J. of Inequal. & Appl., 2000, Vol. 5, pp. 53–61
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Steffensen Pairs and Associated Inequalities

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(Received 22 February 1999; Revised 12 April 1999)

Let $x_1,\dots,x_n$ be positive numbers and $\alpha\geq 2$. It is known that if $\sum_{i=1}^n x_i\leq A$, $\sum_{i=1}^n x_i^\alpha\geq B^\alpha$, then for any k such that $k\geq (A/B)^{1/(\alpha-1)}$, there are k numbers among $x_1,\dots,x_n$ whose sum is bigger than or equal to B. We express this statement saying that a pair of functions $(x^\alpha, x^{1/(\alpha-1)})$ is a Steffensen pair. In this paper we show how to find many Steffensen pairs.

Keywords: Steffensen inequality; Steffensen pair; Convex function; Tchebycheff inequality

1991 Mathematics Subject Classification: Primary 26D15

1. INTRODUCTION

Classical Steffensen's inequality [2] states:

THEOREM A *Let f and g be integrable functions from $[a,b]$ into $\mathbb{R}$ such that f is decreasing, and for every $x\in[a,b]$, $0\leq g(x)\leq 1$. Then*

$$\int_{b-\lambda}^{b} f(x)\,\mathrm{d}x\leq\int_a^b f(x)g(x)\,\mathrm{d}x\leq\int_a^{a+\lambda} f(x)\,\mathrm{d}x,$$

where $\lambda=\int_a^b g(x)\,\mathrm{d}x$.

In [1], the following discrete analogue of Steffensen's inequality was proved:

THEOREM B *Let* $(x_i)_{i=1}^n$ *be a decreasing finite sequence of nonnegative real numbers, and let* $(y_i)_{i=1}^n$ *be a finite sequence of real numbers such that for every* i, $0 \le y_i \le 1$. *Let* $k_1, k_2 \in \{1, \dots, n\}$ *be such that* $k_2 \le y_1 + \cdots + y_n \le k_1$. *Then*

$$\sum_{i=n-k_2+1}^{n} x_i \le \sum_{i=1}^{n} x_i y_i \le \sum_{i=1}^{k_1} x_i.$$

As an immediate consequence of Theorem B, the following proposition was proved in [1]:

PROPOSITION A *Let* $x_1, \dots, x_n$ *be nonnegative real numbers such that the following two conditions are satisfied:* (i) $\sum_{i=1}^n x_i \le A$, (ii) $\sum_{i=1}^n x_i^2 \ge B^2$, *where* A *and* B *are positive real numbers. Let* $k \in \{1, \dots, n\}$ *be such that* $k \ge A/B$. *Then there are* k *numbers among* $x_1, \dots, x_n$ *whose sum is bigger than or equal to* B.

To prove Proposition A we can assume that $B \ge x_1 \ge \cdots \ge x_n$. Set $y_i = x_i/B$. Then $\sum_{i=1}^n y_i \le A/B \le k$. By Theorem B,

$$\sum_{i=1}^{k} x_i \ge \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} \frac{x_i^2}{B} \ge B.$$

Proposition A shows that under certain conditions, a relatively small portion of $x_1, \dots, x_n$ has a relatively large sum. For example, if $\sum_{i=1}^n x_i \le 300$ and $\sum_{i=1}^n x_i^2 \ge 10\,000$, then there are three numbers among $x_1, \dots, x_n$, say x_j, x_k, x_m, such that $x_j + x_k + x_m \ge 100$, i.e. $x_j + x_k + x_m \ge \frac{1}{3}\sum_{i=1}^n x_i$.

We will restate Proposition A using the following definition:

DEFINITION *Let* $\varphi : [c, \infty) \to [0, \infty)$, $c \ge 0$, *and* $\tau : (0, \infty) \to (0, \infty)$ *be two strictly increasing functions. We say that* (φ, τ) *is a Steffensen pair on* $[c, \infty)$ *if the following is satisfied:*

If $x_1, \dots, x_n$ *are real numbers such that* $x_i \ge c$ *for all* i, A *and* B *are positive real numbers, and* (i) $\sum_{i=1}^n x_i \le A$, (ii) $\sum_{i=1}^n \varphi(x_i) \ge \varphi(B)$, *then for any* $k \in \{1, \dots, n\}$ *such that* $k \ge \tau(A/B)$, *there are* k *numbers among* $x_1, \dots, x_n$ *whose sum is bigger than or equal to* B.

Now Proposition A can be reformulated as follows:

PROPOSITION A′ *(x^2, x) is a Steffensen pair on* $[0, \infty)$.

The following more general result was proved in [1]:

PROPOSITION B *If $\alpha \geq 2$, then $(x^\alpha, x^{1/(\alpha-1)})$ is a Steffensen pair on* $[0, \infty)$.

