}$ is

$$k_n(\alpha) := \frac{(n + 2\alpha + 1)_n}{2^n(\alpha + 1)_n} = \frac{\Gamma(2n + 2\alpha + 1)\Gamma(\alpha + 1)}{2^n\Gamma(n + \alpha + 1)\Gamma(n + 2\alpha + 1)}. \tag{2.2}$$

This means that the squared norm of $R_n^{(\alpha)}$ in the space $L_{(1-x^2)^\alpha}^2$ is

$$\|R_n^{(\alpha)}\|_\alpha^2 := \int_{-1}^1 |R_n^{(\alpha)}(x)|^2(1 - x^2)^\alpha \, dx = \frac{2^{2\alpha+1}\Gamma^2(\alpha + 1)n!}{(2n + 2\alpha + 1)\Gamma(n + 2\alpha + 1)} \tag{2.3}$$

(see [8, formula (4.7.15) in p. 81]).

Hence,

$$\|R_n^{(\alpha)}\|_\alpha^2 \approx n^{-(2\alpha+1)}$$

and, as a consequence, for $k \geq 1$

$$\|R_{n-2k}^{(\alpha+2k)}\|_{\alpha+2k}^2 \approx n^{-(2\alpha+4k+1)}. \tag{2.4}$$

This kind of estimates will be very useful in the next section. Notice that these polynomials satisfy a three-term recurrence relation

$$xR_n^{(\alpha)}(x) = \beta_n^{(\alpha)} R_{n+1}^{(\alpha)}(x) + \gamma_n^{(\alpha)} R_{n-1}^{(\alpha)}(x)$$

where

$$\beta_n^{(\alpha)} = \frac{n + 2\alpha + 1}{2n + 2\alpha + 1},$$

$$\gamma_n^{(\alpha)} = \frac{n}{2n + 2\alpha + 1}.$$

We also use (see [8, formula (8.21.10) in p. 196])

$$R_n^{(\alpha)}(\cos \theta) = \frac{n!\Gamma(\alpha + 1)}{n^{1/2}\Gamma(n + \alpha + 1)} k(\theta) \cos(\eta\theta + \gamma) + O(n^{-(\alpha+3/2)}) \tag{2.5}$$

where

$$k(\theta) = \pi^{-1/2} \left(\sin \frac{\theta}{2} \cos \frac{\theta}{2} \right)^{-\alpha - \frac{1}{2}}$$

$$\eta = n + \alpha + \tfrac{1}{2}$$

$$\gamma = -\left(\alpha + \tfrac{1}{2}\right)\pi/2 \quad 0 < \theta < \pi.$$

The bound for the error term holds uniformly in the interval $[\epsilon, \pi - \epsilon]$.

Finally, because of the normalization for the sequence $\{R_n^{(\alpha)}\}_{n \geq 0}$ it follows that

$$\max_{-1 \leq x \leq 1} (1 - x^2)^k |D^{2k} R_n^{(\alpha)}(x)| \leq C_\alpha n^{2k}. \tag{2.6}$$

In terms of the ultraspherical polynomials $\{R_n^{(\alpha)}\}_{n \geq 0}$, Bavinck and Meijer found the following representations for the polynomials $\{B_n^{(\alpha)}\}_{n \geq 0}$ orthogonal with respect to (1.1)

LEMMA 1

$$B_n^{(\alpha)}(x) = \left(a_n x^2 D^2 + b_n x D + c_n \right) R_n^{(\alpha)}(x) \tag{2.7}$$

$$= \left(\frac{a_n}{4(\alpha + 2)(\alpha + 3)} (1 - x^2)^2 D^4 + d_n(1 - x^2)D^2 + e_n \right) R_n^{(\alpha)}(x) \tag{2.8}$$

where

$$a_n = MN \frac{4(2\alpha + 3)_n (2\alpha + 3)_{n-2}}{(\alpha + 1)(\alpha + 2)n!(n - 2)!} + N \frac{2(2\alpha + 3)_{n-1}}{(\alpha + 1)(n - 1)!}, \tag{2.9}$$

$$b_n = -MN \frac{4(2\alpha + 3)_n (2\alpha + 3)_{n-2}(n^2 + (2\alpha + 1)n - 3\alpha - 3)}{(\alpha + 1)(\alpha + 2)(\alpha + 3)n!(n - 2)!}$$

$$- M \frac{2(2\alpha + 2)_{n-1}}{n!} - N \frac{(2\alpha + 3)_{n-1}(n^2 + (2\alpha + 1)n - 4\alpha - 4)}{(\alpha + 1)(n - 1)!}, \tag{2.10}$$

