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Argument Estimates of Meromorphically Multivalent Functions

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The object of the present paper is to obtain some argument properties of meromorphically multivalent functions in the punctured open unit disk. We also derive the integral preserving properties in a sector.

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1. INTRODUCTION

Let $\mathcal{U}=\{z\in\mathbb{C}\colon |z|<1\}$. For f and g which are analytic in $\mathcal{U}$, we say that f is subordinate to g, written $f\prec g$ or $f(z)\prec g(z)$, if there exists a Schwarz function w in $\mathcal{U}$ such that $f(z)=g(w(z))$.

Let Σ_p denote the class of all meromorphic functions of the form

$$f(z)=\frac{1}{z^p}+\frac{a_0}{z^{p-1}}+\cdots+a_{k+p-1}z^k+\cdots\quad (p\in\mathbb{N}=\{1,2,\ldots\})$$

which are analytic in the annulus $\mathcal{D}=\{z\colon 0<|z|<1\}$. We denote by $\Sigma_p^*(\beta)$ the subclass of Σ_p consisting of all functions which are meromorphically starlike of order β in $\mathcal{U}$.

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The Hadamard product or convolution of two functions f and g in Σ_p will be denoted by $f * g$.

Let

$$D^{n+p-1}f(z) = \frac{1}{z^p(1-z)^{n+p}} * f(z) \quad (z \in \mathcal{D}) \tag{1.1}$$

or, equivalently,

$$\begin{aligned} D^{n+p-1}f(z) &= \frac{1}{z^p}\left(\frac{z^{n+2p-1}f(z)}{(n+p-1)!}\right)^{(n+p-1)} \\ &= \frac{1}{z^p} + (n+p)a_0\frac{1}{z^{p-1}} + \frac{(n+p+1)(n+p)}{2!}a_1\frac{1}{z^{p-2}} \\ &\quad + \cdots + \frac{(n+k+2p-1)\cdots(n+p)}{(k+p)!}a_{k+p-1}z^k \\ &\quad + \cdots \quad (z \in \mathcal{D}), \end{aligned}$$

where n is any integer greater than $-p$.

For various interesting developments involving the operators D^{n+p-1} for functions belonging to Σ_p, the reader may be referred to the recent works of author [1], Uralegaddi and Path [7], and others [8,9].

Let

$$\Sigma_p^*[n; A, B] = \left\{f \in \Sigma_p : -\frac{z(D^{n+p-1}f(z))'(z)}{D^{n+p-1}f(z)} \prec p\frac{1+Az}{1+Bz},\ z \in \mathcal{U}\right\}, \tag{1.2}$$

where $-1 \leq B < A \leq 1$. In particular, we note that $\Sigma_p^*[-p+1; 1, -1]$ is the well known class of meromorphically p-valent starlike functions. From (1.2), we observe [6] that a function f is in $\Sigma_p^*[n; A, B]$ if and only if

$$\left|\frac{z(D^{n+p-1}f(z))'(z)}{D^{n+p-1}f(z)} + \frac{p(1-AB)}{1-B^2}\right| < \frac{p(A-B)}{1-B^2} \quad (-1 < B < A \leq 1;\ z \in \mathcal{U}). \tag{1.3}$$

The object of the present paper is to give some argument estimates of meromorphically multivalent functions belonging to Σ_p and the integral preserving properties in connection with the differential operators D^{n+p-1} defined by (1.1).

2. MAIN RESULTS

To establish our main results, we need the following lemmas.