The purpose of this paper is to find more examples of Steffensen pairs.

THEOREM 1 *Let $\psi: [c, \infty) \to [0, \infty)$ where $c \geq 0$, be increasing and convex. Assume that ψ satisfies the following condition:*

$$\psi(xy) \geq \psi(x)g(y) \quad \textit{for all } x \geq c,\ y \geq 1,$$

where $g[1, \infty) \to [0, \infty)$ is strictly increasing. Set $\varphi(x) = x\psi(x)$, $\tau(x) = g^{-1}(x)$, where g^{-1} is the inverse function for g. Then (φ, τ) is a Steffensen pair on $[c, \infty)$.

Example Let $\alpha \geq 2$, $\psi(x) = x^{\alpha-1}$. Then $\psi(xy) = \psi(x)\psi(y)$. Hence $\varphi(x) = x^\alpha$, $\tau(x) = x^{1/(\alpha-1)}$, and we arrive at Proposition B.

THEOREM 2 *Let $f : [0, \infty) \to \mathbb{R}$ be a twice differentiable function on $[0, \infty)$ such that $f'(x) \geq 1$ and $f''(x) \geq 0$ for all $x \geq 0$. Assume that $f(0) = 0$. Then the functions ψ and g from $[1, \infty)$ into $[0, \infty)$ given by*

$$\psi = g = \exp \circ f \circ \ln$$

satisfy the conditions of Theorem 1.

Remark There are many functions satisfying the conditions of Theorem 2. For example, if $f(x) = \sum_{i=1}^{\infty} a_i x^i$ is the sum of a series converging on $[0, \infty)$ and if $a_1 \geq 1$, $a_i \geq 0$ for $i = 2, 3, \ldots$, then $f(x)$ satisfies the conditions of Theorem 2.

PROPOSITION 1 *If $\alpha \geq 1$, then $(x \exp(x^\alpha - 1), (1 + \ln x)^{(1/\alpha)})$ is a Steffensen pair on* $[1, \infty)$.

PROPOSITION 2 *Let a and b be real numbers satisfying the conditions $b > a > 1$ and $\sqrt{ab} \geq e$. Set*

$$\varphi(x) = \begin{cases} (x^{1+\ln b} - x^{1+\ln a})/\ln x, & \textit{if } x > 1, \\ \ln b - \ln a, & \textit{if } x = 1, \end{cases}$$
$$\tau(x) = x^{1/\ln\sqrt{ab}}.$$

Then (φ, τ) is a Steffensen pair on $[1, \infty)$.

Remark Since $\sqrt{ab} \geq e$, $x \geq x^{1/\ln\sqrt{ab}}$ for $x \geq 1$. Therefore it is possible to take $\tau(x) = x$ in Proposition 2.

2. PROOF OF THEOREMS 1, 2 AND PROPOSITIONS 1, 2

Theorem 1 can be deduced easily from Theorem 6.5 in [1]. However the proof of Theorem 6.5 in [1] uses the integration over a general measure space. Because of this reason we give here a direct and elementary proof of Theorem 1 (although it follows closely the ideas of the proof of Theorem 6.5 in [1]).

LEMMA 1 *Assume that* $\psi[c,\infty) \to [0,\infty)$, $c \geq 0$, *is increasing and convex. Set* $\varphi(x) = x\psi(x)$. *Let* $x_1, \ldots, x_r$ *be positive real numbers such that* $x_i \geq c$, $i = 1, \ldots, r$. *Set* $m = \min\{x_1, \ldots, x_r\}$. *Then*

$$\sum_{i=1}^{r} \varphi(x_i) - \psi(m) \sum_{i=1}^{r} x_i \leq \varphi\left(\sum_{i=1}^{r} x_i\right) - \psi(rm) \sum_{i=1}^{r} x_i.$$

Proof Since $\psi(x)$ is convex, it is well known (and easy to prove) that if $x_1 < x_2$ and $\delta \geq 0$, then

$$\psi(x_2) - \psi(x_1) \leq \psi(x_2 + \delta) - \psi(x_1 + \delta).$$

Using this fact we obtain

$$\begin{aligned}
\sum_{i=1}^{r} \varphi(x_i) - \psi(m) \sum_{i=1}^{r} x_i &= \sum_{i=1}^{r} x_i[\psi(x_i) - \psi(m)] \\
&\leq \sum_{i=1}^{r} x_i[\psi(x_i + (r-1)m) - \psi(rm)] \\
&\leq \sum_{i=1}^{r} x_i\left[\psi\left(\sum_{i=1}^{r} x_i\right) - \psi(rm)\right] \\
&= \psi\left(\sum_{i=1}^{r} x_i\right) \sum_{i=1}^{r} x_i - \psi(rm) \sum_{i=1}^{r} x_i \\
&= \varphi\left(\sum_{i=1}^{r} x_i\right) - \psi(rm) \sum_{i=1}^{r} x_i.
\end{aligned}$$

Proof of Theorem 1 Let $x_1, \dots, x_n$ be real numbers such that $x_i \geq c$ for all i. Without loss of generality we can assume that $x_1 \geq \cdots \geq x_n$. Let A and B be positive real numbers, and (i) $\sum_{i=1}^{n} x_i \leq A$, (ii) $\sum_{i=1}^{n} \varphi(x_i) \geq \varphi(B)$. Assume that $k \geq \tau(A/B)$. We will prove that $x_1 + \cdots + x_k \geq B$.