$$c_n = 1 + MN \frac{(2\alpha + 3)_{n+1}(2\alpha + 3)_{n-1}}{(n-1)!(n-3)!(\alpha + 1)(\alpha + 2)^2(\alpha + 3)}$$
$$+ M \frac{2(2\alpha + 3)_{n-1}}{(n-1)!} + N \frac{(2\alpha + 3)_{n-1}(n-2)(n + 2\alpha + 3)}{2(\alpha + 1)(\alpha + 2)(\alpha + 3)(n-1)!}$$
$$\times \left((\alpha + 2)n^2 + (\alpha + 2)(2\alpha + 1)n + 2\alpha + 2 \right), \qquad (2.11)$$

$$d_n = -\frac{N}{2} \frac{(2\alpha + 3)_{n-1}(n-2)(n + 2\alpha + 3)}{(\alpha + 1)(\alpha + 3)(n-1)!} - 2M \frac{(2\alpha + 3)_{n-2}}{n!}, \quad (2.12)$$

$$e_n = B_n^{(\alpha)}(1) = 1 - \frac{N}{2} \frac{(2\alpha + 3)_{n+1}}{(\alpha + 1)(\alpha + 2)(\alpha + 3)(n-3)!}. \qquad (2.13)$$

As a straightforward consequence of the above representations, they deduced that the leading coefficient of $B_n^{(\alpha)}$ is

$$u_n(\alpha) = k_n(\alpha) \left[\frac{n(n-1)(n-2)(n-3)}{4(\alpha + 2)(\alpha + 3)} a_n - n(n-1)d_n + e_n \right].$$

In the next sections we will denote $f(n) \approx g(n)$ when there exists universal constants $C, D \in \mathbb{R}^+$ such that $Cf(n) \leq g(n) \leq Df(n)$ for n large enough.

3. ESTIMATES FOR THE NORM OF THE GEGENBAUER–SOBOLEV TYPE POLYNOMIALS

If we denote $\left\{ \hat{B}_n^\alpha \right\}_{n \geq 0}$ the sequence of polynomials orthonormal with respect to the inner product (1.1), i.e.,

$$\delta_{n,m} = \frac{\Gamma(2\alpha + 2)}{2^{2\alpha+1}\Gamma^2(\alpha + 1)} \int_{-1}^1 \hat{B}_n^\alpha(x)\hat{B}_m^\alpha(x)(1 - x^2)^\alpha \, dx$$
$$+ M \left[\hat{B}_n^\alpha(1)\hat{B}_m^\alpha(1) + \hat{B}_n^\alpha(-1)\hat{B}_m^\alpha(-1) \right]$$
$$+ N \left[\{\hat{B}_n^\alpha\}'(1)\{\hat{B}_m^\alpha\}'(1) + \{\hat{B}_n^\alpha\}'(-1)\{\hat{B}_m^\alpha\}'(-1) \right]$$

then

$$\hat{B}_n^\alpha(x) = \lambda_n B_n^{(\alpha)}(x)$$

where

$$\lambda_n^{-2} = \frac{\Gamma(2\alpha + 2)}{2^{2\alpha+1}\Gamma^2(\alpha + 1)} \int_{-1}^{1} \left[B_n^{(\alpha)}(x)\right]^2 (1 - x^2)^{(\alpha)} \, \mathrm{d}x$$

$$+ 2M\left[B_n^{(\alpha)}(1)\right]^2 + 2N\left[\{B_n^{(\alpha)}\}'(1)\right]^2.$$

First we will compute the integral. In fact, because of (2.8)

$$\int_{-1}^{1} \left[B_n^{(\alpha)}(x)\right]^2 (1 - x^2)^{\alpha} \, \mathrm{d}x$$

$$= \int_{-1}^{1} \frac{a_n^2 (1 - x^2)^4 \left[D^4 R_n^{(\alpha)}(x)\right]^2}{16(\alpha + 2)^2 (\alpha + 3)^2} (1 - x^2)^{\alpha} \, \mathrm{d}x$$

$$+ \int_{-1}^{1} d_n^2 (1 - x^2)^2 \left[D^2 R_n^{(\alpha)}(x)\right]^2 (1 - x^2)^{\alpha} \, \mathrm{d}x$$

$$+ \int_{-1}^{1} e_n^2 \left[R_n^{(\alpha)}(x)\right]^2 (1 - x^2)^{\alpha} \, \mathrm{d}x$$