LEMMA 2.1 [2] *Let h be convex univalent in $\mathcal{U}$ with $h(0)=1$ and* $\mathrm{Re}(\beta h(z)+\gamma)>0$ $(\beta,\gamma\in\mathbb{C})$. *If q is analytic in $\mathcal{U}$ with $q(0)=1$, then*

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}\prec h(z)\quad (z\in\mathcal{U})$$

implies

$$q(z)\prec h(z)\quad (z\in\mathcal{U}).$$

LEMMA 2.2 [4] *Let h be convex univalent in $\mathcal{U}$ and $\lambda(z)$ be analytic in $\mathcal{U}$ with* $\mathrm{Re}\,\lambda(z)\geq 0$. *If q is analytic in $\mathcal{U}$ and $q(0)=h(0)$, then*

$$q(z)+\lambda(z)zq'(z)\prec h(z)\quad (z\in\mathcal{U})$$

implies

$$q(z)\prec h(z)\quad (z\in\mathcal{U}).$$

LEMMA 2.3 [5] *Let q be analytic in $\mathcal{U}$ with $q(0)=1$ and $q(z)\neq 0$ in $\mathcal{U}$. Suppose that there exists a point $z_0\in\mathcal{U}$ such that*

$$|\arg q(z)|<\frac{\pi}{2}\alpha\quad \textit{for } |z|<|z_0| \tag{2.1}$$

and

$$|\arg q(z_0)|=\frac{\pi}{2}\alpha\quad (0<\alpha\leq 1). \tag{2.2}$$

Then we have

$$\frac{z_0q'(z_0)}{q(z_0)}=ik\alpha, \tag{2.3}$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \textit{when} \ \arg q(z_0) = \frac{\pi}{2}\alpha, \tag{2.4}$$

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \textit{when} \ \arg q(z_0) = -\frac{\pi}{2}\alpha \tag{2.5}$$

and

$$q(z_0)^{1/\alpha} = \pm ia \quad (a > 0). \tag{2.6}$$

At first, with the help of Lemma 2.1, we obtain the following

PROPOSITION 2.1 *Let h be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and* $\operatorname{Re} h$ *be bounded in $\mathcal{U}$. If $f \in \Sigma_p$ satisfies the condition*

$$-\frac{z(D^{n+p}f(z))'}{pD^{n+p}f(z)} \prec h(z) \quad (z \in \mathcal{U}),$$

then

$$-\frac{z(D^{n+p-1}f(z))'}{pD^{n+p-1}f(z)} \prec h(z) \quad (z \in \mathcal{U})$$

for $\max_{z \in \mathcal{U}} \operatorname{Re} h(z) < (n + 2p)/p$ *(provided $D^{n+p-1}f(z) \neq 0$ in $\mathcal{U}$).*

Proof Let

$$q(z) = -\frac{z(D^{n+p-1}f(z))'}{pD^{n+p-1}f(z)}.$$

By using the equation

$$z(D^{n+p-1}f(z))' = (n+p)D^{n+p}f(z) - (n+2p)D^{n+p-1}f(z), \tag{2.7}$$

we get

$$q(z) - \frac{n+2p}{p} = -\frac{(n+p)D^{n+1}f(z)}{pD^{n+p-1}f(z)}. \tag{2.8}$$

Taking logarithmic derivatives in both sides of (2.8) and multiplying by z, we have

$$\frac{zq'(z)}{-pq(z)+n+2p}+q(z)=-\frac{z(D^{n+p}f(z))'}{pD^{n+p}f(z)}\prec h(z) \quad (z\in\mathcal{U}).$$

From Lemma 2.1, it follows that $q(z)\prec h(z)$ for $\mathrm{Re}(-h(z)+(n+2p)/p)>0$ $(z\in\mathcal{U})$, which means

$$-\frac{z(D^{n+p-1}f(z))'}{pD^{n+p-1}f(z)}\prec h(z) \quad (z\in\mathcal{U})$$

for $\max_{z\in\mathcal{U}}\mathrm{Re}\,h(z)<(n+2p)/p$.

