J. of Inequal. & Appl., 2001, Vol. 6, pp. 99–117
Reprints available directly from the publisher
Photocopying permitted by license only

Special Solutions of Neutral Functional Differential Equations

ISTVÁN GYŐRI and MIHÁLY PITUK*

Department of Mathematics and Computing, University of Veszprém, P.O. Box 158, H-8201, Veszprém, Hungary

(Received 11 October 1999; Revised 5 November 1999)

For a system of n nonlinear neutral functional differential equations we prove the existence of an n-parameter family of "special solutions" which characterize the asymptotic behavior of all solutions at infinity. For retarded functional differential equations the special solutions used in this paper were introduced by Ryabov.

Keywords: Neutral equation; Special solution; Asymptotic behavior

AMS No.: 34K15; 34K25

1. INTRODUCTION

Let $\mathbb{R}$ and $\mathbb{R}^n$ denote the real line and the n-dimensional space of real column vectors, respectively. Let $|\cdot|$ denote any convenient norm of a vector or the associated induced norm of a square matrix.

Consider the neutral functional differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}Dx(t) = f(t, x(t), x(t-\tau)), \quad Dx(t) = x(t) + A(t)x(t-\sigma), \quad (1.1)$$

where σ, τ are positive constants, $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ is a continuous matrix function and $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector function.

* Corresponding author. E-mail: pitukm@almos.vein.hu.

Let $r = \max\{\sigma, \tau\}$ and $t_0 \in \mathbb{R}$. A function x is said to be a *solution* of Eq. (1.1) on $[t_0 - r, \infty)$ if x is defined and continuous on $[t_0 - r, \infty)$, Dx (defined by (1.1)) is differentiable on $[t_0, \infty)$ and Eq. (1.1) is satisfied for $t \geq t_0$.

It is well known (see, e.g., [9, Chap. 12] or [10, Chap. 2]) that if f is locally Lipschitzian in its last two variables, then for every continuous function $\phi : [t_0 - r, t_0] \to \mathbb{R}^n$ Eq. (1.1) has a unique solution x on $[t_0 - r, \infty)$ satisfying the *initial condition*

$$x(t) = \phi(t) \quad \text{for } t_0 - r \leq t \leq t_0. \tag{1.2}$$

The purpose of this paper is to establish conditions under which Eq. (1.1) possesses an n-parameter family of "special solutions" which characterize the asymptotic behavior of all solutions as $t \to \infty$. We shall see that this situation occurs if, e.g., $\sup_{t \in \mathbb{R}} |A(t)| < 1$, f is Lipschitzian in its second and third variable (see condition (2.2) below) and the delays σ and τ are "small". For a class of delay differential equations (Eq. (1.1) with $A(t) \equiv 0$), the special solutions used in this paper were introduced by Ryabov in [19].

The paper is organized as follows. In Section 2, we introduce the class of special solutions of Eq. (1.1) and prove the existence and uniqueness of the special solutions. The main result of the paper is formulated in Theorem 3 in Section 3. It shows that under appropriate assumptions every solution of Eq. (1.1) is asymptotic to some member of the n-parameter family of special solutions. In Section 4, we apply the results to linear systems. Finally, in Section 5, we compare our results to some previous ones in the literature.

2. SPECIAL SOLUTIONS

Throughout the paper we shall assume that there exist positive constants K, L, M, N and λ_0 such that the following conditions are satisfied:

$$|A(t)| \leq K \quad \text{for } t \in \mathbb{R}, \tag{2.1}$$

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq L|u_1 - u_2| + M|v_1 - v_2| \quad \text{for } t \in \mathbb{R},\ u_i, v_i \in \mathbb{R}^n,\ i = 1, 2 \tag{2.2}$$

$$|f(t,0,0)| \le Ne^{-\lambda_0 t} \quad \text{for } t \le 0 \tag{2.3}$$

$$K\lambda_0 e^{\lambda_0\sigma} + L + Me^{\lambda_0\tau} < \lambda_0. \tag{2.4}$$

Hypothesis (2.2) and the continuity of A and f imply that the *initial value problem* (1.1) and (1.2), formulated in Section 1, has a unique solution on $[t_0 - r, \infty)$.

Remark 1 The crucial assumption in our investigations is:

(H) inequality (2.4) has a positive solution λ_0.

Observe that for $\sigma = \tau = 0$ inequality (2.4) has the form $K\lambda_0 + L + M < \lambda_0$, which certainly holds if $K < 1$ and λ_0 is sufficiently large. From this, using a continuity argument, it follows that assumption (H) is satisfied if $K < 1$ and the delays σ and τ are sufficiently small. More precise necessary and sufficient conditions for the validity of assumption (H) follow from the following consideration.

Introduce

$$h(\lambda) = K\lambda e^{\lambda\sigma} + L + Me^{\lambda\tau} - \lambda \quad \text{for } \lambda \in [0,\infty).$$

Assumption (H) can be written equivalently as

$$h(\lambda_0) < 0 \quad \text{for some } \lambda_0 > 0. \tag{2.5}$$

By simple calculation, $h''(\lambda) > 0$ for $\lambda > 0$. From this, it follows that a necessary condition for assumption (H) is that

$$h'(0) = K + M\tau - 1 < 0. \tag{2.6}$$

Indeed, since h' is strictly increasing on $[0,\infty)$, the inequality $h'(0) \ge 0$ would imply that $h'(\lambda) \ge h'(0) \ge 0$ for $\lambda \ge 0$ and hence $h(\lambda) \ge h(0) = L + M > 0$ for all $\lambda > 0$, contradicting (2.5).

