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On the Boundedness of the Solutions of the Continuous Riccati Equation

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In this paper, an inequality on the solution of the continuous-time Riccati equation which is used by many authors in the field of control theory is shown to be false: a counter-example is given. Moreover correct bounds for the solution of the Riccati equation are given.

Keywords and Phrases: Riccati equation; Bound; Continuous time systems

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1. INTRODUCTION

It is well known that the continuous and discrete Riccati equations are widely applied to various engineering areas such as signal processing and, especially, control theory. In the area of control system analysis, these equations play important roles in system stability analysis, optimal controllers and filters design, *etc.* In addition, in the last few years some authors developed techniques to construct observers for

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nonlinear systems [1, 2]. Like the Kalman filter, these observers involve the solution of a continuous Riccati equation and the convergence of this kind of observer is ensured provided that a gain-parameter θ is taken sufficiently large. How large this parameter must be chosen depends on the bounds of the solution of the Riccati equation. In order to get an estimate of these bounds, the authors base on an inequality which is stated in many works on the Riccati equation (see *e.g.* [3, 4]).

The goal of this paper is to show that this inequality is false and to give a correct estimate of the bounds of the Riccati equation.

In the first section, this inequality will be clearly stated and we explain where the usual proof seems to be false. In Section 3, a counter example will be given and, finally in the last section, new bounds are proposed.

2. THE RICCATI EQUATION – DESCRIPTION OF THE PROBLEM

We consider the continuous-time Riccati equation (the notations used are those of [3]):

$$\begin{cases} \dot{P}(t) = F(t)P(t) + P(t)F^T(t) - P(t)H^T(t)R^{-1}(t)H(t)P(t) \\ \qquad\qquad + G(t)Q(t)G^T(t) \\ P(t_0) = \Gamma \geq 0 \end{cases} \tag{1}$$

where all the matrices are supposed to be smooth, $F(t) \in \mathbb{R}^{n \times n}$, $G(t) \in \mathbb{R}^{n \times m}$ and $H(t) \in \mathbb{R}^{p \times n}$. Moreover $Q(t)$ and $R(t)$ are symmetric nonnegative. Classically, it can be shown, under the assumption $P(t_0) \geq 0$, that the solution of (1) is defined for all $t \geq t_0$ and is nonnegative (see *e.g.* [5]).

Now, in order to bound the solutions of (1), most authors base on an inequality which can be found in [3] and in [4]. Before stating this inequality, we have to introduce some notations. To Eq. (1) is associated the time-variant linear system:

$$\begin{cases} \dot{x} = F(t)x + G(t)u \\ y = H(t)x \end{cases} \tag{2}$$

and the two following matrices are related to this system:

$$C_Q(t, t-\Delta) = \int_{t-\Delta}^{t} \Phi(t,s)\; G(s)\; Q(s)\; G^T(s)\; \Phi^T(t,s)\; ds \qquad (3)$$

$$W_R(t, t-\Delta) = \int_{t-\Delta}^{t} \Phi^T(s,t)\; H^T(s)\; R^{-1}(s)\; H(s)\; \Phi(s,t)\; ds \qquad (4)$$

where Δ is the interval of observability ($\Delta \leq t - t_0$), and $\Phi(t,s)$ is the fundamental matrix of $F(t)$. It is known that system (2) is observable if and only if the matrix W_R is positive definite. If in addition, there exist positive number β_1 and β_2 such that:

$$\beta_1\; \mathbf{I_d} \leq W_R(t, t-\Delta) \leq \beta_2\; \mathbf{I_d} \qquad (5)$$

for all t, then system (2) is said uniformly R-observable. Similarly, system (2) is said Q-controllable if and only if there exist positive numbers α_1 and α_2 such that

$$\alpha_1\; \mathbf{I_d} \leq C_Q(t, t-\Delta) \leq \alpha_2\; \mathbf{I_d} \qquad (6)$$

Under the assumption of uniform R-observability, it is claimed that every solution $P(t)$ of Eq. (1) satisfies the inequality:

$$P(t) \leq C_Q(t, t-\Delta) + W_R^{-1}(t, t-\Delta) \qquad (7)$$

provided that $\Delta < t - t_0$.