PROPOSITION 2.2 *Let h be convex univalent in $\mathcal{U}$ with $h(0)=1$ and* $\mathrm{Re}\,h$ *be bounded in $\mathcal{U}$. Let F be the integral operator defined by*

$$F(z)=\frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)\,\mathrm{d}t \quad (c>0). \tag{2.9}$$

If $f\in\Sigma_p$ satisfies the condition

$$-\frac{z(D^{n+p-1}f(z))'}{pD^{n+p-1}f(z)}\prec h(z) \quad (z\in\mathcal{U}),$$

then

$$-\frac{z(D^{n+p-1}F(z))'}{pD^{n+p-1}F(z)}\prec h(z) \quad (z\in\mathcal{U})$$

for $\max_{z\in\mathcal{U}}\mathrm{Re}\,h(z)<(c+p)/p$ *(provided $D^{n+p-1}F(z)\neq 0$ in $\mathcal{U}$).*

Proof From (2.9), we have

$$z(D^{n+p-1}F(z))'=cD^{n+p-1}f(z)-(c+p)D^{n+p-1}F(z). \tag{2.10}$$

Let

$$p(z)=-\frac{z(D^{n+p-1}F(z))'}{pD^{n+p-1}F(z)}.$$

Then, by using (2.10), we get

$$q(z)-(c+p)=-c\frac{D^{n+p-1}f(z)}{D^{n+p-1}F(z)}. \tag{2.11}$$

Taking logarithmic derivatives in both sides of (2.11) and multiplying by z, we have

$$\frac{zq'(z)}{-pq(z)+(c+p)}+q(z)=-\frac{z(D^{n+p-1}f(z))'}{pD^{n+p-1}f(z)}\prec h(z)\quad (z\in\mathcal{U}).$$

Therefore, by Lemma 2.1, we have

$$-\frac{z(D^{n+p-1}F(z))'}{pD^{n+p-1}F(z)}\prec h(z)\quad (z\in\mathcal{U})$$

for $\max_{z\in\mathcal{U}}\operatorname{Re}h(z)<(c+p)/p$ (provided $D^{n+p-1}F(z)\neq 0$ in $\mathcal{U}$).

Remark Taking $p=1$ and $h(z)=(1+z)/(1-z)$ in Propositions 2.1 and 2.2, we have the results obtained by Ganigi and Uralegaddi [3].

Applying Lemmas 2.2, 2.3 and Proposition 2.1, we now derive

THEOREM 2.1 *Let $f\in\Sigma_p$. Choose an integer n such that*

$$n\geq\frac{p(1+A)}{1+B}-2p,$$

where $-1<B<A\leq 1$ and $p\in\mathbb{N}$. If

$$\left|\arg\left(-\frac{z(D^{n+p}f(z))'}{D^{n+p}g(z)}-\gamma\right)\right|<\frac{\pi}{2}\delta\quad (0\leq\gamma<p;\ 0<\delta\leq 1)$$

for some $g\in\Sigma_p^*[n+1;A,B]$, then

$$\left|\arg\left(-\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)}-\gamma\right)\right|<\frac{\pi}{2}\alpha,$$

where α $(0<\alpha\leq 1)$ *is the solution of the equation*

$$\delta=\alpha+\frac{2}{\pi}\tan^{-1}\Big((\alpha\sin\pi/2(1-t(A,B))) \Big/(((n+2p)(1-B)+A-1)/(1-B)) + \alpha\cos\pi/2(1-t(A,B))\Big) \tag{2.12}$$

when

$$t(A,B)=\frac{2}{\pi}\sin^{-1}\left(\frac{p(A-B)}{(n+2p)(1-B^2)-p(1-AB)}\right). \tag{2.13}$$

Proof Let

$$q(z)=-\frac{1}{p-\gamma}\left(\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)}+\gamma\right).$$

By (2.7), we have

$$(p-\gamma)zq'(z)D^{n+p-1}g(z)+(1-\gamma)q(z)z(D^{n+p-1}g(z))' - (n+2p)z(D^{n+p-1}f(z))' = -(n+p)z(D^{n+p}f(z))'-\gamma z(D^{n+p-1}g(z))'(z). \tag{2.14}$$