The facts that h' is strictly increasing, $h'(0) < 0$ and $\lim_{\lambda\to\infty} h'(\lambda) = \infty$ imply that there exists a unique $\lambda_* > 0$ such that

$$h'(\lambda_*) = Ke^{\lambda_*\sigma}(1 + \lambda_*\sigma) + M\tau e^{\lambda_*\tau} - 1 = 0. \tag{2.7}$$

Furthermore, condition (2.5) and thus assumption (H) is fulfilled if and only if

$$h(\lambda_*) < 0. \tag{2.8}$$

In that case the equation $h(\lambda) = 0$, or equivalently,

$$K\lambda e^{\lambda\sigma} + L + Me^{\lambda\tau} = \lambda \tag{2.9}$$

has exactly two positive roots λ_1, λ_2, $\lambda_1 < \lambda_* < \lambda_2$. Moreover, inequality (2.4) is satisfied if and only if $\lambda_0 \in (\lambda_1, \lambda_2)$.

In the "nonneutral" case when $K=0$ (i.e. $A(t) \equiv 0$), Eq. (2.7) can be solved explicitly, $\lambda_* = -1/\tau \ln(M\tau)$, and (2.8) gives the following explicit necessary and sufficient condition for the validity of assumption (H):

$$M\tau e^{1+L\tau} < 1. \tag{2.10}$$

Note that if $K=0$ and (2.10) holds, then inequality (2.4) is satisfied with $\lambda_0 = 1/\tau + L$.

Definition *A function* $\tilde{x} : \mathbb{R} \to \mathbb{R}^n$ *is said to be a* special solution *of Eq.* (1.1) *if* $\tilde{x}$ *is a solution of Eq.* (1.1) *on the whole interval* $(-\infty, \infty)$ *and*

$$\sup_{t \le 0} |\tilde{x}(t)| e^{\lambda_0 t} < \infty. \tag{2.11}$$

Remark 2 For the delay differential equation

$$\frac{dx}{dt}(t) = f(t, x(t), x(t-\tau)), \tag{2.12}$$

a special case of Eq. (1.1) when $A(t) \equiv 0$, assuming condition (2.10) with $L=0$ and taking $\lambda_0 = 1/\tau$, the above special solutions coincide with those introduced by Ryabov [19] and investigated by several authors (see, e.g., [3–7,11,16–18]).

In the following theorem we prove the existence of the special solutions. We also show that they are uniquely determined if we prescribe the value of the difference operator Dx at some point $t_0 \in \mathbb{R}$.

THEOREM 1 *Suppose conditions* (2.1)–(2.4) *hold. Then for every* $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$, *Eq.* (1.1) *has a unique special solution* $\tilde{x}$ *such that*

$$D\tilde{x}(t_0) = x_0. \tag{2.13}$$

The special solution $\tilde{x}$ of Eq. (1.1) satisfying (2.13) will be denoted by $\tilde{x} = \tilde{x}(t_0, x_0)$.

Proof Let B denote the space of those continuous functions $x : (-\infty, t_0] \to \mathbb{R}^n$ for which

$$\|x\| \stackrel{\text{def}}{=} \sup_{t \le t_0} |x(t)| e^{\lambda_0 (t - t_0)} < \infty. \tag{2.14}$$

$(B, \|\cdot\|)$ is a Banach space.

For $x \in B$, $t \le t_0$, define

$$(\mathcal{F}x)(t) = x_0 - A(t)x(t-\sigma) - \int_t^{t_0} f(s, x(s), x(s-\tau))\, ds.$$

Evidently, $\mathcal{F}x$ is continuous on $(-\infty, t_0]$, and for some $N_1 \ge N$,†

$$\begin{aligned}
|(\mathcal{F}x)(t)| &\le |x_0| + K|x(t-\sigma)| \\
&\quad + \int_t^{t_0} [|f(s,0,0)| + |f(s, x(s), x(s-\tau)) - f(s,0,0)|]\, ds \\
&\le |x_0| + K\|x\| e^{\lambda_0 (t_0 - t + \sigma)} \\
&\quad + \int_t^{t_0} [N_1 e^{-\lambda_0 s} + L\|x\| e^{\lambda_0 (t_0 - s)} + M\|x\| e^{\lambda_0 (t_0 - s + \tau)}]\, ds \\
&\le |x_0| + K e^{\lambda_0 \sigma} \|x\| e^{\lambda_0 (t_0 - t)} \\
&\quad + N_1 \lambda_0^{-1} e^{-\lambda_0 t} + \lambda_0^{-1} (L + M e^{\lambda_0 \tau}) \|x\| e^{\lambda_0 (t_0 - t)}.
\end{aligned}$$

Hence

$$\begin{aligned}
|(\mathcal{F}x)(t)| e^{\lambda_0 (t - t_0)} &\le |x_0| e^{\lambda_0 (t - t_0)} + K e^{\lambda_0 \sigma} \|x\| \\
&\quad + N_1 \lambda_0^{-1} e^{-\lambda_0 t_0} + \lambda_0^{-1} (L + M e^{\lambda_0 \tau}) \|x\|,
\end{aligned}$$

which is bounded on $(-\infty, t_0]$. Thus, $\mathcal{F}(B) \subset B$.

† If $t_0 \le 0$, then $N_1 = N$, while for $t_0 > 0$, $N_1 = \max\{N, \max_{0 \le s \le t_0} |f(s,0,0)| e^{\lambda_0 s}\}$.

