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Polar Decomposition Approach To Reid's Inequality

C.-S. LIN*

Dedicated to Professor Gustavus E. Huige on his retirement

Department of Mathematics, Bishop's University, Lennoxville, PQ J1M 1Z7, Canada

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If $S \geq 0$ and SK is Hermitian, then $|(SKx, x)| \leq \|K\|(Sx, x)$ holds for all $x \in H$, which is known as Reid's inequality and was sharpened by Halmos in which $\|K\|$ was replaced by $r(K)$, the spectral radius of K. In this article we present generalizations of Reid's and Halmos' inequalities via polar decomposition approach. Conditions on S and SK are relaxed. Theorem 1 regards Reid-type inequalities, and Theorem 2 contains Halmos-type inequalities.

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Throughout the paper we use capital letters to denote bounded linear operators on a Hilbert space H. T is positive (written $T \geq O$) in case $(Tx, x) \geq 0$ for all $x \in$ H. If S and T are Hermitian, we write $T \geq S$ in case $T - S \geq O$. $T = U|T|$ is the polar decomposition of T with U the partial isometry such that $N(U) = N(T)$ ($N(A)$ means the null space of A), and $|T|$ the positive square root of the positive operator T^*T, *i.e.*, $|T| = (T^*T)^{1/2}$. Also, we have $T^* = |T|U^*$ and $|T^*| = (TT^*)^{1/2}$ with $N(U^*) = N(T^*)$. Recall that if $S \geq O$ and SK is Hermitian, then the inequality $|(SKx, x)| \leq \|K\|(Sx, x)$ holds for all $x \in$ H. This is known as Reid's inequality [7], and was sharpened by Halmos

* E-mail: plin@ubishops.ca

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[2] in which $\|K\|$ was replaced by $r(K)$, the spectral radius of K. Recently, the sharpened inequality was extended in [4], and the equivalence relation with the Furuta inequality appeared in [5] in which it is assumed that $S \geq O$ and SK is Hermitian in every result.

We shall prove in this paper the inequality by the polar decomposition approach, which also enables us to relax conditions on S and SK. In other words, we present generalizations of Reid's and Halmos' inequalities. More precisely, Theorem 1 regards Reid-type inequalities, and Theorem 2 contains Halmos-type inequalities. In the proof we require the Löwner–Heinz formula, *i.e.*, $A^r \geq B^r$ holds for $r \in [0, 1]$ if $A \geq B \geq O$ [3], but the inequality does not hold in general for $r > 1$. We also need some basic properties of the polar decomposition, *i.e.*, if $T = U|T|$ as in above, then $U^*U = I$, the identity operator, and $|T^*|^c = U|T|^cU^*$ for $c > 0$. Our basic tool is the next result which is interesting by itself. In spite of our simple proof by direct replacements, (ii) in Lemma 1 below was shown without the bound in [1, Theorem 1], and equality conditions were discussed depending on the value of α.

LEMMA 1 *For an arbitrary operator T and for $a, b, x, y \in \mathrm{H}$ and $\alpha \in [0, 1]$, the following are equivalent.*

(i) $|(a, b)| \leq \|a\|\|b\|$ *(Cauchy–Schwarz inequality).*
Equality holds if and only if $a = \delta b$ for suitable δ. Moreover, the bound of inequality is

$$\frac{\|a\|^2\|b\|^2 - |(a, b)|^2}{\|a\|^2} \leq \frac{\|\beta b - a\|^2}{\beta^2}$$

for any real number $\beta \neq 0$ and $a \neq 0$.

(ii) $|(Tx, y)|^2 \leq (|T|^{2\alpha}x, x)(|T^*|^{2(1-\alpha)}y, y)$.
Equality holds if and only if $U|T|^{\alpha}x = \delta|T^*|^{1-\alpha}y$ for suitable δ. Moreover, the bound of inequality is

$$\frac{(|T|^{2\alpha}x, x)(|T^*|^{2(1-\alpha)}y, y) - |(Tx, y)|^2}{(|T|^{2\alpha}x, x)} \leq \frac{\|\beta|T^*|^{1-\alpha}y - U|T|^{\alpha}x\|^2}{\beta^2}$$

for any real number $\beta \neq 0$ and $|T|^{\alpha}x \neq 0$.

