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Spherical Derivative of Meromorphic Function with Image of Finite Spherical Area

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Let Ω be a domain in the complex plane $\mathbf{C}$ with the Poincaré metric $P_\Omega(z)|\mathrm{d}z|$ which is $|\mathrm{d}z|/(1-|z|^2)$ if Ω is the open unit disk. Suppose that the Riemann sphere $\mathbf{C}\cup\{\infty\}$ of radius 1/2, so that it has the area π and let $0<\beta<\pi$. Let $\alpha_{\Omega,\beta}(z)$, $z\in\Omega$, be the supremum of the spherical derivative $|f'(z)|/(1+|f(z)|^2)$ of f meromorphic in Ω such that the spherical area of the image $f(\Omega)\subset\mathbf{C}\cup\{\infty\}$ is not greater than β. Then

$$\alpha_{\Omega,\beta}(z)\le\sqrt{\frac{\beta}{\pi-\beta}}P_\Omega(z),\quad z\in\Omega.$$

The equality holds if and only if Ω is simply connected.

Keywords: Chordal distance; Spherical derivative; Spherical area; Poincaré metric

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1. INTRODUCTION

The complex plane $\mathbf{C}=\mathbf{R}^2$, together with the point ∞ at infinity, is identified with the Riemann sphere $\mathbf{C}^*$ of radius 1/2 touching $\mathbf{C}$ from above at the origin with the aid of the stereographic projection viewed from the north pole of $\mathbf{C}^*$. The sphere $\mathbf{C}^*=\mathbf{C}\cup\{\infty\}$ is the metric subspace of the Euclidean space $\mathbf{R}^3$, so that it has the distance $X(z,w)$

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which can be expressed by

$$X(z,w)=\frac{|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}},\quad z,w\in\mathbf{C};$$

$$X(z,\infty)=X(\infty,z)=\frac{1}{\sqrt{1+|z|^2}},\quad z\in\mathbf{C};\qquad X(\infty,\infty)=0.$$

The spherical area of a set $E\subset\mathbf{C}^*$ is given by the double integral

$$A(E)=\iint_{E\setminus\{\infty\}}\frac{\mathrm{d}x\,\mathrm{d}y}{(1+|z|^2)^2},\quad z=x+\mathrm{i}y;$$

here E is identified with its projection. For example, the spherical cap $E(a,r)=\{z\in\mathbf{C}^*; X(z,a)<r\}$ of center $a\in\mathbf{C}^*$ and radius, r, $0<r\le 1$, has the area πr^2, so that $A(\mathbf{C})=A(E(0,1))=\pi$. Actually, $A(E(a,r))=A(E(0,r))$.

Let $\mathcal{M}(\Omega)$ be the family of all the meromorphic functions in a domain $\Omega\subset\mathbf{C}$; the constant function ∞ is regarded as a member of $\mathcal{M}(\Omega)$. The spherical derivative of $f\in\mathcal{M}(\Omega)$ at $z\in\Omega$ is defined by

$$f^{\#}(z)=\lim_{|w-z|\to 0}\frac{X(f(w),f(z))}{|w-z|};$$

hence $f^{\#}(z)=|f'(z)|/(1+|f(z)|^2)$ if $f(z)\ne\infty$ and $f^{\#}(z)=|(1/f)'(z)|$ if $f(z)=\infty$. Note that $(\infty)^{\#}(z)\equiv 0$. The set of $w\in\mathbf{C}^*$ assumed by $f\in\mathcal{M}(\Omega)$ at least once in Ω is denoted by $f(\Omega)$; hence $f(\Omega)\subset\mathbf{C}^*$ and $w\in f(\Omega)$ if and only if $w=f(z)$ for a $z\in\Omega$. For a constant β, $0<\beta<\pi$, we let $\mathcal{F}(\Omega,\beta)$ be the set of all $f\in\mathcal{M}(\Omega)$ such that $A(f(\Omega))\le\beta$. Note that $A(f(\Omega))\le\iint_\Omega f^{\#}(z)^2\,\mathrm{d}x\,\mathrm{d}y$; the right-hand side integral may possibly be $+\infty$.

We suppose that Ω is hyperbolic, namely, $\mathbf{C}\setminus\Omega$ contains at least two points. Let ϕ be a universal covering projection from $D=\{w;\ |w|<1\}$ onto Ω, $\phi\in\mathrm{Proj}(\Omega)$ in notation; ϕ is holomorphic and ϕ' is zero-free. The Poincaré density P_Ω is then the function in Ω defined by

$$P_\Omega(z)=\frac{1}{(1-|w|^2)|\phi'(w)|},\quad z\in\Omega,$$

where $z=\phi(w)$; the choice of $\phi \in \mathrm{Proj}(\Omega)$ and w is immaterial as far as $z=\phi(w)$ is satisfied.