Further, for $x, y \in B$ and $t \leq t_0$, we have

$$\begin{aligned}
&|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)| \\
&= \left| A(t)[x(t-\sigma) - y(t-\sigma)] \right. \\
&\quad \left. + \int_t^{t_0} [f(s, x(s), x(s-\tau)) - f(s, y(s), y(s-\tau)]\,\mathrm{d}s \right| \\
&\leq K|x(t-\sigma) - y(t-\sigma)| \\
&\quad + \int_t^{t_0} [L|x(s) - y(s)| + M|x(s-\tau) - y(s-\tau)|]\,\mathrm{d}s \\
&\leq K\|x - y\|\mathrm{e}^{\lambda_0(t_0 - t + \sigma)} + \|x - y\| \\
&\quad \times \int_t^{t_0} [L\mathrm{e}^{\lambda_0(t_0 - s)} + M\mathrm{e}^{\lambda_0(t_0 - s + \tau)}]\,\mathrm{d}s \\
&\leq [K\mathrm{e}^{\lambda_0\sigma} + \lambda_0^{-1}(L + M\mathrm{e}^{\lambda_0\tau})]\|x - y\|\mathrm{e}^{\lambda_0(t_0 - t)}.
\end{aligned}$$

Hence

$$\|\mathcal{F}x - \mathcal{F}y\| \leq [K\mathrm{e}^{\lambda_0\sigma} + \lambda_0^{-1}(L + M\mathrm{e}^{\lambda_0\tau})]\|x - y\|,$$

and, by virtue of (2.4), $K\mathrm{e}^{\lambda_0\sigma} + \lambda_0^{-1}(L + M\mathrm{e}^{\lambda_0\tau}) < 1$. Thus, $\mathcal{F} : B \to B$ is a contraction mapping. The unique solution $\tilde{x} \in B$ of $\mathcal{F}x = x$ gives a solution of Eq. (1.1) on $(-\infty, t_0]$ satisfying conditions (2.11) and (2.13). The solution $\tilde{x}$ can be uniquely extended to all $\mathbb{R}$ by the existence theorem mentioned in the Introduction.

COROLLARY *Under the hypotheses of Theorem* 1 *the totality of special solutions is an n-parameter family.*

Proof By Theorem 1, the map $\tilde{x} \mapsto D\tilde{x}(0)$ is a one-to-one correspondence between the set of special solutions and $\mathbb{R}^n$.

The next theorem provides an estimate for the distance of two special solutions.

THEOREM 2 *Suppose conditions* (2.1)–(2.4) *hold. Let* $t_0 \in \mathbb{R}, x_1, x_2 \in \mathbb{R}^n$. *Then for every* $t \leq t_0$,

$$|\tilde{x}(t_0, x_1)(t) - \tilde{x}(t_0, x_2)(t)| \leq C|x_1 - x_2|\mathrm{e}^{\lambda_1(t_0 - t)}, \tag{2.15}$$

where λ_1 *is the smallest positive root of Eq.* (2.9) (*see Remark* 1) *and* C *is a constant defined by*

$$C = \lambda_1 (L + M e^{\lambda_1 \tau})^{-1}. \tag{2.16}$$

Proof Define

$$u_0(t) = x_1,$$
$$u_{i+1}(t) = x_1 - A(t)u_i(t-\sigma) - \int_t^{t_0} f(s, u_i(s), u_i(s-\tau))\,\mathrm{d}s,$$

and

$$v_0(t) = x_2,$$
$$v_{i+1}(t) = x_2 - A(t)v_i(t-\sigma) - \int_t^{t_0} f(s, v_i(s), v_i(s-\tau))\,\mathrm{d}s,$$

for $t \le t_0$, $i = 0, 1, 2, \ldots$ We shall show by induction on i that

$$|u_i(t) - v_i(t)| \le C|x_1 - x_2| e^{\lambda_1 (t_0 - t)}, \quad t \le t_0, \tag{2.17}$$

for $i = 0, 1, \ldots$

Since

$$\lambda_1^{-1}(L + M e^{\lambda_1 \tau}) \le K e^{\lambda_1 \sigma} + \lambda_1^{-1}(L + M e^{\lambda_1 \tau}) = 1 \quad \text{(see (2.9))},$$

it follows that $C \ge 1$. Consequently,

$$|u_0(t) - v_0(t)| = |x_1 - x_2| \le C e^{\lambda_1 (t_0 - t)} |x_1 - x_2|, \quad t \le t_0.$$

Thus, (2.17) holds for $i = 0$. Now suppose that (2.17) holds for some $i \ge 0$. Then for $t \le t_0$,

$$\begin{aligned}
&|u_{i+1}(t) - v_{i+1}(t)| \\
&\quad \le |x_1 - x_2| + K|u_i(t-\sigma) - v_i(t-\sigma)| \\
&\qquad + \int_t^{t_0} [L|u_i(s) - v_i(s)| + M|u_i(s-\tau) - v_i(s-\tau)|]\,\mathrm{d}s \\
&\quad \le |x_1 - x_2| \Big\{ 1 + KC e^{\lambda_1 (t_0 - t + \sigma)} \\
&\qquad + C \int_t^{t_0} [L e^{\lambda_1 (t_0 - s)} + M e^{\lambda_1 (t_0 - s + \tau)}]\,\mathrm{d}s \Big\} \\
&\quad = |x_1 - x_2| \{ 1 - C\lambda_1^{-1}(L + M e^{\lambda_1 \tau}) \\
&\qquad + C e^{\lambda_1 (t_0 - t)} [K e^{\lambda_1 \sigma} + \lambda_1^{-1}(L + M e^{\lambda_1 \tau})] \} \\
&\quad = |x_1 - x_2| C e^{\lambda_1 (t_0 - t)},
\end{aligned}$$

the last relation being a consequence of (2.16) and (2.9) with $\lambda=\lambda_1$. Thus, we have proved that (2.17) holds for all $i=0,1,2,\ldots$ Referring to the proof of Theorem 1, for every $t\leq t_0$, $u_i(t)\to\tilde{x}(t_0,x_1)(t)$ and $v_i(t)\to\tilde{x}(t_0,x_2)(t)$ as $i\to\infty$. Inequality (2.15) now follows from (2.17) by letting $i\to\infty$.

3. MAIN RESULT

Our main result is the following theorem which shows that the special solutions characterize the asymptotic behavior of all solutions as $t\to\infty$. More precisely, every solution of (1.1) approaches exponentially some special solution as $t\to\infty$.