The inequality $k \geq \tau(A/B)$ implies that $g(k) \geq A/B$. Hence

$$A\psi(x_k) = \psi(x_k)\frac{A}{B}B \leq \psi(x_k)g(k)B.$$

Since $\psi(xy) \geq \psi(x)g(y)$, we obtain

$$A\psi(x_k) \leq \psi(kx_k)B. \tag{1}$$

Now we have

$$\begin{aligned}
\varphi(B) \leq \sum_{i=1}^{n} \varphi(x_i) &= \sum_{i=1}^{k} \varphi(x_i) + \sum_{i=k+1}^{n} x_i\,\psi(x_i) \\
&\leq \sum_{i=1}^{k} \varphi(x_i) + \psi(x_k) \sum_{i=k+1}^{n} x_i \\
&= \sum_{i=1}^{k} \varphi(x_k) + \psi(x_k)\left(\sum_{i=1}^{n} x_i - \sum_{i=1}^{k} x_i\right) \\
&\leq \sum_{i=1}^{k} \varphi(x_k) - \psi(x_k)\sum_{i=1}^{k} x_i + A\psi(x_k).
\end{aligned}$$

Lemma 1 implies that

$$\varphi(B) \leq \varphi\left(\sum_{i=1}^{k} x_i\right) - \psi(kx_k)\sum_{i=1}^{k} x_i + A\psi(x_k).$$

By (1), we obtain that

$$\begin{aligned}
\varphi(B) - \varphi\left(\sum_{i=1}^{k} x_i\right) &\leq -\psi(kx_k)\sum_{i=1}^{k} x_i + \psi(kx_k)B \\
&= \psi(kx_k)\left(B - \sum_{i=1}^{k} x_i\right).
\end{aligned}$$

Assume that the conclusion of the theorem is wrong, that is, assume that $B - \sum_{i=1}^{k} x_i > 0$. Then we have

$$\varphi(B) - \varphi\left(\sum_{i=1}^{k} x_i\right) \le \psi\left(\sum_{i=1}^{k} x_i\right)\left(B - \sum_{i=1}^{k} x_i\right)$$
$$= B\psi\left(\sum_{i=1}^{k} x_i\right) - \varphi\left(\sum_{i=1}^{k} x_i\right).$$

This implies that $\varphi(B) \le B\psi(\sum_{i=1}^{k} x_i)$. Hence $\psi(B) \le \psi(\sum_{i=1}^{k} x_i)$. It follows that $B \le \sum_{i=1}^{k} x_i$, which contradicts the above assumption.

Proof of Theorem 2 For $x > 1$,

$$\psi'(x) = \psi(x) f'(\ln x)\frac{1}{x} > 0,$$
$$\psi''(x) = \psi(x) f'(\ln x)\frac{1}{x^2}[f'(\ln x) - 1] + \psi(x) f''(\ln x)\frac{1}{x^2} \ge 0.$$

Therefore ψ is increasing and convex. Let $y \ge 0$, be a fixed number. For $x \ge 0$, set

$$F(x) = f(x+y) - f(x) - f(y).$$

Then

$$F'(x) = f'(x+y) - f'(x) \ge 0, \quad F(0) = 0.$$

Hence $F(x) \ge 0$ for all $x \ge 0$. Thus

$$f(x+y) \ge f(x) + f(y)$$

for all $x, y \ge 0$. Therefore, for $x, y \ge 1$, we obtain

$$\psi(xy) = \exp(f(\ln xy)) = \exp(f(\ln x + \ln y))$$
$$\ge \exp[f(\ln x) + f(\ln y)]$$
$$= \exp(f(\ln x)) \cdot \exp(f(\ln y)) = \psi(x)\psi(y).$$

Proof of Proposition 1 For $\alpha \ge 1$, set $f(x) = e^{\alpha x} - 1$. Then $f(0) = 0$ and for all $x \ge 0, f'(x) \ge 1, f''(x) \ge 0$. Therefore by Theorem 2, functions ψ and g from $[1, \infty)$ into $[0, \infty)$ given by $\psi(x) = g(x) = \exp(e^{\alpha \ln x} - 1) = \exp(x^\alpha - 1)$ satisfy the conditions of Theorem 1. It follows by Theorem 1, that (φ, τ), where $\varphi(x) = x\psi(x) = x\exp(x^\alpha - 1)$ and $\tau(x) = g^{-1}(x) = (1 + \ln x)^{1/\alpha}$ is a Steffensen pair on $[1, \infty)$.