We begin with

THEOREM 1 *For each $f \in \mathcal{F}(\Omega, \beta)$ for hyperbolic Ω, the estimate holds:*

$$f^{\#}(z) \leq \sqrt{\frac{\beta}{\pi-\beta}} P_{\Omega}(z) \tag{1.1}$$

at each $z \in \Omega$. If the equality holds in (1.1) *at a point $z \in \Omega$, then f maps Ω univalently onto a spherical cap. Conversely if Ω is mapped by a meromorphic function $f \in \mathcal{F}(\Omega, \beta)$ univalently onto a spherical cap, then there exists exactly one point $z \in \Omega$ where the equality holds in* (1.1).

THEOREM 2 *The family $\mathcal{F}(\Omega, \beta)$ for hyperbolic Ω is compact. Namely, given $f_n \in \mathcal{F}(\Omega, \beta), n=1,2,\ldots$, we have a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ and an $f \in \mathcal{F}(\Omega, \beta)$ such that*

$$\max_{z \in E} X(f_{n_j}(z), f(z)) \to 0 \quad \text{as } n_j \to \infty \tag{1.2}$$

for each compact set E (in $\mathbf{C}$) comprised in Ω.

Set

$$c_{\Omega,\beta}(z) = \sup\{f^{\#}(z); f \in \mathcal{F}(\Omega, \beta)\}, \quad z \in \Omega.$$

It then follows from Theorem 2 that the supremum is the maximum; $f^{\#}(z)=c_{\Omega,\beta}(z)$ for an $f \in \mathcal{F}(\Omega, \beta)$. For this extremal function we set

$$g(w) = \begin{cases} \dfrac{f(w)-f(z)}{e^{i \arg f'(z)}(1+\overline{f(z)}f(w))}, & \text{if } f(z) \neq \infty; \\ \dfrac{1}{e^{i \arg(1/f)'(z)} f(w)}, & \text{if } f(z) = \infty. \end{cases}$$

Then $g \in \mathcal{F}(\Omega, \beta)$ with $g(z)=0$ and $0 \leq g'(z) = c_{\Omega,\beta}(z)$. Again, g is extremal.

THEOREM 3 *If Ω is hyperbolic, then*

$$c_{\Omega,\beta}(z) \leq \sqrt{\frac{\beta}{\pi-\beta}} P_{\Omega}(z) \tag{1.3}$$

at each $z \in \Omega$. *If the equality holds in* (1.3) *at a point* $z \in \Omega$, *then* Ω *is simply connected. If* Ω *is simply connected, then the equality holds in* (1.3) *everywhere in* Ω.

We omit the detailed proof of Theorem 3 because we have only to apply Theorem 2 to an extremal function.

2. PROOF OF THEOREM 1

LEMMA 1 *For* $f \in \mathcal{F}(D, \beta)$ *the inequality holds:*

$$f^{\#}(0)^2 \leq \frac{\beta}{\pi - \beta}. \tag{2.1}$$

The equality holds in (2.1) *if and only if*

$$f(z) \equiv \frac{az + b}{1 - \bar{b}az}, \quad z \in D, \tag{2.2}$$

where $a \in \mathbf{C}$ *and* $b \in \mathbf{C}^*$ *are constants with*

$$|a| = \sqrt{\frac{\beta}{\pi - \beta}}.$$

Read $f(z) = -1/(az)$ *in case* $b = \infty$.

This lemma is Dufresnoy's, the case $r_0 = 1$ in [2, Lemma I, (2)]. The equality condition in the present "if and only if" form is obtained in the similar manner as in [1, pp. 219–220]. Note that Dufresnoy adopted the unit sphere of center at the origin in $\mathbf{R}^3$ as the Riemann sphere, so that we need obvious changes.

Proof of Theorem 1 We choose $\phi \in \mathrm{Proj}(\Omega)$ with $z = \phi(0)$ and we observe that $f \circ \phi \in \mathcal{F}(D, \beta)$. Since

$$(f \circ \phi)^{\#}(0) = f^{\#}(z)|\phi'(0)| = f^{\#}(z)/P_{\Omega}(z),$$

the inequality (1.1) is a consequence of (2.1). The equality holds in (1.1) at z if and only if

$$(f\circ\phi)(w)=\frac{aw+b}{1-\bar{b}aw},\quad w\in D,$$

where $|a|=\sqrt{\beta/(\pi-\beta)}$. Hence f is univalent in Ω, so that ϕ is univalent in D. The image $f(\Omega)$ is $\{(aw+b)/(1-\bar{b}aw);\ w\in D\}$ which is the image of the cap $E(0,|a|/\sqrt{(1+|a|^2)})$ by the rotation of $\mathbf{C}^*$: $T(\zeta)=(\zeta+b)/(1-\bar{b}\zeta)$, so that $f(\Omega)$ is a spherical cap.

