

# A Reversed Poincaré Inequality for Monotone Functions

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We derive a family of sharp reversed Poincaré inequalities for positive monotone functions defined on a finite interval of the real line.

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The purpose of this article is to derive a new family of sharp inequalities for positive monotone functions defined on a finite interval of the real line. The method used is based on recent results obtained for eigenvalue problems of conservative Hamiltonian systems and other nonlinear eigenvalue problems of relevance in physics [3].

The inequalities we present here are embodied in the following.

**THEOREM** *Let  $g$  be a nonnegative nondecreasing function defined on  $(0, b)$ . Assume  $g \in C^1(0, b)$ , and denote its derivative by  $g'$ . Let  $q > 2$  and  $m \geq 1$ . Then,*

$$\frac{\left(\int_0^b (g'(u))^{1/q} du\right)^q}{\int_0^b g(u)u^{m-1} du} \leq \frac{b^{q-m-1}}{m^{q-2}} B\left(\frac{1}{m}, \frac{q-2}{q-1}\right)^{q-1}, \quad (1)$$

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where  $B$  denotes the Beta function, i.e.,  $B(n, m) \equiv \Gamma(n)\Gamma(m)/\Gamma(n+m)$ . Equality is obtained in (1) for a unique function  $g$ , up to a multiplicative constant.

*Remarks* (1) The case  $q=3, m=2$ , is of particular interest in the derivation of a variational characterization of the principal branch of the two point boundary value problem

$$-u'' = \lambda u + N(u), \quad \text{in } (0, 1)$$

with  $u'(0) = 0$  and  $u(1) = 0$  [3]. For this case, (1) reads,

$$\frac{\left(\int_0^b (g'(u))^{1/3} du\right)^3}{\int_0^b g(u)u du} \leq \frac{\pi^2}{2}.$$

(2) Another interesting case is obtained for  $m=2$  and  $q=\infty$ . Then, (1) reads

$$\exp\left[\frac{1}{b}\int_0^b \log g'(u) du\right] \leq \frac{e^2}{2b^3}\int_0^b g(u)u du.$$

(3) For simplicity we have considered intervals of the form  $(0, b)$  with  $0 < b < \infty$ . By an appropriate translation our result can be extended to intervals of the form  $(a, b)$  with  $-\infty < a < b < \infty$ .

*Proof* We will assume  $\int_0^b g(u)u^{m-1} du < \infty$  since otherwise the inequality is trivial. Using that  $g$  is increasing and nonnegative we have  $0 \leq g(u)(b^m - u^m) < m \int_u^b s^{m-1} g(s) ds$ . Hence,  $\lim_{u \rightarrow b} g(u)(b^m - u^m) = 0$ , a fact that we will use in the sequel. Let  $n = q - 1 > 1$ , and consider the following two point boundary value problem:

$$-\frac{d}{dx} \left( \left| \frac{du}{dx} \right|^{n-2} \frac{du}{dx} \right) = \lambda |u|^{m-2} u \quad \text{on } (0, 1), \quad (2)$$

with  $u'(0) = 0$  and  $u(1) = 0$ . Denote  $u(0) = b$ . For every constant  $b > 0$  this two point boundary value problem has a unique positive solution if and only if

$$\lambda = \lambda(b) = \frac{n-1}{n} \frac{b^{n-m}}{m^{n-1}} B(1/m, 1 - 1/n)^n \quad (3)$$

(see e.g. [5], p. 357; for completeness we have included this result in the Lemma below). This positive solution is at least  $C^1(0, 1)$  and it is monotonic decreasing, i.e.  $u'(x) < 0$ . Henceforth, for a given  $b > 0$  we fix  $\lambda = \lambda(b)$ , given by (3). Let  $p = -du/dx > 0$ . Equation (2) is equivalent to the system of first order equations

$$\begin{aligned} u' &= -p, \\ p' &= \frac{\lambda u^{m-1}}{(n-1)p^{n-2}}. \end{aligned} \tag{4}$$

The functions  $u(x), p(x)$  corresponding to a solution of (4) trace out a trajectory in the  $(u, p)$ -plane (usually called the phase plane). Such a trajectory has slope

$$\frac{dp}{du} = -\frac{\lambda u^{m-1}}{(n-1)p^{n-1}}, \tag{5}$$

at any point where  $p \neq 0$ . Now, let  $g \in C^1(0, b)$  with  $g' \geq 0$  and  $g \geq 0$  but otherwise arbitrary. Multiplying (5) by  $np^{n-1}g(u)$  and integrating in  $u$  from 0 to  $b$  we get,

$$\begin{aligned} \frac{n}{n-1} \lambda \int_0^b u^{m-1} g(u) \, du &= -n \int_0^b p^{n-1} \frac{dp}{du} g(u) \, du \\ &\geq \int_0^b p^n g'(u) \, du \\ &= \int_0^b p^{q-1} g'(u) \, du, \end{aligned} \tag{6}$$

where the last inequality follows by integrating by parts, using  $g(0) \geq 0$ , and the fact that  $\lim_{u \rightarrow b} g(u)(b^m - u^m) = 0$ . Note that in the neighborhood of  $u = b$ ,  $p(u) \approx (b^m - u^m)^{1/n}$  (see the Lemma below). Using Hölder's inequality we obtain,

$$\left( \int_0^b (g'(u))^{1/q} \, du \right)^q \leq \left( \int_0^b \frac{du}{p} \right)^{q-1} \int_0^b p^{q-1} g'(u) \, du. \tag{7}$$