Unfortunately, this inequality is false as we will see in the next section. In the proof of (7), it seems that the authors assumed that if A and B are symmetric matrices then $A \leq B$ implies $A^2 \leq B^2$. Now, consider the matrices:

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & 10/\alpha \end{pmatrix} \quad B = \begin{pmatrix} 2\alpha & 3 \\ 3 & 20/\alpha \end{pmatrix}$$

with α a positive number. An easy calculation will convince the reader that $A \leq B$ and det $(A^2 - B^2) = (-9/\alpha^2)(\alpha^2 - 4)(\alpha^2 - 25)$ which is negative if $\alpha < 4$ or $\alpha > 5$; thus we don't have $A^2 \leq B^2$ if α is sufficiently large. Notice also the following: by using the method of Lagrange multipliers, we know that $\inf\{\lambda | A^2 \leq \lambda B^2\}$ is equal to the greatest eigenvalues of $B^{-2}A^2$, the trace of this last matrix being equal to $(9\alpha^2+800+(900/\alpha^2)/3611)$. Thus, we can see that the greatest eigenvalue of $B^{-2}A^2$ can be made arbitrarily large; consequently, in general, the knowledge of the inequality $A \leq B$ cannot help us in finding the lowest λ which satisfies $A^2 \leq \lambda B^2$.

3. A COUNTER-EXAMPLE

For the construction of our counter example, we will work with constant matrices; namely, we take:

$$F = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

$$H = (1 \quad 1), \quad R = 1 \quad \text{with } a, b > 0$$

Notice that with this choice of matrices, the pair (F, H) is observable and, that it is possible to find a squared-root $\bar{Q}$ of $Q(Q = \bar{Q}\,\bar{Q})$ such that the pair $(F, \bar{Q})$ is controllable $\left(\text{take } \bar{Q} = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{b} \end{pmatrix}\right)$.

We consider now the continuous algebraic Riccati equation:

$$F\,P + P\,F^T - P\,H^T\,H\,P + Q = 0 \tag{8}$$

As a consequence of the above-mentioned properties of observability and controllability, we known that this last equation admits a positive definite solution P_∞ such that $P_\infty = \lim_{t\to\infty} P(t)$ where $P(t)$ is the solution of the continuous-time Riccati equation

$$\dot{P}(t) = F\,P(t) + P(t)\,F^T - P(t)\,H^T\,H\,P(t) + Q \tag{9}$$

satisfying $P(0) = 0$. Thus, if we are able to find a and b such that the inequality $P_\infty \leq C_Q(t, t-\Delta) + W_R^{-1}(t, t-\Delta)$ does not hold then the inequality $P(t) \leq C_Q + W_R^{-1}$ does not more hold for all t sufficiently large.

The manipulation of the solution of (8) in the case where a and b are nonzero is somewhat tedious, so we prefer to deal with the case $a > 0$, $b = 0$ and we will conclude thanks to the continuity properties of the solution of the continuous algebraic Riccati equation (see [6]); notice that in this case it is not possible to find a squared-root of Q such that the pair $(F,\ \bar{Q})$ is controllable.

Consider the matrix:

$$P_\infty = \begin{pmatrix} \frac{4a-1+\sqrt{1+a}}{9} & -\frac{4}{9}(a-1+\sqrt{1+a}) \\ -\frac{4}{9}(a-1+\sqrt{1+a}) & \frac{4}{9}(a+5+4\sqrt{1+a}) \end{pmatrix}$$

the reader will easily check that:

- P_∞ is positive definite,
- P_∞ is solution of Eq. (8),
- $F^T - H^T H P_\infty$ is Hurwitz,

and this proves that P_∞ is the maximal solution of (8). Concerning the matrices C_Q and W_R, an easy computation gives:

$$C_Q = \begin{pmatrix} (a/2)(1-e^{-2s}) & 0 \\ 0 & 0 \end{pmatrix}$$

$$W_R = \begin{pmatrix} (1/2)(e^{2s}-1) & 1-e^{-s} \\ 1-e^{-s} & (1/4)(1-e^{-2s}) \end{pmatrix}$$

The (2, 2) element of W_R^{-1}, denoted by w_{22}, is equal to

$$w_{22} = \frac{e^{2s}-1}{(e^s-1)^2(((e^s+e^{-s})/2)-1)(((e^s+e^{-s})/2)+2)}$$

so $\lim_{s\to\infty} w_{22} = 4$.

On the other hand, the (2, 2)-element of P_∞ tends to infinity with a. It is therefore possible to find a and Δ such that the (2, 2)-element of the matrix $P_\infty - (C_Q + W_R^{-1})$ is positive: to be more concrete, if we take $\Delta = 2$ and $a = 1$, we find that the (2, 2)-element of $P_\infty - (C_Q + W_R^{-1})$ is approximatively equal to 0.67. We conclude that for $\Delta = 2$ and $a = 1$ we don't have $P_\infty \leq (C_Q + W_R^{-1})$; now the maximal solution of the continuous algebraic Riccati equation being continuous with respect to the elements of matrix Q (see [6]), if b is chosen sufficiently small the solution of (8) $P_\infty(b)$ will be closed to P_∞ and will not satisfy inequality $P_\infty(b) \leq (C_Q + W_R^{-1})$. As a numerical application, if we take $\Delta = 2$, $a = 1$ and $b = (1/3000)$, the (2, 2)-element of the matrix $P_\infty(b) - (C_Q + W_R^{-1})$ is approximatively equal to 0.43.