$$+ \frac{2a_n d_n}{4(\alpha + 2)(\alpha + 3)} \int_{-1}^{1} (1 - x^2)^3 D^4 R_n^{(\alpha)}(x) D^2 R_n^{(\alpha)}(x) (1 - x^2)^{\alpha} \, \mathrm{d}x$$

$$+ \frac{2a_n e_n}{4(\alpha + 2)(\alpha + 3)} \int_{-1}^{1} (1 - x^2)^2 D^4 R_n^{(\alpha)}(x) R_n^{(\alpha)}(x) (1 - x^2)^{\alpha} \, \mathrm{d}x$$

$$+ 2d_n e_n \int_{-1}^{1} (1 - x^2) D^2 R_n^{(\alpha)}(x) R_n^{(\alpha)}(x) (1 - x^2)^{\alpha} \, \mathrm{d}x$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \tag{3.1}$$

Taking into account (2.1)

$$I_1 = \frac{a_n^2 (-n)_4^2 (n + 2\alpha + 1)_4^2}{16^3 (\alpha + 2)^2 (\alpha + 3)^2 (\alpha + 1)_4^2} \int_{-1}^{1} \left[R_{n-4}^{(\alpha+4)}(x)\right]^2 (1 - x^2)^{\alpha+4} \, \mathrm{d}x$$

$$= \frac{a_n^2 (-n)_4^2 (n + 2\alpha + 1)_4^2}{16^3 (\alpha + 2)^2 (\alpha + 3)^2 (\alpha + 1)_4^2} \| R_{n-4}^{(\alpha+4)} \|_{\alpha+4}^2.$$

In a similar way, using (2.1)

$$
\begin{aligned}
I_2 &= \frac{d_n^2(-n)_2^2(n+2\alpha+1)_2^2}{16(\alpha+1)_2^2} \int_{-1}^1 \left[R_{n-2}^{(\alpha+2)}(x)\right]^2 (1-x^2)^{\alpha+2}\,dx \\
&= \frac{d_n^2(-n)_2^2(n+2\alpha+1)_2^2}{16(\alpha+1)_2^2} \|R_{n-2}^{(\alpha+2)}\|_{\alpha+2}^2,
\end{aligned}
$$

$$
I_3 = e_n^2 \|R_n^{(\alpha)}\|_\alpha^2,
$$

$$
\begin{aligned}
I_4 &= \frac{a_n d_n(-n)_2(n+2\alpha+1)_2}{8(\alpha+2)_2(\alpha+1)_2} \\
&\quad \times \int_{-1}^1 (1-x^2)\left[D^4 R_n^{(\alpha)}(x)\right] R_{n-2}^{(\alpha+2)}(x)(1-x^2)^{\alpha+2}\,dx \\
&= -\frac{a_n d_n}{2(\alpha+2)_2}\, \frac{n(n-1)(n-2)(n-3)(-n)_2(n+2\alpha+1)_2}{4(\alpha+1)_2} \\
&\quad \times \frac{k_n(\alpha)}{k_{n-2}(\alpha+2)} \|R_{n-2}^{(\alpha+2)}\|_{\alpha+2}^2 \\
&= -\frac{a_n d_n n^2(n-1)^2(n-2)(n-3)(n+2\alpha+1)_2^2}{32(\alpha+2)_2(\alpha+1)_2^2} \|R_{n-2}^{(\alpha+2)}\|_{\alpha+2}^2,
\end{aligned}
$$

$$
\begin{aligned}
I_5 &= \frac{2a_n e_n}{4(\alpha+2)(\alpha+3)} n(n-1)(n-2)(n-3) \int_{-1}^1 \left[R_n^{(\alpha)}(x)\right]^2 (1-x^2)^\alpha \\
&= \frac{2a_n e_n}{4(\alpha+2)(\alpha+3)} n(n-1)(n-2)(n-3) \|R_n^{(\alpha)}\|_\alpha^2,
\end{aligned}
$$