Dividing (2.14) by $D^{n+p-1}g(z)$ and simplifying, we get

$$q(z)+\frac{zq'(z)}{-r(z)+n+2p}=-\frac{1}{p-\gamma}\left(\frac{z(D^{n+p}f(z))'}{D^{n+p}g(z)}+\gamma\right), \tag{2.15}$$

where

$$r(z)=-\frac{z(D^{n+p-1}g(z))'}{D^{n+p-1}g(z)}.$$

Since $g\in\Sigma_p^*[n+1;A,B]$, from Proposition 2.1, we have

$$r(z)\prec p\frac{1+Az}{1+Bz}.$$

Using (1.3), we have

$$-r(z)+n+2p=\rho e^{i(\pi/2)}\phi,$$

where

$$\frac{(n+2p)(1+B)-(1+A)}{1+B}<\rho<\frac{(n+2p)(1-B)+A-1}{1-B},$$
$$-t(A,B)<\phi<t(A,B)$$

when $t(A,B)$ is given by (2.13). Let h be a function which maps $\mathcal{U}$ onto the angular domain $\{w\colon |\arg w|<(\pi/2)\delta\}$ with $h(0)=1$. Applying Lemma 2.2 for this h with $\lambda(z)=1/(-r(z)+n+2p)$, we see that $\operatorname{Re} q(z)>0$ in $\mathcal{U}$ and hence $q(z)\neq 0$ in $\mathcal{U}$.

If there exists a point $z_0\in\mathcal{U}$ such that the conditions (2.1) and (2.2) are satisfied, then (by Lemma 2.3) we obtain (2.3) under the restrictions (2.4), (2.5) and (2.6).

At first, suppose that $q(z_0)^{1/\alpha}=ia(a>0)$. Then we obtain

$$\begin{aligned}
&\arg\left[-\frac{1}{p-\gamma}\left(\frac{z_0(D^{n+p}f(z_0)'}{D^{n+p}g(z_0)}+\gamma\right)\right]\\
&=\arg\left(q(z_0)+\frac{z_0q'(z_0)}{-r(z_0)+n+2p}\right)\\
&=\frac{\pi}{2}\alpha+\arg(1+i\alpha k(\rho e^{i(\pi/2)\phi})^{-1})\\
&=\frac{\pi}{2}\alpha+\tan^{-1}\left(\frac{\eta k\sin\pi/2(1-\phi)}{\rho+\alpha k\cos\pi/2(1-\phi)}\right)\\
&\geq\frac{\pi}{2}\alpha+\tan^{-1}\left((\alpha\sin\pi/2(1-t(A,B)))\Big/(((n+2p)(1-B)+A-1)\right.\\
&\quad\left./(1-B)+\alpha\cos\pi/2(1-t(A,B)))\right)\\
&=\frac{\pi}{2}\delta,
\end{aligned}$$

where δ and $t(A,B)$ are given by (2.12) and (2.13), respectively. This is a contradiction to the assumption of our theorem.

Next, suppose that $p(z_0)^{1/\alpha} = -ia$ $(a > 0)$. Applying the same method as the above, we have

$$\begin{aligned}
&\arg\left[-\frac{1}{p-\gamma}\left(\frac{z_0(D^{n+p}f(z_0))'}{D^{n+p}g(z_0)}+\gamma\right)\right] \\
&\le -\frac{\pi}{2}\alpha - \tan^{-1}\Big((\alpha \sin \pi/2(1-t(A,B))) \\
&\quad \Big/(((n+2p)(1-B)+A-1) \\
&\quad /(1-B)+\alpha\cos\pi/2(1-t(A,B)))\Big) \\
&= -\frac{\pi}{2}\delta,
\end{aligned}$$

where δ and $t(A, B)$ are given by (2.12) and (2.13), respectively, which contradicts the assumption. Therefore we complete the proof of our theorem.