Now, noticing that  $1 = \int_0^1 dx = \int_0^b du/p(u)$ , from (6) and (7) we get

$$\frac{n}{n-1} \lambda \int_0^b u^{m-1} g(u) \, du \geq \left( \int_0^b (g'(u))^{1/q} \, du \right)^q. \tag{8}$$

Then (1) follows from (3) and (8). To conclude with the proof of the theorem we need to discuss the case of equality. Equality in (1) is attained if and only if

$$g'(u) = cp^{-q}, \tag{9}$$

where  $c$  is an arbitrary positive constant. Notice that  $p(u)$  is given by  $p(u) = (n\lambda(b^m - u^m)/(m(n-1)))^{1/n}$ , which follows from Eq. (5) and  $p(b) = 0$ . Equation (9) has a unique solution satisfying the initial condition  $g(0) = 0$ , which is given by

$$\hat{g}(u) = \tilde{c} \int_0^u (b^m - s^m)^{-(n+1)/n} ds. \tag{10}$$

This  $\hat{g}$  is unique up to the multiplicative constant  $\tilde{c}$ , it is increasing and positive for  $u \in (0, b)$ . Clearly  $\hat{g} \in C^1(0, b)$ . As  $u \rightarrow b$ ,  $\hat{g}(u) \approx (b-u)^{-1/n}$  and therefore  $\int u^{m-1} \hat{g}(u) du < \infty$  if and only if  $n > 1$ , i.e.,  $q > 2$ .

*Remark* The positive solution of the boundary value problem given by (2) and (3) is the maximizing function of the one-dimensional analog of the Sobolev inequality. See [5], p. 357 for details. The existence and uniqueness of the solution to this boundary value problem can be proved using standard techniques of the calculus of variations. For completeness we give the necessary facts in the Lemma below. The extension of this boundary value problem to higher dimensions has been studied in [4].

**LEMMA** (Talenti [5]) *For the two point boundary value problem given by (2), for  $n > 1$ ,  $m \geq 1$ , with  $u'(0) = u(1) = 0$  and  $u(0) = b > 0$  there exists a unique positive solution if and only if  $\lambda = \lambda(b)$  is given by (3). Moreover  $u$  is decreasing and*

$$u'(x) = - \left[ \left( \frac{n\lambda(b)}{m(n-1)} \right) (b^m - u^m) \right]^{1/n}. \tag{11}$$

*Proof* By an elementary computation we have that for any  $n > 1, m \geq 1$ ,

$$\begin{aligned} \int_0^b \frac{du}{(b^m - u^m)^{1/n}} &= \frac{2}{m} b^{(n-m)/n} \int_0^{\pi/2} (\sin \theta)^{(2-m)/m} (\cos \theta)^{(n-2)/n} d\theta \\ &= \frac{b^{(n-m)/n}}{m} B\left(\frac{1}{m}, \frac{n-1}{n}\right). \end{aligned} \tag{12}$$

Now, for  $\lambda(b)$  given by (3), define implicitly the function  $u: [0, 1] \rightarrow [0, b]$  by

$$\int_{u(x)}^b \frac{dv}{(b^m - v^m)^{1/n}} = \left[ \frac{\lambda(b)n}{m(n-1)} \right]^{1/n} x. \quad (13)$$

It follows from (13) that  $u(0) = 0$  and that

$$u'(x) = - \left[ \left( \frac{\lambda(b)n}{m(n-1)} \right) (b^m - u^m) \right]^{1/n} \leq 0. \quad (14)$$

Now, from (3), (12), and (13) we have  $u(1) = 0$ . Also, from (14) and  $u(0) = b$  we have  $u'(0) = 0$ . Finally, it follows directly from (14) that  $u$  satisfies (2). From (12) and a standard bootstrap argument one can show that  $u \in C^1(0, 1)$ . In fact one can show that  $u$  is real analytic away from the origin. Uniqueness can be established using the corresponding minimization problem and convexity arguments.

Phase-space techniques similar to the ones we have used here to prove (1) have been used previously by us. In fact, we have followed the same type of ideas to prove a variational characterization for the speed of propagating fronts of a certain class of nonlinear diffusion equations [1]. We have also employed similar methods to find a variational characterization for the principal eigenvalue associated to nonlinear, one-dimensional, Hamiltonian systems [2]. For a review see [3].

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