THEOREM 3 *Suppose conditions* (2.1)–(2.4) *hold. Let* x *be a solution of Eq.* (1.1) *on* $[t_0-r,\infty)$. *Then there exists a unique special solution* $\tilde{x}$ *of Eq.* (1.1) *such that*

$$\sup_{t\geq t_0-r}|x(t)-\tilde{x}(t)|e^{\lambda_0 t}<\infty. \tag{3.1}$$

Specially,

$$|x(t)-\tilde{x}(t)|\to 0 \quad \textit{exponentially as } t\to\infty. \tag{3.2}$$

Proof Let B be the vector space of those functions $y:[t_0-r,\infty)\to\mathbb{R}^n$ for which

$$\|y\|\stackrel{\text{def}}{=}\sup_{t\geq t_0-r}|y(t)|e^{\lambda_0(t-t_0)}<\infty. \tag{3.3}$$

$(B,\|\cdot\|)$ is a Banach space.

Denote by S the set of those functions $y\in B$ which are continuous on the intervals $[t_0-r,t_0)$, $[t_0+k\sigma,t_0+(k+1)\sigma)$ for $k=0,1,2,\ldots$ We shall show that S is a closed subset of B. To this aim, consider a sequence $\{y_i\}_{i=1}^{\infty}$ of elements in S which converges to some $y\in B$, i.e., $\|y_i-y\|\to 0$ as $i\to\infty$. We have to show that y also belongs to S. From the definition of the norm (3.3), it follows that $y_i\to y$ as $i\to\infty$ uniformly on finite subintervals of $[t_0-r,\infty)$. Consequently, since each y_i is continuous on the intervals $[t_0 - r, t_0)$, $[t_0 + k\sigma, t_0 + (k+1)\sigma)$, $k = 0, 1, 2, \ldots$, so is y. Thus, y belongs to S and therefore S is a closed subset of B.

For $y \in S$, define

$$(\mathcal{F}y)(t) = \begin{cases} -A(t)y(t-\sigma) + (\mathcal{G}y)(t), & \text{for } t \geq t_0, \\ x(t) - \tilde{x}(t_0, Dx(t_0) - (\mathcal{G}y)(t_0))(t), & \text{for } t_0 - r \leq t < t_0, \end{cases}$$

where

$$(\mathcal{G}y)(t) = -\int_t^\infty [f(s, x(s), x(s-\tau)) \\ - f(s, x(s) - y(s), x(s-\tau) - y(s-\tau))]\,\mathrm{d}s, \quad t \geq t_0.$$

By virtue of (2.2) and (3.3), we have for $t \geq t_0$,

$$\begin{aligned} |(\mathcal{G}y)(t)| &\leq \int_t^\infty [L|y(s)| + M|y(s-\tau)|]\,\mathrm{d}s \\ &\leq \|y\| \int_t^\infty [L\mathrm{e}^{\lambda_0(t_0 - s)} + M\mathrm{e}^{\lambda_0(t_0 - s + \tau)}]\,\mathrm{d}s \\ &= \|y\| \lambda_0^{-1}(L + M\mathrm{e}^{\lambda_0 \tau})\mathrm{e}^{\lambda_0(t_0 - t)}, \end{aligned}$$

which shows that operator $\mathcal{G}$ and thus $\mathcal{F}$ is well defined. The last estimate, (2.1) and (3.3) imply that

$$|(\mathcal{F}y)(t)|\mathrm{e}^{\lambda_0(t - t_0)} \leq [K\mathrm{e}^{\lambda_0 \sigma} + \lambda_0^{-1}(L + M\mathrm{e}^{\lambda_0 \tau})]\|y\| \quad \text{for } t \geq t_0. \tag{3.4}$$

Taking into account that $\mathcal{G}y$ is continuous on $[t_0, \infty)$, it follows from the definitions of operator $\mathcal{F}$ and the set S that $\mathcal{F}y$ is continuous on the intervals $[t_0 - r, t_0)$, $[t_0 + k\sigma, t_0 + (k+1)\sigma)$, $k = 0, 1, 2, \ldots$ This, together with (3.4), implies that $\mathcal{F}$ maps S into itself.

Now we show that $\mathcal{F} : S \to S$ is a contraction mapping. Let $y, z \in S$. In order to estimate $|(\mathcal{F}y)(t) - (\mathcal{F}z)(t)|$, we distinguish two cases.

Case 1 Assume that $t \geq t_0$. By similar estimates as in the proof of (3.4), it follows that

$$|(\mathcal{F}y)(t) - (\mathcal{F}z)(t)|\mathrm{e}^{\lambda_0(t - t_0)} \leq \varkappa_1 \|y - z\|,$$

where

$$\varkappa_1 = K\mathrm{e}^{\lambda_0\sigma} + \lambda_0^{-1}(L + M\mathrm{e}^{\lambda_0\tau}) < 1 \quad \text{(see (2.4))}. \tag{3.5}$$

Consequently,

$$\sup_{t\geq t_0} |(\mathcal{F}y)(t) - (\mathcal{F}z)(t)|\mathrm{e}^{\lambda_0(t-t_0)} \leq \varkappa_1 \|y - z\|. \tag{3.6}$$

Case 2 Let $t_0 - r \leq t < t_0$. By Theorem 2,

$$\begin{aligned}|(\mathcal{F}y)(t) - (\mathcal{F}z)(t)| &= |\tilde{x}(t_0, Dx(t_0) - (\mathcal{G}y)(t_0))(t) \\ &\quad - \tilde{x}(t_0, Dx(t_0) - (\mathcal{G}z)(t_0))(t)| \\ &\leq C|(\mathcal{G}y)(t_0) - (\mathcal{G}z)(t_0)|\mathrm{e}^{\lambda_1(t_0-t)},\end{aligned}$$

where λ_1 is the smallest positive root of Eq. (2.9) and C is the constant given in (2.16).