4. BOUNDS FOR THE SOLUTION OF THE RICCATI EQUATION

The problem of finding bounds for the solution of the Riccati equation has been investigated by many authors; we can cite [7] for the discrete Riccati equation and [8] for the continuous time Riccati equation. Notice that in this last reference, the matrices involved in the Riccati equation are constant and have to satisfy a rank assumption. The following proposition gives bounds involving matrices C_Q and W_R for the solution of the Riccati equation.

PROPOSITION 1 *If system* (2) *is uniformly R-observable and uniformly Q-conrollable, we have the following inequality for $P(t)$, the solution of system* (1):

$$(C_Q^{-1} + C_Q^{-1}\ I_1\ C_Q^{-1})^{-1} \leq P(t) \leq W_R^{-1} + W_R^{-1}\ I_2\ W_R^{-1} \qquad (10)$$

where I_1 and I_2 are symetric definite positives matrix given in the proof.

Proof First, let's prove the right hand side of inequality (10). We use the observability assumption, so, t being fixed, we consider the linear system

$$\frac{dx}{d\tau} = \tilde{F}^T(\tau)x(\tau) + \tilde{H}^T(\tau)u(\tau) \qquad (11)$$

and the minimizing problem

$$J(u) = \int_{t-\Delta}^{t} (u^T(s)\ \tilde{R}(s)\ u(s) + x^T(s)\ \tilde{Q}(s)\ x(s))ds + x^T(t)\ Q_f\ x(t) \qquad (12)$$

where $\tilde{F}(\tau) = F(2t - \Delta - \tau)$, $\tilde{H}(\tau) = H(2t - \Delta - \tau)$, $\tilde{R}(\tau) = R(2t- \Delta - \tau)$, $\tilde{Q}(\tau) = Q(2t - \Delta - \tau)$ and $Q_f = P(t-\Delta)$.

It is well known that the least value of $J(u)$ (over all measurable u), is equal to $x^T(t-\Delta)\ \Pi(t-\Delta)\ x(t-\Delta)$ where Π satisfies the following Riccati equation:

$$\frac{d\Pi}{d\tau} = -\tilde{F}(\tau)\ \Pi(\tau) - \Pi(\tau)\ \tilde{F}^T(\tau) - \tilde{Q}(\tau) + \Pi(\tau)\ \tilde{H}^T(\tau)\ \tilde{R}^{-1}(\tau)\ \tilde{H}(\tau)\ \Pi(\tau) \qquad (13)$$

with the final condition $\Pi(t) = Q_f = P(t-\Delta)$.

Letting $\tilde{P}(\tau) = P(2t - \Delta - \tau)$, we can see that $\tilde{P}$ satisfies the same differential equation than Π, and, that $\tilde{P}(t) = P(t-\Delta) = \Pi(t)$, so we can conclude that $\tilde{P}(\tau) = \Pi(\tau)$ for all $\tau \in [t-\Delta, t]$, and as, for all u, $x^T(t-\Delta)\, \Pi(t-\Delta)\, x(t-\Delta) \leq J(u)$, we have also

$$x^T(t-\Delta)\, P(t)\, x(t-\Delta) \leq J(u) \tag{14}$$

which is true for all values of the initial conditions $x(t-\Delta)$.

Now, take $u(\tau) = -\tilde{R}^{-1}(\tau)\, \tilde{H}(\tau)\, \tilde{\Phi}^T(t-\Delta, \tau)\, W_R^{-1} x(t-\Delta)$ where $\tilde{\Phi}$ is the fundamental matrix of $\tilde{F}^T$ of system (11). First, we have

$$x(s) = \tilde{\Phi}(s, t-\Delta)\left(x(t-\Delta) + \int_{t-\Delta}^{s} \tilde{\Phi}(t-\Delta, \tau)\, \tilde{H}^T(\tau)\, u(\tau)\, d\tau\right). \tag{15}$$

We will see that $x(t) = 0$, so consider the integral

$$J = \int_{t-\Delta}^{t} \tilde{\Phi}(t-\Delta, \tau)\, \tilde{H}^T(\tau)\, u(\tau)\, d\tau \tag{16}$$

after the change of variable $\tau' = 2t - \Delta - \tau$, J becomes:

$$J = \int_{t-\Delta}^{t} \tilde{\Phi}(t-\Delta, 2t-\Delta-\tau')\, \tilde{H}^T(2t-\Delta-\tau')\, u(2t-\Delta-\tau')\, d\tau'. \tag{17}$$