$$
I_6 = -2d_n e_n n(n-1)\|R_n^{(\alpha)}\|_\alpha^2.
$$

As a conclusion, (3.1) can be expressed

$$
\begin{aligned}
&= a_n^2 \frac{n^2(n-1)^2(n-2)^2(n-3)^2(n+2\alpha+1)_4^2}{16^3(\alpha+2)^2(\alpha+3)^2(\alpha+1)_4^2} \|R_{n-4}^{(\alpha+4)}\|_{\alpha+4}^2 \\
&\quad + \frac{d_n n^2(n-1)^2(n+2\alpha+1)_2^2}{16(\alpha+1)_2^2}\left[d_n - \frac{(n-2)(n-3)a_n}{2(\alpha+2)_2}\right] \|R_{n-2}^{(\alpha+2)}\|_{\alpha+2}^2 \\
&\quad + e_n\left[e_n - 2n(n-1)d_n + \frac{n(n-1)(n-2)(n-3)}{2(\alpha+2)(\alpha+3)}a_n\right] \|R_n^{(\alpha)}\|_\alpha^2
\end{aligned}
$$

$$= \left\{ \frac{n(n-1)(n-2)(n-3)}{16} \frac{(n+2\alpha+1)_4}{(\alpha+2)^2(\alpha+3)^2} a_n^2 \right.$$

$$+ d_n n(n-1)(n+2\alpha+1)_2 \left[d_n - \frac{(n-2)(n-3)a_n}{2(\alpha+2)_2} \right]$$

$$\left. + e_n \left[e_n - 2n(n-1)d_n + \frac{n(n-1)(n-2)(n-3)}{2(\alpha+2)(\alpha+3)} a_n \right] \right\} \|R_n^{(\alpha)}\|_\alpha^2.$$

On the other hand

$$B_n^{(\alpha)}(1) = e_n, \tag{3.2}$$

$$\{B_n^{(\alpha)}\}'(1) = -2d_n\{R_n^{(\alpha)}\}''(1) + e_n\{R_n^{(\alpha)}\}'(1)$$

$$= \{R_n^{(\alpha)}\}'(1) + \frac{(2\alpha+3)_n}{(\alpha+1)(\alpha+2)(n-2)!} M$$

$$= \frac{n}{2} \frac{(n+2\alpha+1)}{\alpha+1} R_{n-1}^{(\alpha+1)}(1) + \frac{(2\alpha+3)_n}{(\alpha+1)(\alpha+2)(n-2)!} M$$

$$= \frac{n}{2} \frac{(n+2\alpha+1)}{\alpha+1} + \frac{(2\alpha+3)_n}{(\alpha+1)(\alpha+2)(n-2)!} M. \tag{3.3}$$

Thus

$$\lambda_n^{-2} = 2Me_n^2 + 2N\left(\frac{n(n+2\alpha+1)}{2(\alpha+1)} + \frac{(2\alpha+3)_n}{(\alpha+1)(\alpha+2)(n-2)!} M \right)^2$$

$$+ \frac{\Gamma(2\alpha+2)n!}{(2n+2\alpha+1)\Gamma(n+2\alpha+1)}$$

$$\times \left\{ \frac{n(n-1)(n-2)(n-3)}{16} \frac{(n+2\alpha+1)_4}{(\alpha+2)^2(\alpha+3)^2} a_n^2 \right.$$

$$+ n(n-1)(n+2\alpha+1)_2 d_n^2$$

$$- \frac{n(n-1)(n-2)(n-3)}{2(\alpha+2)_2} (n+2\alpha+1)_2 a_n d_n$$

$$\left. + e_n^2 - 2n(n-1)e_n d_n + \frac{n(n-1)(n-2)(n-3)}{2(\alpha+2)(\alpha+3)} a_n e_n \right\}.$$

$$\tag{3.4}$$

In order to estimate λ_n^{-2}, we will distinguish the following three cases:

1. $M > 0$, $N > 0$, then

$$a_n = 4MN \frac{n(n-1)}{(\alpha+1)(\alpha+2)} \frac{2\alpha+n+2}{2\alpha+n+1} \alpha_n^2 + 2N \frac{n}{\alpha+1} \alpha_n,$$

$$d_n = -\frac{N}{2} \frac{n(n-2)}{(\alpha+1)(\alpha+3)} (n+2\alpha+3)\alpha_n - 2M \frac{1}{2\alpha+n+1} \alpha_n,$$

$$e_n = 1 - \frac{N}{2} \frac{n(n-1)(n-2)(n+2\alpha+3)(n+2\alpha+2)}{(\alpha+1)(\alpha+2)(\alpha+3)} \alpha_n$$

where

$$\alpha_n = \frac{\Gamma(2\alpha+n+2)}{\Gamma(2\alpha+3)\Gamma(n+1)}.$$

But, according to the asymptotic behavior of the gamma function (see [7, formula 8.16 in p. 88])