On the other hand,

$$\begin{aligned}|(\mathcal{G}y)(t_0) - (\mathcal{G}z)(t_0)| &\leq \|y - z\| \int_{t_0}^{\infty} [L\mathrm{e}^{\lambda_0(t_0-s)} + M\mathrm{e}^{\lambda_0(t_0-s+\tau)}]\,\mathrm{d}s \\ &= \lambda_0^{-1}(L + M\mathrm{e}^{\lambda_0\tau})\|y - z\|.\end{aligned}$$

Consequently,

$$|(\mathcal{F}y)(t) - (\mathcal{F}z)(t)| \leq \varkappa_2\|y - z\|\mathrm{e}^{\lambda_1(t_0-t)},$$

where

$$\varkappa_2 = C\lambda_0^{-1}(L + M\mathrm{e}^{\lambda_0\tau}).$$

Using the fact that $\lambda_1 < \lambda_0$ (see Remark 1), the last inequality implies

$$|(\mathcal{F}y)(t) - (\mathcal{F}z)(t)| \leq \varkappa_2\| y - z\|\mathrm{e}^{\lambda_0(t_0-t)},$$

and hence

$$\sup_{t_0-r\leq t<t_0} |(\mathcal{F}y)(t) - (\mathcal{F}z)(t)|\mathrm{e}^{\lambda_0(t-t_0)} \leq \varkappa_2\|y - z\|. \tag{3.7}$$

By virtue of (2.4),

$$\lambda_0^{-1}(L + Me^{\lambda_0\tau}) < 1 - Ke^{\lambda_0\sigma}.$$

Consequently,

$$\begin{aligned}\varkappa_2 &= C\lambda_0^{-1}(L + Me^{\lambda_0\tau}) \\ &< C(1 - Ke^{\lambda_0\sigma}) < C(1 - Ke^{\lambda_1\sigma}) = 1,\end{aligned} \tag{3.8}$$

the last and the last but one relation being a consequence of Eqs. (2.9) (with $\lambda = \lambda_1$), (2.16) and the inequality $\lambda_1 < \lambda_0$, respectively.

From (3.6) and (3.7), it follows that

$$\|\mathcal{F}y - \mathcal{F}z\| \le \varkappa\|y - z\| \quad \text{for all } y, z \in S,$$

where $\varkappa = \max\{\varkappa_1, \varkappa_2\} < 1$ (see (3.5) and (3.8)). Thus, $\mathcal{F} : S \to S$ is a contraction mapping.

Let $\tilde{y}$ be the unique fixed point of operator $\mathcal{F}$ in S. We know from the definition of S that $\tilde{y}$ is continuous on the intervals $[t_0 - r, 0)$, $[t_0 + k\sigma, t_0 + (k+1)\sigma)$, $k = 0, 1, 2, \ldots$ Our next aim is to show that $\tilde{y}$ is continuous on the whole interval $[t_0 - r, \infty)$. To do this, we have to show that $\tilde{y}$ is continuous from the left at each point $t_0 + k\sigma$, $k = 0, 1, 2, \ldots$ Using the continuity of A and $\mathcal{G}\tilde{y}$, it follows easily from the definition of operator $\mathcal{F}$ that if $\mathcal{F}\tilde{y}(= \tilde{y})$ is continuous from the left at some point $t_0 + k\sigma$ for some nonnegative integer k, then it is also continuous from the left at $t_0 + (k+1)\sigma$. Consequently, it suffices to show that $\tilde{y}$ is continuous from the left at t_0.

For brevity, we shall write

$$\tilde{x} = \tilde{x}(t_0, Dx(t_0) - (\mathcal{G}\tilde{y})(t_0)). \tag{3.9}$$

By the definition of the special solutions,

$$D\tilde{x}(t_0) = Dx(t_0) - (\mathcal{G}\tilde{y})(t_0).$$

From this, using the definition of the difference operator (see (1.1)), we find

$$x(t_0) - \tilde{x}(t_0) = -A(t_0)[x(t_0 - \sigma) - \tilde{x}(t_0 - \sigma)] + (\mathcal{G}\tilde{y})(t_0). \tag{3.10}$$

We have

$$\begin{aligned}\lim_{t\to t_0-}\tilde{y}(t) &= \lim_{t\to t_0-}(\mathcal{F}\tilde{y})(t) = \lim_{t\to t_0-}[x(t)-\tilde{x}(t)]\\ &= x(t_0)-\tilde{x}(t_0)\\ &= -A(t_0)[x(t_0-\sigma)-\tilde{x}(t_0-\sigma)]+(\mathcal{G}\tilde{y})(t_0)\\ &= -A(t_0)(\mathcal{F}\tilde{y})(t_0-\sigma)+(\mathcal{G}\tilde{y})(t_0)\\ &= -A(t_0)\tilde{y}(t_0-\sigma)+(\mathcal{G}\tilde{y})(t_0)=(\mathcal{F}\tilde{y})(t_0)=\tilde{y}(t_0),\end{aligned}$$

where we have used the fact that $\tilde{y}$ is a fixed point of $\mathcal{F}$, the definition of $\mathcal{F}$, the continuity of x and $\tilde{x}$ at t_0 and relation (3.10). Thus, $\tilde{y}$ is continuous from the left at t_0. As we have mentioned, this implies that $\tilde{y}$ is continuous on the whole interval $[t_0-r,\infty)$.