Letting $A = 1$, $B = 0$ and $\delta = 1$ in Theorem 2.1, we have

COROLLARY 2.1 *Let $f \in \Sigma$. If*

$$-\mathrm{Re}\left\{\frac{z(D^{n+p}f(z))'}{D^{n+p}g(z)}\right\} > \gamma \quad (0 \le \gamma < p)$$

for some $g \in \Sigma_p$ satisfying the condition

$$\left|\frac{z(D^{n+p}g(z))'}{D^{n+p}g(z)} + p\right| < p,$$

then

$$-\mathrm{Re}\left\{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)}\right\} > \gamma.$$

Taking $A = 1$, $B = 0$ and $g(z) = 1/z^p$ in Theorem 2.1, we have

COROLLARY 2.2 *Let $f \in \Sigma_p$. If*

$$|\arg[-z^{p+1}(D^{n+p}f(z))' - \gamma]| < \frac{\pi}{2}\delta \quad (0 \le \gamma < p;\ 0 < \delta \le 1),$$

then

$$|\arg[-z^{p+1}(D^{n+p-1}f(z))' - \gamma]| < \frac{\pi}{2}\delta.$$

Making $n = 0, p = 1$ and $\delta = 1$ in Corollary 2.2, we have

COROLLARY 2.3 *Let* $f \in \Sigma_1$. *If*

$$-\mathrm{Re}\{z^2(zf''(z) + 3f'(z))\} > \gamma \quad (0 \leq \gamma < 1),$$

then

$$-\mathrm{Re}\{z^2 f'(z)\} > \gamma.$$

By the same techniques as in the proof of Theorem 2.1, we obtain

THEOREM 2.2 *Let* $f \in \Sigma$. *Choose an integer n such that*

$$n \geq \frac{p(1+A)}{1+B} - 2p,$$

where $-1 < B < A \leq 1$ *and* $p \in \mathbb{N}$. *If*

$$\left|\arg\left(\frac{z(D^{n+p}f(z))'}{(D^{n+p}g(z))} + \gamma\right)\right| < \frac{\pi}{2}\delta \quad (\gamma > p,\ 0 < \delta \leq 1)$$

for some $g \in \Sigma_p^*[n+1; A, B]$, *then*

$$\left|\arg\left(\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)} + \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where α $(0 < \alpha \leq 1)$ *is the solution of the equation given by* (2.12).

Next, we prove

THEOREM 2.3 *Let* $f \in \Sigma_p$ *and choose a positive number c such that*

$$c \geq \frac{1+A}{1+B} - p,$$

where $-1 < B < A \leq 1$ *and* $p \in \mathbb{N}$. *If*

$$\left|\arg\left(-\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta \quad (0 \leq \gamma < p;\ 0 < \delta \leq 1)$$

for some $g \in \Sigma_p^*[n; A, B]$, *then*

$$\left|\arg\left(-\frac{z(D^{n+p-1}F(z))'}{D^{n+p-1}G(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where F is the integral operator given by (2.9),

$$G(z) = \frac{c}{z^{c+p}}\int_0^z t^{c+p-1}g(t)\,\mathrm{d}t \quad (c > 0), \tag{2.16}$$

and α $(0 < \alpha \leq 1)$ *is the solution of the equation*

$$\begin{aligned}\delta = \alpha + \frac{2}{\pi}\tan^{-1}\Big(&((\alpha \sin \pi/2(1 - t(A, B, c)))\\ &\Big/(((c+p)(1-B) + A - 1)\\ &/(1-B) + \alpha\cos\pi/2(1 - t(A, B, c)))\Big)\end{aligned} \tag{2.17}$$

when

$$t(A, B, c) = \frac{2}{\pi}\sin^{-1}\left(\frac{p(A-B)}{(c+p)(1-B^2) - p(1-AB)}\right).$$

Proof Let

$$q(z) = -\frac{1}{p-\gamma}\left(\frac{z(D^nF(z))'}{D^nG(z)} + \gamma\right).$$

Since $g \in \Sigma_p^*[n; A, B]$, from Proposition 2.2, $g \in \Sigma_p^*[n; A, B]$. Using (2.10), we have

$$\begin{aligned}&(p-\gamma)q(z)D^{n+p-1}G(z) - (c+p)D^{n+p-1}F(z)\\ &\quad = -cD^{n+p-1}f(z) - \gamma D^{n+p-1}G(z).\end{aligned}$$