Suppose that $f\in\mathcal{F}(\Omega,\beta)$ maps Ω univalently onto a spherical cap $E(a,\sqrt{\beta/\pi})$, $a\in\mathbf{C}^*$ and set $\rho=\sqrt{\beta/(\pi-\beta)}$. Then

$$\phi(w)=f^{-1}\left(\frac{\rho w+a}{1-\bar{a}\rho w}\right)$$

is in $\mathrm{Proj}(\Omega)$. Since

$$\frac{f^{\#}(\phi(w))}{P_\Omega(\phi(w))}=\frac{\rho(1-|w|^2)}{1+\rho^2|w|^2},$$

it follows that $f^{\#}(\phi(w))=\rho P_\Omega(\phi(w))$ if and only if $w=0$. Hence the equality holds in (1.1) at exactly one point $\phi(0)$, the inverse of a by f.

If $z\neq w$, then

$$X(z,w)\leq\arctan\left|\frac{z-w}{1+\bar{z}w}\right|=\int_\Gamma \mathrm{d}X(\zeta),$$

where $\mathrm{d}X(\zeta)=|\mathrm{d}\zeta|/(1+|\zeta|^2)$ and Γ is the projection of the shorter of the great circle passing through, and bisected by z,w; in case $z\bar{w}=-1$ we have many Γ. Here $0<\arctan\rho\leq\pi/2$ for $0<\rho\leq+\infty$.

Suppose that $\beta=A(\Omega)<\pi$ for a domain $\Omega\subset\mathbf{C}$. We then have

$$\mathrm{d}X(z)\leq\sqrt{\frac{\beta}{\pi-\beta}}P_\Omega(z)|\mathrm{d}z|,\quad z\in\Omega.$$

More precisely,

$$\frac{1}{1+|z|^2}\leq\sqrt{\frac{A(\Omega)}{\pi-A(\Omega)}}P_\Omega(z),\quad z\in\Omega. \tag{2.3}$$

The equality holds in (2.3) at a point $z \in \Omega$ if and only if Ω is a spherical cap

$$\left\{\frac{a\zeta + z}{1 - \bar{z}a\zeta};\ \zeta \in D\right\},$$

where $a \in \mathbf{C}$ and $|a| = \sqrt{A(\Omega)/(\pi - A(\Omega))}$. For the proof we have only to follow that of Theorem 1 with $f(\zeta) \equiv \zeta, f \in \mathcal{F}(\Omega, \beta)$.

3. PROOF OF THEOREM 2

We begin with the case $\Omega = D$. Since $f^{\#}$ of $f \in \mathcal{F}(D, \beta)$ is bounded by

$$\sqrt{\frac{\beta}{\pi - \beta}} \cdot \frac{1}{1 - r^2}$$

on $\{|z| < r\}$, $0 < r < 1$, it follows that $\mathcal{F}(D, \beta)$ is equicontinuous in X on $\{|z| < r\}$. Hence $\mathcal{F}(D, \beta)$ is normal in that sense that given $\{f_n\} \subset \mathcal{F}(D, \beta)$, we have a subsequence $\{f_{n_j}\} \subset \{f_n\}$ and $f \in \mathcal{M}(D)$ such that (1.2) holds for each compact set E comprised in D. To prove that $f \in \mathcal{F}(D, \beta)$ we may suppose that f is nonconstant.

For simplicity we suppose that

$$\max_{z \in E} X(f_n(z), f(z)) \to 0 \quad \text{as } n \to \infty$$

for $f_n \in \mathcal{F}(D, \beta)$, $n = 1, 2, \ldots$ We shall then prove that for each $b \in D$ we have $r = r(b) > 0$ and a natural number $N = N(b)$ such that $f(\Delta) \subset f_n(\Omega)$ for all $n > N$, where

$$\Delta = \{z;\ |z - b| < r\} \subset D. \tag{3.1}$$

Then for each compact set $K \subset D$ we have n such that $f(K) \subset f_n(\Omega)$, so that $A(f(K)) \leq \beta$, whence $f \in \mathcal{F}(\Omega, \beta)$.