$$\Gamma(x) \approx e^{-x} x^x \sqrt{\frac{2\pi}{x}}$$

then

$$\alpha_n \approx n^{2\alpha+1}, \tag{3.5}$$

and

$$a_n \approx n^{4\alpha+4}, \tag{3.6}$$

$$d_n \approx n^{2\alpha+4}, \tag{3.7}$$

$$e_n \approx n^{2\alpha+6}. \tag{3.8}$$

Thus

$$\lambda_n^{-2} = 2Me_n^2 + 2N\left(\frac{n(n+2\alpha+1)}{2(\alpha+1)} + \frac{(n+2\alpha+2)n(n-1)}{(\alpha+1)_2}M\alpha_n\right)^2$$

$$+ \frac{1}{2\alpha+2}\frac{n+2\alpha+1}{2n+2\alpha+1}\frac{1}{\alpha_n}\left\{\frac{(n-3)_4}{16}\frac{(n+2\alpha+1)_4}{(\alpha+2)_2^2}a_n^2\right.$$

$$+ n(n-1)(n+2\alpha+1)_2 d_n^2 - \frac{(n-3)_4}{2(\alpha+2)_2}(n+2\alpha+1)_2 a_n d_n$$

$$+ e_n^2 - 2n(n-1)e_n d_n + \frac{(n-3)_4}{2(\alpha+2)_2}a_n e_n\right\} \approx n^{6\alpha+15}. \tag{3.9}$$

2. $M = 0$, $N > 0$, then

$$a_n = \frac{2Nn}{(\alpha+1)}\alpha_n,$$

$$d_n = -\frac{N}{2}\frac{n(n-2)}{(\alpha+1)(\alpha+3)}(n+2\alpha+3)\alpha_n,$$

$$e_n = 1 - \frac{N}{2}\frac{(n-2)_3(n+2\alpha+2)_2}{(\alpha+1)_3}\alpha_n.$$

Thus

$$a_n \approx n^{2\alpha+2}, \tag{3.10}$$

$$d_n \approx n^{2\alpha+4}, \tag{3.11}$$

$$e_n \approx n^{2\alpha+6}. \tag{3.12}$$

and

$$\lambda_n^{-2} \approx n^{2\alpha+11}. \tag{3.13}$$

3. $M > 0$, $N = 0$, then

$$a_n = 0, \tag{3.14}$$

$$d_n = -\frac{2M\alpha_n}{2\alpha + n + 1}, \tag{3.15}$$

$$e_n = 1. \tag{3.16}$$

Thus

$$\lambda_n^{-2} \approx n^{2\alpha+3}. \tag{3.17}$$

As a conclusion

PROPOSITION 1

$$\lambda_n = \|B_n^{(\alpha)}\|_2^{-1} \approx \begin{cases} n^{-3\alpha-15/2} & M > 0 \quad N > 0, \\ n^{-\alpha-11/2} & M = 0 \quad N > 0, \\ n^{-\alpha-3/2} & M > 0 \quad N = 0. \end{cases}$$

COROLLARY 1   *The leading coefficient $v_n(\alpha)$ of $\hat{B}_n^{(\alpha)}(x)$ satisfies*

$$v_n(\alpha) \approx 2^n.$$

Proof   The leading coefficient of $\hat{B}_n^{(\alpha)}(x)$ is

$$\lambda_n k_n(\alpha) \left[\frac{n(n-1)(n-2)(n-3)}{4(\alpha+2)(\alpha+3)} a_n - n(n-1)d_n + e_n \right].$$

Thus we will distinguish the following three cases:

1. $M > 0$, $N > 0$

$$v_n(\alpha) \approx n^{-3\alpha-15/2} 2^{n+2\alpha+1/2} n^{-\alpha-1/2} n^{4\alpha+8} = 2^{n+2\alpha+1/2}.$$

2. $M = 0$, $N > 0$

$$v_n(\alpha) \approx n^{-\alpha-11/2} 2^{n+2\alpha+1/2} n^{-\alpha-1/2} n^{2\alpha+6} = 2^{n+2\alpha+1/2}.$$

3. $M > 0$, $N = 0$

$$v_n(\alpha) \approx n^{-\alpha-3/2} 2^{n+2\alpha+1/2} n^{-\alpha-1/2} n^{2\alpha+2} = 2^{n+2\alpha+1/2}.$$