Proof of Proposition 2 We prove this proposition using Theorem 1 and recent results from [3]. Let $b > a > 1$ and $\sqrt{ab} \geq e$. Set

$$h(x) = \begin{cases} (b^x - a^x)/x, & \text{if } x \neq 0, \\ \ln b - \ln a, & \text{if } x = 0. \end{cases}$$

By Proposition 3 in [3], $h'(x) > 0$.

LEMMA 2 *$h''(x) \geq h'(x)$ for $x \geq 0$.*

Proof It is easy to see that

$$h^{(n)}(x) = \int_a^b (\ln t)^n t^{x-1}\, \mathrm{d}t. \tag{2}$$

We will use the following Tchebycheff inequality.

Let $p, q : [a, b] \to \mathbb{R}$ be integrable increasing functions and let $r : [a, b] \to [0, \infty)$ be an integrable function. Then

$$\int_a^b r(t)p(t)\, \mathrm{d}t \int_a^b r(t)q(t)\, \mathrm{d}t \leq \int_a^b r(t)\, \mathrm{d}t \int_a^b r(t)p(t)q(t)\, \mathrm{d}t.$$

Taking $p(t) = q(t) = \ln t$, $r(t) = t^{x-1}$, we get

$$\left(\int_a^b \ln t \cdot t^{x-1}\, \mathrm{d}t\right)^2 \leq \int_a^b t^{x-1}\, \mathrm{d}t \int_a^b (\ln t)^2 t^{x-1}\, \mathrm{d}t.$$

By (2), we obtain that for all x,

$$[h'(x)]^2 \leq h(x)h''(x). \tag{3}$$

By Proposition 4 in [3], for every $y \geq 0$, $F(x) = h(x+y)/h(x)$ is increasing as a function of x. Therefore

$$F'(x) = \frac{h'(x+y)h(x) - h(x+y)h'(x)}{[h(x)]^2} \geq 0.$$

Hence

$$h'(x+y)h(x) - h(x+y)h'(x) \geq 0 \tag{4}$$

for all x and all $y \geq 0$.

Taking $x=0$ in (4), we obtain

$$h'(y)h(0) - h(y)h'(0) \geq 0 \tag{5}$$

for all $y \geq 0$.

$$h(0) = \ln b - \ln a$$

$$h'(0) = \lim_{x\to 0} \frac{1}{x}\left[\frac{b^y - a^y}{x} - (\ln b - \ln a)\right] = \frac{1}{2}[(\ln b)^2 - (\ln a)^2].$$

Hence $h'(0) = h(0)\ln\sqrt{ab}$. Since $\sqrt{ab} \geq e$, we obtain that $h'(0) \geq h(0)$. It follows from (5) that $h'(y) \geq h(y)$ for $y \geq 0$. Therefore, by (3) and (5),

$$h(x)h''(x) \geq [h'(x)]^2 \geq h(x)h'(x)$$

for $x \geq 0$. Thus $h''(x) \geq h'(x)$ for all $x \geq 0$. That proves the lemma.

Set $\psi(x) = h(\ln x)$ for $x \geq 1$. Then $\psi'(x) = h'(\ln x)(1/x) > 0$, $\psi''(x) = (1/x^2)[h''(\ln x) - h'(\ln x)] \geq 0$. Hence $\psi(x)$ is increasing and convex. In addition,

$$\frac{\psi(xy)}{\psi(x)} = \frac{h(\ln(xy))}{h(\ln x)} = \frac{h(\ln x + \ln y)}{h(\ln x)}.$$

By Proposition 5 in [3], we have that for $x, y \geq 0$,

$$\frac{h(x+y)}{h(x)} \geq (\sqrt{ab})^y.$$

Therefore, for $x, y \geq 1$,

$$\frac{\psi(xy)}{\psi(x)} \geq (\sqrt{ab})^{\ln y}.$$

Set $g(x) = (\sqrt{ab})^{\ln x}$. Then $g^{-1}(x) = x^{1/\ln\sqrt{ab}}$.

By Theorem 1 (φ, τ), where

$$\varphi(x) = x\psi(x)$$

$$= \begin{cases} x(b^{\ln x} - a^{\ln x})/\ln x = (x^{1+\ln b} - x^{1+\ln a})/\ln x, & \text{if } x > 1, \\ \ln b - \ln a, & \text{if } x = 1, \end{cases}$$

$$\tau(x) = x^{1/\ln\sqrt{ab}},$$

is a Steffensen pair.

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