We first suppose that $f(b) \neq \infty$. Then we have $r_1 > 0$ and a constant $M_1 > 0$ such that f is pole-free and bounded, $|f| < M_1$, on $\Delta_1 = \{z; |z - b| \leq r_1\} \subset D$. We then find a constant $M_2 \geq M_1$ and a natural number N_1 such that f_n is pole-free and bounded, $|f_n| < M_2$, on Δ_1 for

$n \geq N_1$. Hence

$$\max_{z \in \Delta_1} |f_n(z) - f(z)| \to 0 \quad \text{as } n \to \infty. \tag{3.2}$$

Consequently, for Δ of (3.1) with $0 < r < r_1$, we have

$$\sup_{z \in \Delta} |f_n'(z) - f'(z)| \to 0 \quad \text{as } n \to \infty. \tag{3.3}$$

To prove that the present Δ is the requested we suppose that there exists $q \in \Delta$ such that $p = f(q) \notin f_{n_j}(\Omega)$ for infinitely many n_j, $n_1 < n_2 < \cdots$ Choose $R > 0$ such that

$$\{z;\ |z - q| \leq R\} \subset \Delta \quad \text{and} \quad p \notin f(\{z;\ 0 < |z - q| \leq R\}),$$

and set

$$c = \{z;\ |z - q| = R\}.$$

The argument principle then shows that

$$1 \leq \frac{1}{2\pi \mathrm{i}} \int_c \frac{f'(\zeta)}{f(\zeta) - p} \mathrm{d}\zeta.$$

The right-hand side integral is, with the aid of (3.2) and (3.3), the limit

$$\lim_{n_j \to \infty} \frac{1}{2\pi \mathrm{i}} \int_c \frac{f_{n_j}'(\zeta)}{f_{n_j}(\zeta) - f_{n_j}(q)} \mathrm{d}\zeta = 0.$$

This is a contradiction.

In the case $f(b) = \infty$ we consider $\{1/f_n\} \subset \mathcal{F}(\Omega, \beta)$ with $1/f \in \mathcal{M}(\Omega)$ and arrive at a contradiction again.

For general Ω we fix $\phi \in \mathrm{Proj}(\Omega)$. Then, for each compact set E in Ω we may find a compact set $E_1 \subset D$ such that $\phi(E_1) = E$. Furthermore, ϕ is automorphic with respect to the universal covering transformation group $\mathcal{G} : \phi = \phi \circ T,\ T \in \mathcal{G}$. Since $f_n \circ \phi \in \mathcal{F}(D, \beta)$, we have a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ and $g \in \mathcal{F}(D, \beta)$ such that $f_{n_j} \circ \phi$ converges to g on each compact set in D. Since g is then automorphic with respect to $\mathcal{G}$, we have $f \in \mathcal{M}(\Omega)$ such that $f \circ \phi = g$ and this f is the requested.

4. CONFORMAL INVARIANT $c_{\Omega,\beta}(z)$

Let Σ be another domain in $\mathbf{C}$ and let f be holomorphic in Ω with $f(\Omega) \subset \Sigma$. Then

$$c_{\Sigma,\beta}(f(z))|f'(z)| \leq c_{\Omega,\beta}(z), \quad z \in \Omega.$$

In particular, if f is univalent and $f(\Omega) = \Sigma$, then

$$c_{\Sigma,\beta}(f(z))|f'(z)| = c_{\Omega,\beta}(z), \quad z \in \Omega,$$

so that $c_{\Omega,\beta}(z)|dz|$ is conformally invariant.

Let $\mathcal{B}_z(\Omega)$ be the family of all f holomorphic, bounded, $|f| < 1$, in Ω, and further, $f(z) = 0$, $z \in \Omega$. Then

$$\gamma_\Omega(z) = \sup\{|f'(z)|; f \in \mathcal{B}_z(\Omega)\}$$

is called the analytic capacity of Ω at z. Then $\gamma_\Omega(z)$ is the maximum $\gamma_\Omega(z) = |f'(z)| = f'(z)$ for a unique $f \in \mathcal{B}_z(\Omega)$ called the Ahlfors function of Ω at z. See [3, p. 110]. Since $\mathcal{B}_z(\Omega) \subset \mathcal{F}(\Omega, \pi/2)$, it follows that

$$\gamma_\Omega(z) \leq c_{\Omega,\pi/2}(z), \quad z \in \Omega.$$

On the other hand, it follows from (1.3) that

$$c_{\Omega,\pi/2}(z) \leq P_\Omega(z), \quad z \in \Omega.$$

If $f \in \mathcal{F}(\Omega, \beta)$, then $\mathbf{C}^* \setminus f(\Omega)$ is of positive spherical area, so that this set is of positive capacity. Hence f is of uniformly bounded characteristic in Ω; see [5, Theorem 1] and also [4,6]. Suppose that each function of uniformly bounded characteristic in Ω is constant. Then $f \in \mathcal{F}(\Omega, \beta)$ is a constant.

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