4. ASYMPTOTICS FOR GEGENBAUER–SOBOLEV TYPE POLYNOMIALS

Now we are going to deduce a formula similar to (2.5) for the polynomial $B_n^{(\alpha)}$. Using (2.1) and (2.8) it holds

$$B_n^{(\alpha)}(x) = \frac{a_n}{4(\alpha+2)(\alpha+3)} \frac{(n+2\alpha+1)_4(-n)_4}{2^4(\alpha+1)_4} (1-x^2)^2 R_{n-4}^{\alpha+4}(x)$$

$$+ d_n \frac{(n+2\alpha+1)_2(-n)_2}{2^2(\alpha+1)_2} (1-x^2) R_{n-2}^{\alpha+2}(x)$$

$$+ e_n R_n^{\alpha}(x).$$

Putting $x = \cos\theta$ and taking into account that $\sqrt{1-x^2} = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$, we have

$$B_n^{(\alpha)}(\cos\theta) = \frac{n!\Gamma(\alpha+1)}{4\Gamma(n+\alpha+1)} k(\theta) \cos(\eta\theta+\gamma)$$

$$\times \left[a_n \frac{(n+2\alpha+1)_4}{(\alpha+2)_2(n-4)^{1/2}} - d_n \frac{(n+2\alpha+1)_2}{(n-2)^{1/2}} + e_n \frac{4}{n^{1/2}} \right]$$

$$+ a_n O(n^{-\alpha+5/2}) + d_n O(n^{-\alpha+1/2}) + e_n O(n^{-\alpha-3/2}).$$

Now we distinguish the following three cases:

1. $M > 0$, $N > 0$. Using that

$$a_n = 4MN \frac{(2\alpha+3)_n (2\alpha+3)_{n-2}}{(\alpha+1)(\alpha+2) n!(n-2)!} + O(n^{2\alpha+2}),$$

$$d_n \approx n^{2\alpha+4},$$

$$e_n \approx n^{2\alpha+6},$$

we get

$$B_n^{(\alpha)}(\cos\theta) = MN \frac{\Gamma(\alpha+1) n^{3\alpha+15/2}}{(\alpha+1)(\alpha+2)^2(\alpha+3)(\Gamma(2\alpha+3))^2}$$

$$\times k(\theta) \cos(\eta\theta+\gamma) + O(n^{\beta})$$

where

$$\beta = \begin{cases} 3\alpha + \frac{13}{2} & \alpha \geq -\frac{1}{2}, \\ \alpha + \frac{11}{2} & -1 \leq \alpha \leq -\frac{1}{2}, \end{cases}$$

and N, γ are the same constants as in (2.5). The bound for the error term holds uniformly in the interval $[\epsilon, \pi - \epsilon]$.

2. $M = 0, N > 0$. Doing the same kind of calculations it holds

$$B_n^{(\alpha)}(\cos\theta) = N\frac{\Gamma(\alpha+1)n^{\alpha+11/2}}{8(\alpha+1)(\alpha+3)\Gamma(2\alpha+3)}k(\theta)\,\cos(\eta\theta+\gamma) \\ + O(n^{\alpha+9/2})$$

and N, γ are the same constants as in (2.5). The bound for the error term holds uniformly in the interval $[\epsilon, \pi - \epsilon]$.

3. $M > 0, N = 0$. In this case we have

$$B_n^{(\alpha)}(\cos\theta) = M\frac{\Gamma(\alpha+1)n^{\alpha+3/2}}{2\Gamma(2\alpha+3)}k(\theta)\,\cos(\eta\theta+\gamma) + O(n^\beta)$$

where

$$\beta = \begin{cases} \alpha + \frac{1}{2} & \alpha \geq -\frac{1}{2} \\ -(\alpha + \frac{1}{2}) & -1 \leq \alpha \leq -\frac{1}{2} \end{cases}$$

and N, γ are the same constants as in (2.5). The bound for the error term holds uniformly in the interval $[\epsilon, \pi - \epsilon]$.

Since the polynomials $\{D^4 R_n^{(\alpha)}\}_{n\geq 4}$ are orthogonal with respect to the weight function $(1 - x^2)^{\alpha+4}$, using (2.5) we get

$$(1 - x^2)^2|D^4 R_n^{(\alpha)}(x)| \leq C_\alpha n^{-\alpha+7/2}(1 - x^2)^{-(\alpha/2+1/4)} \qquad (4.1)$$

where the constant C_α is independent of $n \in \mathbb{N}$ and $x \in (-1, 1)$. In the same way

$$(1 - x^2)|D^2 R_n^{(\alpha)}(x)| \leq C_\alpha n^{-\alpha+3/2}(1 - x^2)^{-(\alpha/2+1/4)} \qquad (4.2)$$

where, again, the constant $C_\alpha > 0$ is independent of $n \in \mathbb{N}$ and $x \in (-1, 1)$.