Using the fact that $\tilde{y}$ is a fixed point of $\mathcal{F}$ and notation (3.9), we have

$$\tilde{y}(t)=x(t)-\tilde{x}(t)\quad \text{for } t_0-r\le t<t_0, \tag{3.11}$$

while for $t\ge t_0$,

$$\begin{aligned}\tilde{y}(t)+A(t)\tilde{y}(t-\sigma) = -\int_t^\infty &[f(s,x(s),x(s-\tau))\\ &-f(s,x(s)-\tilde{y}(s),x(s-\tau)-\tilde{y}(s-\tau))]\,\mathrm{d}s.\end{aligned} \tag{3.12}$$

Observe that the integrand on the right-hand side of (3.12) is continuous, and, in view of the continuity of $\tilde{y}$, x and $\tilde{x}$, relation (3.11) holds also for $t=t_0$. From this, it follows that $\tilde{y}$ is a solution of the initial value problem

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}Dy(t) = &f(t,x(t),x(t-\tau))\\ &-f(t,x(t)-y(t),x(t-\tau)-y(t-\tau))\quad \text{for } t\ge t_0,\end{aligned} \tag{3.13}$$

$$y(t)=x(t)-\tilde{x}(t)\quad \text{for } t_0-r\le t\le t_0. \tag{3.14}$$

Evidently, the function

$$y(t)=x(t)-\tilde{x}(t),\quad t\ge t_0-r$$

is also a solution of the initial value problem (3.13) and (3.14). Since the right-hand side of Eq. (3.13) (the function $g(t,u,v)=f(t,x(t),x(t-\tau))-f(t,x(t)-u,x(t-\tau)-v)$) is continuous and Lipschitzian in the last two variables (see (2.2)), the solution of Eqs. (3.13) and (3.14) is unique on $[t_0 - r, \infty)$. Consequently,

$$\tilde{y}(t) = x(t) - \tilde{x}(t) \quad \text{for all } t \geq t_0 - r.$$

Hence

$$\sup_{t \geq t_0 - r} |x(t) - \tilde{x}(t)| \mathrm{e}^{\lambda_0(t-t_0)} = \sup_{t \geq t_0 - r} |\tilde{y}(t)| \mathrm{e}^{\lambda_0(t-t_0)} = \|\tilde{y}\| < \infty.$$

Thus, the special solution $\tilde{x}$ of Eq. (1.1), defined by relation (3.9), satisfies condition (3.1). The proof of the theorem is complete.

The following theorem provides further information about the special solution $\tilde{x}$ described in Theorem 3.

THEOREM 4 *Suppose conditions* (2.1)–(2.4) *hold and let x and $\tilde{x}$ have the meaning from Theorem* 3. *Then for every* $t_1 \in \mathbb{R}$,

$$\tilde{x}(t_1) = \lim_{t \to \infty} \tilde{x}(t, Dx(t))(t_1). \tag{3.15}$$

Proof By Theorem 3,

$$H \stackrel{\text{def}}{=} \sup_{t \geq t_0 - r} |x(t) - \tilde{x}(t)| \mathrm{e}^{\lambda_0(t-t_0)} < \infty. \tag{3.16}$$

Let $t \geq \max\{t_0, t_1\}$. Since $\tilde{x} = \tilde{x}(t, D\tilde{x}(t))$, from Theorem 2, we obtain

$$\begin{aligned}
&|\tilde{x}(t_1) - \tilde{x}(t, Dx(t))(t_1)| \\
&\quad \leq C\mathrm{e}^{\lambda_1(t-t_1)} |D\tilde{x}(t) - Dx(t)| \\
&\quad = C\mathrm{e}^{\lambda_1(t-t_1)} |\tilde{x}(t) - x(t) + A(t)[\tilde{x}(t-\sigma) - x(t-\sigma)]| \\
&\quad \leq C\mathrm{e}^{\lambda_1(t-t_1)} \{|\tilde{x}(t) - x(t)| + |A(t)||\tilde{x}(t-\sigma) - x(t-\sigma)|\}.
\end{aligned}$$

Using (2.1) and (3.16), the last inequality implies

$$\begin{aligned}
|\tilde{x}(t_1) - \tilde{x}(t, Dx(t))(t_1)| &\leq C\mathrm{e}^{\lambda_1(t-t_1)} \{H\mathrm{e}^{-\lambda_0(t-t_0)} + KH\mathrm{e}^{-\lambda_0(t-\sigma-t_0)}\} \\
&= CH(1 + K\mathrm{e}^{\lambda_0\sigma})\mathrm{e}^{-\lambda_0(t_1-t_0)}\mathrm{e}^{(\lambda_1-\lambda_0)(t-t_1)}.
\end{aligned}$$

Since $\lambda_1 < \lambda_0$, the right-hand side of the last inequality tends to zero as $t \to \infty$, which completes the proof.

Remark 3 Theorems 1–4 remain valid for the equation with time-dependent delays

$$\frac{\mathrm{d}}{\mathrm{d}t}Dx(t) = f(t, x(t), x(t-\tau(t))), \quad Dx(t) = x(t) + A(t)x(t-\sigma(t)),$$

provided the following assumptions are satisfied:

(i) $A : \mathbb{R} \to \mathbb{R}^{n\times n}$, $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous,
(ii) there exist positive constants K, L, M, N and λ_0 such that conditions (2.1)–(2.3) are satisfied,
(iii) $\sigma, \tau : \mathbb{R} \to [0, \infty)$ are continuous,
(iv) the function $g(t) \overset{\mathrm{def}}{=} t - \sigma(t)$, $t \in \mathbb{R}$, is strictly increasing,
(v) $0 < \sigma(t) \le \sigma_0$ and $0 \le \tau(t) \le \tau_0$ for $t \in \mathbb{R}$, where σ_0 and τ_0 are positive constants,
(vi) $K\lambda_0 e^{\lambda_0\sigma_0} + L + Me^{\lambda_0\tau_0} < \lambda_0$.