Then, by a simple calculation, we get

$$\begin{aligned}(p-\gamma)(zq'(z)+q(z)(-r(z)+c+p))+\gamma(-r(z)+c+p)\\ =-\frac{cz(D^{n+p-1}f(z))'}{D^{n+p-1}G(z)},\end{aligned}$$

where

$$r(z)=-\frac{z(D^{n+p-1}G(z))'}{D^{n+p-1}G(z)}.$$

Hence we have

$$q(z)+\frac{zq'(z)}{-r(z)+c+p}=-\frac{1}{p-\gamma}\left(\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)}+\gamma\right).$$

The remaining part of the proof is similar to that of Theorem 2.1 and so we omit it.

Letting $n=-p+1$, $A=1$, $B=0$ and $\delta=1$ in Theorem 2.3, we have

COROLLARY 2.4 *Let* $c>0$ *and* $f\in\Sigma$. *If*

$$-\mathrm{Re}\left\{\frac{zf'(z)}{g(z)}\right\}>\gamma\quad(0\leq\gamma<p)$$

for some $g\in\Sigma_p$ *satisfying the condition*

$$\left|\frac{zg'(z)}{g(z)}+p\right|<p,$$

then

$$-\mathrm{Re}\left\{\frac{zF'(z)}{G(z)}\right\}>\gamma,$$

where F *and* G *are given by* (2.9) *and* (2.16), *respectively*.

Taking $n=0$, $B\to A$ and $g(z)=1/z^p$ in Theorem 2.3, we have

COROLLARY 2.5 *Let* $c>0$ *and* $f\in\Sigma_p$. *If*

$$|\arg(-z^{p+1}f'(z)-\gamma)|<\frac{\pi}{2}\delta,\quad(0\leq\gamma<p;\ 0<\delta\leq1)$$

then

$$|\arg(-z^{p+1}F'(z)-\gamma)|<\frac{\pi}{2}\alpha,$$

where F is the integral operator given by (2.9) *and* α $(0<\alpha\leq 1)$ *is the solution of the equation*

$$\delta=\alpha+\frac{2}{\pi}\tan^{-1}\left(\frac{\alpha}{c+p-1}\right).$$

By using the same methods as in proving Theorem 2.3, we have

THEOREM 2.4 *Let $f\in\Sigma_p$ and choose a positive number c such that*

$$c\geq\frac{1+A}{1+B}-p,$$

where $-1<B<A\leq 1$ and $p\in\mathbb{N}$. If

$$\left|\arg\left(\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)}+\gamma\right)\right|<\frac{\pi}{2}\delta\quad(\gamma>p;\ 0<\delta\leq 1)$$

for some $g\in\Sigma_p^*[n;\ A,B]$, then

$$\left|\arg\left(\frac{z(D^{n+p-1}F(z))'}{D^{n+p-1}G(z)}+\gamma\right)\right|<\frac{\pi}{2}\alpha,$$

where F and G are given by (2.9) *and* (2.16), *respectively, and* α $(0<\alpha\leq 1)$ *is the solution of the equation given by* (2.17).

Finally, we derive

THEOREM 2.5 *Let $f\in\Sigma_p$. Choose an integer n such that*

$$n\geq\frac{p(1+A)}{1+B}-2p,$$

where $-1<B<A\leq 1$ and $p\in\mathbb{N}$. If

$$\left|\arg\left(-\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)}-\gamma\right)\right|<\frac{\pi}{2}\delta\quad(0\leq\gamma<p;\ 0<\delta\leq 1)$$

for some $g \in \Sigma_p^*[n; A, B]$, *then*

$$\left|\arg\left(-\frac{z(D^{n+p}F(z))'}{D^{n+p}G(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta,$$

where F *and* G *are given by* (2.9) *and* (2.16) *with* $c = n + p$, *respectively.*

Proof From (2.7) and (2.8) with $c = n+p$, we have $D^{n+p-1}f(z) = D^{n+p}F(z)$. Therefore

$$\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)} = \frac{z(D^{n+p}F(z))'}{D^{n+p}G(z)}$$

and the result follows.

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