Using (2.8), (4.1) and (4.2), we have the following estimate.

PROPOSITION 2

$$|B_n^{(\alpha)}(x)| \le C_\alpha \left\{ \frac{a_n}{(\alpha+2)(\alpha+3)} n^{-\alpha+7/2} \right.$$
$$\left. + |d_n| n^{-\alpha+3/2} + |e_n| n^{-\alpha-1/2} \right\} (1-x^2)^{-\alpha/2-1/4}$$

where the constant $C_\alpha > 0$ is independent of $n \in \mathbb{N}$ and $x \in (-1, 1)$.

Next, we will deduce the uniform estimates for the orthonormalized polynomials $\{\hat{B}_n^{(\alpha)}\}_{n \ge 0}$. Taking into account the expression of $\{\hat{B}_n^{(\alpha)}\}_{n \ge 0}$ in terms of $\{B_n^{(\alpha)}\}_{n \ge 0}$

$$|\hat{B}_n^{(\alpha)}(x)| = \lambda_n |B_n^{(\alpha)}(x)|$$
$$\le \lambda_n \left\{ \frac{a_n}{4(\alpha+2)(\alpha+3)} \max_{-1 \le x \le 1} (1-x^2)^2 |D^4 R_n^{(\alpha)}(x)| \right.$$
$$+ |d_n| \max_{-1 \le x \le 1} (1-x^2)|D^2 R_n^{(\alpha)}(x)|$$
$$\left. + |e_n| \max_{-1 \le x \le 1} |R_n^{(\alpha)}(x)| \right\}.$$

But, from (2.6), we get

$$|\hat{B}_n^{(\alpha)}(x)| \le C_\alpha \lambda_n \left\{ a_n n^4 + |d_n| n^2 + |e_n| \right\}.$$

Thus

PROPOSITION 3

$$\max_{-1 \le x \le 1} |\hat{B}_n^{(\alpha)}(x)| \le C_\alpha n^{\alpha+1/2}, \quad n \in \mathbb{N}.$$

Proof As in the above propositions, we must distinguish three cases.
1. $M > 0$, $N > 0$, then

$$\max_{-1 \le x \le 1} |\hat{B}_n^{(\alpha)}(x)| \le C_\alpha n^{-(3\alpha+15/2)} \{n^{4\alpha+8} + n^{2\alpha+6}\} = C_\alpha n^{\alpha+1/2}.$$

2. $M = 0$, $N > 0$, then

$$\max_{-1 \le x \le 1} |\hat{B}_n^{(\alpha)}(x)| \le C_\alpha n^{-(\alpha+11/2)} n^{2\alpha+6} = C_\alpha n^{\alpha+1/2}.$$

3. $M > 0$, $N = 0$, then

$$\max_{-1 \le x \le 1} |\hat{B}_n^{(\alpha)}(x)| \le C_\alpha n^{-(\alpha+3/2)} n^{2\alpha+2} = C_\alpha n^{\alpha+1/2}.$$

Finally, the comparison of the above two estimates leads to the analysis of the behavior of $|\hat{B}_n^\alpha(1)|$ as well as $|\{\hat{B}_n^\alpha\}'(1)|$.

PROPOSITION 4

$$|\hat{B}_n^\alpha(1)| \approx \begin{cases} n^{-\alpha-3/2} & \text{if } M > 0, N > 0, \text{ or } M > 0, \ N = 0, \\ n^{\alpha+1/2} & \text{if } M = 0, N > 0. \end{cases}$$

$$|\{\hat{B}_n^\alpha\}'(1)| \approx \begin{cases} n^{-\alpha-7/2} & \text{if } M > 0, N > 0, \text{ or } M = 0, \ N > 0, \\ n^{\alpha+5/2} & \text{if } M > 0, N = 0. \end{cases}$$

Notice that for $\alpha = 0$ we recover Proposition 3.4 in [6].