The only modification needed in the proof of Theorem 3 is to replace the sequence $t_0, t_0+\sigma, t_0+2\sigma, t_0+3\sigma, \ldots$ with the sequence t_0, $t_1 = g^{-1}(t_0)$, $t_2 = g^{-1}(t_1)$, $t_3 = g^{-1}(t_2), \ldots$, where g^{-1} denotes the inverse function of g.

4. LINEAR SYSTEMS

In this section, we apply our results to the linear equation

$$\frac{\mathrm{d}}{\mathrm{d}t}Dx(t) = B(t)x(t) + C(t)x(t-\tau), \quad Dx(t) = x(t) + A(t)x(t-\sigma), \tag{4.1}$$

where σ, τ are positive constants, and $A, B, C : \mathbb{R} \to \mathbb{R}^{n\times n}$ are continuous matrix functions.

Equation (4.1) is a special case of (1.1) when $f(t, u, v) = B(t)u + C(t)v$. Conditions (2.1)–(2.4) reduce to

$$|A(t)| \le K \quad \text{for } t \in \mathbb{R}, \tag{4.2}$$

$$|B(t)| \le L \quad \text{for } t \in \mathbb{R}, \tag{4.3}$$

$$|C(t)| \leq M \quad \text{for } t \in \mathbb{R}, \tag{4.4}$$

$$K\lambda_0 e^{\lambda_0 \sigma} + L + M e^{\lambda_0 \tau} < \lambda_0 \quad \text{for some } \lambda_0 > 0. \tag{4.5}$$

For $i = 1, 2, \ldots, n$, let e_i denote the ith column of the $n \times n$ unit matrix I. Let $t_0 \in \mathbb{R}$ be fixed and let $\tilde{x}_i$ $(i = 1, 2, \ldots, n)$ be the (unique) special solution of Eq. (4.1) for which $D\tilde{x}_i(t_0) = e_i$.

Let $\tilde{X}$ be the $n \times n$ matrix function defined by

$$\tilde{X}(t) = (\tilde{x}_1(t), \tilde{x}_2(t), \ldots, \tilde{x}_n(t)) \quad \text{for } t \in \mathbb{R}. \tag{4.6}$$

THEOREM 5 *Suppose conditions* (4.2)–(4.5) *hold. The matrix function* $\tilde{X}$ *is a solution of the matrix equation*

$$\frac{\mathrm{d}}{\mathrm{d}t} DX(t) = B(t)X(t) + C(t)X(t-\tau), \quad DX(t) = X(t) + A(t)X(t-\sigma), \tag{4.7}$$

on $(-\infty, \infty)$ *such that*

$$D\tilde{X}(t_0) = I \tag{4.8}$$

and

$$\sup_{t \leq 0} |\tilde{X}(t)| e^{\lambda_0 t} < \infty. \tag{4.9}$$

This matrix function $\tilde{X}$, *called the special matrix solution of Eq.* (4.1), *has the following properties: the special solution* $\tilde{x}(t_0, x_0)$ *of Eq.* (4.1) *described in Theorem* 1 *is given by*

$$\tilde{x}(t_0, x_0)(t) = \tilde{X}(t) x_0 \quad \textit{for } t \in \mathbb{R},$$

and for every $t \in \mathbb{R}$,

$$\det D\tilde{X}(t) \neq 0. \tag{4.10}$$

Proof All properties except for (4.10) are immediate consequences of Theorem 1. We shall show (4.10). Let $t \in \mathbb{R}$ be fixed. From the uniqueness of the special solutions, it follows that the only special solution $\tilde{x}$

for which $D\tilde{x}(t) = 0$ is $\tilde{x} \equiv 0$. Since the special solutions of (4.1) are of the form $\tilde{x}(t) = \tilde{X}(t)x_0$, $x_0 \in \mathbb{R}^n$, this shows that the only solution of the equation $D\tilde{X}(t)x_0 = 0$ is $x_0 = 0$, which is equivalent to (4.10).

Theorem 3 about the asymptotic behavior of all solutions can be reformulated as follows.

THEOREM 6 *Suppose conditions* (4.2)–(4.5) *hold. Let x be a solution of Eq.* (4.1) *on* $[t_0 - r, \infty)$. *Then the limit*

$$l = \lim_{t \to \infty} ([D\tilde{X}(t)]^{-1} Dx(t)) \tag{4.11}$$

exists in $\mathbb{R}^n$, *and*

$$\sup_{t \geq t_0 - r} |x(t) - \tilde{X}(t)l| e^{\lambda_0 t} < \infty. \tag{4.12}$$

Specially,

$$|x(t) - \tilde{X}(t)l| \to 0 \quad \textit{exponentially as } t \to \infty. \tag{4.13}$$

Moreover, the constant vector l given in (4.11) *is the only one satisfying condition* (4.12).

Proof By Theorem 3, Eq. (4.1) has a unique special solution $\tilde{x}$ satisfying (3.1). We know from Theorem 5 that this special solution $\tilde{x}$ has the form

$$\tilde{x}(t) = \tilde{X}(t)x_0, \quad \text{where } x_0 = D\tilde{x}(t_0). \tag{4.14}$$

By Theorem 4,

$$\tilde{x}(t_0) = \lim_{s \to \infty} \tilde{x}(s, Dx(s))(t_0)$$

and

$$\tilde{x}(t_0 - \sigma) = \lim_{s \to \infty} \tilde{x}(s, Dx(s))(t_0 - \sigma).$$

From this,

$$x_0 = D\tilde{x}(t_0) = \lim_{s \to \infty} D\tilde{x}(s, Dx(s))(t_0). \tag{4.15}$$

It is easily seen that for every $t_1 \in \mathbb{R}$, $x_1 \in \mathbb{R}^n$,

$$\tilde{x}(t_1, x_1)(t) = \tilde{X}(t)[D\tilde{X}(t_1)]^{-1}x_1.$$

Hence

$$\tilde{x}(s, Dx(s))(t) = \tilde{X}(t)[D\tilde{X}(s)]^{-1}Dx(s).$$

From this and (4.8), we find

$$D\tilde{x}(s, Dx(s))(t_0) = [D\tilde{X}(s)]^{-1}Dx(s).$$

This, together with (4.15), implies that $x_0 = l$, completing the proof.