Now notice that $\tilde{\Phi}(t-\Delta, 2t-\Delta-\tau') = \Phi^T(\tau', t)$ (these two matrices are equal at $\tau' = t$ and satisfy the same differential equation in $(d/d\tau)$). So

$$\begin{aligned} u(2t-\Delta-\tau') &= -\tilde{R}^{-1}(2t-\Delta-\tau')\, \tilde{\phi}^T(t-\Delta, 2t-\Delta-\tau') W_R^{-1}\, x(t-\Delta) \\ &= -\tilde{R}^{-1}(2t-\Delta-\tau')\, \tilde{\phi}^T(\tau', t)\, W_R^{-1}\, x(t-\Delta) \end{aligned}$$

and

$$J = -\underbrace{\left(\int_{t-\Delta}^{t} \Phi^T(\tau', t) H^T(\tau')\, R^{-1}(\tau)\, H(\tau')\, \Phi(\tau', t)\, d\tau'\right)}_{W_R} W_R^{-1}\, x(t-\delta)$$

$$= -x(t-\delta).$$

So $x(t) = \tilde{\Phi}(s, t-\Delta)\ (x(t-\Delta) - x(t-\Delta)) = 0$. Notice that, this allows us to write $x(s)$ in a more compact form:

$$x(s) = -\tilde{\Phi}(s, t-\Delta) \int_s^t \tilde{\Phi}(t-\Delta, \tau)\ \tilde{H}^T(\tau)\ u(\tau)\ d\tau \tag{18}$$

Replacing $u(\tau)$, we can write:

$$x(s) = \tilde{\Phi}(s, t-\Delta) \left(\int_s^t \tilde{\Phi}(t-\Delta, \tau)\ \tilde{H}(\tau)\ \tilde{R}^{-1}(\tau)\ \tilde{\Phi}(t-\Delta, \tau)\ d\tau \right) \times W_R^{-1}\ x(t-\Delta) \tag{19}$$

$$= \tilde{\Phi}(s, t-\Delta)\Gamma(s)\ W_R^{-1}\ x(t-\Delta) \tag{20}$$

following our notations. So

$$\int_{t-\Delta}^t x^T(s)\ \tilde{Q}(s)\ x(s)\ ds = x^T(t-\Delta)\ W_R^{-1}\ I_2\ W_R^{-1}\ x(t-\Delta)$$

with

$$I_2 = \int_{t-\Delta}^t \Gamma(s)\ \tilde{\Phi}(s, t-\Delta)\ \tilde{Q}(s)\ \tilde{\Phi}(s, t-\Delta)\ \Gamma(s)\ ds \tag{21}$$

and we have

$$x^T(t-\Delta)\ P(t)\ x(t-\Delta) \leq x^T(t-\Delta)\ (W_R^{-1} + W_R^{-1}\ I_2\ W_R^{-1})\ x(t-\Delta) \tag{22}$$

whatever the choice of the initial condition $x(t-\Delta)$ in system. This prove the right hand side of inequality (10)

In order to obtain the left inequality, we remark that P^{-1} satisfies the Riccati equation:

$$\frac{d}{d\tau} P^{-1} = -P^{-1}\ F - F^T\ P^{-1} + H^T\ R^{-1}\ H - P^{-1}\ G\ Q\ G^T\ P^{-1}$$

So the proof of the left hand side of inequality (10) will be the same as the previous one using now the contollability assumption. So we have

$$I_2 = \int_{t-\delta}^t \Omega(s)\ \tilde{\Phi}^T(s, t-\Delta)\ \tilde{R}^{-1}\ \tilde{\Phi}(s, t-\delta)\ \Omega(s)\ ds \tag{23}$$

where

$$\Omega(s) = \int_s^t \tilde{\Phi}(t-\Delta,\tau)\ \tilde{G}(\tau)\ \tilde{Q}(\tau)\ \tilde{G}^T(\tau)\ \tilde{\Phi}(t-\Delta,\tau)\ d\tau \qquad (24)$$

$\tilde{\Phi}$ being now the fundamental matrix of $\tilde{F}$ of the linear system

$$\frac{d_x}{d\tau} = \tilde{F}(\tau)\ x(\tau) + \tilde{G}(\tau)\ u(\tau). \qquad (25)$$

To finish the proof, observe that if the inputs of the matrix $F(t)$ are bounded, it is a trivial matter to show that the same is true for the fondamental matrix. More preciselly, if $\|F(t)\| \leq k$ for all t, we have that $\|\Phi(t,t_0)\| \leq \alpha\ e^{k(t-t_0)}(\alpha = \|I_d\|)$. It is then easy to give an estimation of matrix I_1 and I_2. ■

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