Proof   Taking into account (3.2), and (3.3),

$$|\hat{B}_n^\alpha(1)| = \lambda_n e_n$$

$$|\{\hat{B}_n^\alpha\}'(1)| = \lambda_n \left[\frac{n(n+2\alpha+1)}{2(\alpha+1)} + \frac{(2\alpha+3)_n}{(\alpha+1)(\alpha+2)(n-2)!} M \right].$$

Thus

  (i)   If $M > 0$, $N > 0$, then

$$|\hat{B}_n^\alpha(1)| \approx n^{-\alpha-3/2},$$

$$|\{\hat{B}_n^\alpha\}'(1)| \approx n^{-\alpha-7/2}.$$

  (ii)   If $M = 0$, $N > 0$, then

$$|\hat{B}_n^\alpha(1)| \approx n^{\alpha+1/2},$$

$$|\{\hat{B}_n^\alpha\}'(1)| \approx n^{-\alpha-7/2}.$$

(iii) If $M > 0$, $N = 0$, then

$$|\hat{B}_n^\alpha(1)| \approx n^{-\alpha-3/2},$$
$$|\{\hat{B}_n^\alpha\}'(1)| \approx n^{\alpha+5/2}.$$

As a last step we deduce the uniform asymptotic behavior of the sequence $\{B_n^{(\alpha)}\}_{n\geq 0}$ in compact subsets of the domain $\mathbb{C}\backslash[-1, 1]$.
Setting $l_n^{(\alpha)} = R_n^{(\alpha)}/\|R_n^{(\alpha)}\|_\alpha$ and dividing (2.7) by $\|R_n^{(\alpha)}\|_\alpha$ we have

$$\frac{B_n^{(\alpha)}(x)}{\|R_n^{(\alpha)}\|_\alpha} = \frac{a_n(n+2\alpha+1)_2 n(n-1)}{2^2(\alpha+1)_2} \frac{\|R_{n-2}^{(\alpha+2)}\|_{\alpha+2}}{\|R_n^{(\alpha)}\|_\alpha} x^2 \, l_{n-2}^{(\alpha+2)}(x)$$

$$+ b_n \frac{(n+2\alpha+1)n}{2(\alpha+1)} \frac{\|R_{n-2}^{(\alpha+2)}\|_{\alpha+2}}{\|R_n^{(\alpha)}\|_\alpha} x \, l_{n-1}^{(\alpha+1)}(x) + c_n l_n^{(\alpha)}(x).$$

The following asymptotic formula for $\{l_n^{(\alpha)}\}$ (see [8, formula (12.1.3) in p. 297]) is well known

$$l_n^{(\alpha)}\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right) = \frac{1}{\sqrt{2\pi}} \frac{1}{z^n} \frac{1}{D_\omega(z)} [1 + o(1)], \quad |z| < 1 \qquad (4.3)$$

where $o(1)$ holds uniformly inside the unit disk and D_ω is the so-called Szegö's function which is defined in the following way:

$$D_\omega(z) = \exp\left\{\frac{1}{4\pi} \int_{-\pi}^{+\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} \ln(\omega(\cos t)|\sin t|)\, dt\right\}$$

where

$$\omega(x) = (1 - x^2)^\alpha.$$

The Szegö's function satisfies

(a) $D_\omega(z) \in H_2$.
(b) $D_\omega(e^{i\theta}) = \lim_{r \to 1^-} D_\omega(re^{i\theta})$ exists a.e. and $|D_\omega(e^{i\theta})|^2 = \omega(\cos(\theta)) |\sin(\theta)|$.
(c) $D_\omega(z) \neq 0$, $|z| < 1$ and $D_\omega(0) > 0$.

Now using (4.3) we distinguish the following three cases, and for each of them the result holds uniformly in compact subsets of the domain $\mathbb{C}\backslash[-1, 1]$.

1. $M > 0, N > 0$

$$\frac{B_n^{(\alpha)}(x)}{n^{4\alpha+8}\|R_n^{(\alpha)}\|_\alpha} \approx l_n^{(\alpha)}(x)$$

or, equivalently,

$$\frac{B_n^{(\alpha)}(x)}{n^{4\alpha+8}R_n^{(\alpha)}(x)} \approx 1.$$

2. $M > 0, N = 0$

$$\frac{B_n^{(\alpha)}(x)}{n^{2\alpha+2}\|R_n^{(\alpha)}\|_\alpha} \approx l_n^{(\alpha)}(x)$$

or, equivalently,

$$\frac{B_n^{(\alpha)}(x)}{n^{2\alpha+2}R_n^{(\alpha)}(x)} \approx 1.$$

3. $M = 0, N > 0$

$$\frac{B_n^{(\alpha)}(x)}{n^{2\alpha+6}\|R_n^{(\alpha)}\|_\alpha} \approx l_n^{(\alpha)}(x)$$

or, equivalently,

$$\frac{B_n^{(\alpha)}(x)}{n^{2\alpha+6}R_n^{(\alpha)}(x)} \approx 1.$$

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