5. DISCUSSION

Previous papers on this subject were restricted mostly to the case of delay differential equations, i.e., when $A(t) \equiv 0$. Let us briefly mention some of them.

As we have mentioned, the special solutions were introduced by Ryabov [19] for Eq. (2.12) and some more general equations in the case $\lambda_0 = 1/\tau$. For linear delay differential systems the asymptotic relation (4.11) was obtained by Driver [5]. (Note that if $A(t) \equiv 0$ then $Dx(t) = x(t)$.) The asymptotic characterization of all solutions (Theorem 3) for general nonlinear delay differential equations was proved by Jarník and Kurzweil [11]. Under some additional assumptions, the asymptotic behavior of the special solutions was investigated by the first author in [6] and by the second author in [17]. For general linear delay differential equations, without any further restrictions, the special solutions were described by Arino and the authors [4]. In [4, Theorem 2.4] it is shown that the special matrix solution is a fundamental matrix for a linear homogeneous ordinary differential equation whose coefficient matrix can be expressed by certain infinite series. Using this series representation, the authors obtained sharp stability criteria (see [7]). In the recent paper [3], Arino and the second author showed that for linear delay differential equations the value of the limit (4.11) can be computed explicitly in terms of the initial function of a given solution.

Finally, we mention the recent papers by Arino and Bourad [1], Arino and the second author [2], Krisztin [12], Krisztin and Wu [13–15], Wu [20] and Wu and Freedman [21] on neutral functional differential equations which are relevant to our study. For example, in [15, Theorem 4.1] Krisztin and Wu showed that for certain scalar periodic neutral functional differential equations "almost every" solution is asymptotic to some member of a one-parameter family of periodic solutions, which is a qualitative result similar to ours.

Acknowledgments

I. Győri was partially supported by Hungarian National Foundation for Scientific Research Grant No. T 019846 and by Hungarian Ministry of Culture and Education Grant No. FKFP 1024/1997. M. Pituk was partially supported by Hungarian National Foundation for Scientific Research Grant Nos. F 023772 and T 019846.

References

[1] O. Arino and F. Bourad, On the asymptotic behavior of the solution of a class of scalar neutral equations generating a monotone semi-flow, *J. Differential Equations*, **87** (1990), 84–95.

[2] O. Arino and M. Pituk, Asymptotic constancy for neutral functional differential equations, *Differential Equations Dynam. Systems*, **6** (1998), 261–273.

[3] O. Arino and M. Pituk, More on linear differential systems with small delays, *J. Differential Equations* (to appear).

[4] O. Arino, I. Győri and M. Pituk, Asymptotically diagonal delay differential systems, *J. Math. Anal. Appl.*, **204** (1996), 701–728.

[5] R.D. Driver, Linear differential systems with small delays, *J. Differential Equations*, **21** (1976), 149–167.

[6] I. Győri, On existence of the limits of solutions of functional differential equations, *Colloq. Math. Soc. János Bolyai, 30. Qualitative theory of Differential Equations*, Szeged (Hungary) (1979), North Holland Publ. Company (1980), pp. 325–362.

[7] I. Győri and M. Pituk, Stability criteria for linear delay differential equations, *Differential Integral Equations*, **10** (1997), 841–852.

[8] I. Győri and J. Wu, A neutral equation arising from compartmental systems with pipes, *J. Dynam. Differential Equations*, **3** (1991), 289–311.

[9] J. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.

[10] J.K. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993.

[11] J. Jarník and J. Kurzweil, Ryabov's special solutions of functional differential equations, *Boll. Un. Mat. Ital.*, **11** (1975), 198–218.

[12] T. Krisztin, Monotone semiflows generated by neutral functional differential equations, *Proceedings of the Second Marrakesh International Conference on Differential Equations* (to appear).

[13] T. Krisztin and J. Wu, Monotone semiflows generated by neutral equations with different delays in neutral and retarded parts, *Acta Math. Univ. Comenianae*, **63** (1994), 207–220.

[14] T. Krisztin and J. Wu, Asymptotic periodicity, monotonicity and oscillation of solutions of scalar neutral functional differential equations, *J. Math. Anal. Appl.*, **199** (1996), 502–525.

[15] T. Krisztin and J. Wu, Asymptotic behaviors of solutions of scalar neutral functional differential equations, *Differential Equations Dynam. Systems*, **4** (1996), 351–366.

[16] M. Pituk, Asymptotic characterization of solutions of functional differential equations, *Boll. Un. Mat. Ital. B*, **7** (1993), 653–689.

[17] M. Pituk, Asymptotic behavior of solutions of a differential equation with asymptotically constant delay, *Nonlinear Anal.*, **30** (1997), 1111–1118.

[18] M. Pituk, Special solutions of functional differential equations, *Proceedings of the Second Marrakesh International Conference on Differential Equations* (to appear).

[19] Yu.A. Ryabov, Certain asymptotic properties of linear systems with small time lag, *Trudy Sem. Teor. Differencial. Uravnenii s Otklon. Argumentom Univ. Druzby Narodov Patrica Lumumby*, **3** (1965), 153–164 (in Russian).

[20] J. Wu, Asymptotic periodicity of solutions to a class of neutral functional differential equations, *Proc. Amer. Math. Soc.*, **113** (1993), 355–363.

[21] J. Wu and H.I. Freedman, Monotone semiflows generated by neutral equations and application to compartmental systems, *Canadian J. Math.*, **43** (1991), 1098–1120.