Proof Remark that the bound in (i) was proved in [6]. (i) implies (ii). All we have to do is replacing a and b in (i) by $U|T|^{\alpha}x$ and $|T^*|^{1-\alpha}y$, respectively, and simplifying them due to the basic properties of the polar decomposition. More precisely,

$$\begin{aligned}(a, b) &= (U|T|^{\alpha}x, |T^*|^{1-\alpha}y) = (U|T|^{\alpha}x, U|T|^{1-\alpha}U^*y)\\ &= (U|T|x, y) = (Tx, y);\end{aligned}$$

and

$$\begin{aligned}\|a\|^2\|b\|^2 &= (U|T|^{\alpha}x, U|T|^{\alpha}x)(|T^*|^{1-\alpha}y, |T^*|^{1-\alpha}y)\\ &= (|T|^{2\alpha}x, x)(|T^*|^{2(1-\alpha)}y, y).\end{aligned}$$

(ii) implies (i). Let $T = I$, $x = a$ and $y = b$ in (ii).

A different proof of (ii) in Lemma 1 is possible by letting $a = |T|^{\alpha}x$ and $b = |T|^{1-\alpha}U^*y$. Incidentally, from (ii) in Lemma 1 we have $|(Tx, x)| = (|T|x, x)$ for any Hermitian operator T and any $x \in$ H. Notice that the Cauchy–Schwarz inequality for positive S is the relation $|(Sx, y)|^2 \leq (Sx, x)(Sy, y)$, which is obviously a special case of (ii) in Lemma 1. If $\alpha = 1/2$ in particular, inequality (ii) is precisely Problem 138 in [2].

LEMMA 2 *Let $SK = V|SK|$ be the polar decomposition. Then the following inequalities hold for every $x \in$ H and $\alpha \in [0, 1]$.*

(1) $(|SK|^{2\alpha}x, x) \leq \|S\|^{2\alpha}(|K|^{2\alpha}x, x)$.
(2) $(|(SK)^*|^{2\alpha}x, x) \leq \|K\|^{2\alpha}(|S^*|^{2\alpha}x, x)$.
(3) $(|SK|^{2\alpha}x, x) \leq \|K\|^{2\alpha}(|S^*|^{2\alpha}x, x)$ *if SK is Hermitian.*
(4) $(|SK|^{2\alpha}x, x) \leq \|K\|^{2\alpha}(|S|^{2\alpha}x, x)$ *if both S and SK are Hermitian.*
(5) $(|SK|^{2\alpha}x, x) \leq \|K\|^{2\alpha}(S^{2\alpha}x, x)$ *if $S \geq O$ and SK is Hermitian.*

Moreover, the power 2α in above inequalities may be replaced by the power $2(1-\alpha)$ without changing inequalities.

Proof (1) Since the operator $S/\|S\|$ is a contraction, *i.e.*, $S^*S \leq \|S\|^2$,

$$0 \leq \frac{|SK|^2}{\|S\|^2} = \frac{K^*S^*SK}{\|S\|^2} \leq K^*K = |K|^2,$$

so that $0 \leq |SK|^2 \leq \|S\|^2|K|^2$. It follows that $|SK|^{2\alpha} \leq \|S\|^{2\alpha}|K|^{2\alpha}$ by the Löwner–Heinz formula, and we have inequality (1).

(2) The proof is similar to (1) if we start with $KK^* \leq \|K\|^2$ since $K/\|K\|$ is a contraction. The relations

$$0 \leq \frac{|(SK)^*|^2}{\|K\|^2} = \frac{SKK^*S^*}{\|K\|^2} \leq SS^* = |S^*|^2$$

imply (2).

It is easily seen that all (3), (4) and (5) follow from (2), and the last statement is clear.

THEOREM 1 *Let $SK = V|SK|$ be the polar decomposition. Then the following inequalities hold for every $x, y \in \mathrm{H}$ and $\alpha \in [0, 1]$.*

(1) $$\begin{aligned}|(SKx, y)|^2 &\leq \|K\|^{2(1-\alpha)}(|SK|^{2\alpha}x, x)(|S^*|^{2(1-\alpha)}y, y)\\ &\leq \|S\|^{2\alpha}\|K|^{2(1-\alpha)}(|K|^{2\alpha}x, x)(|S^*|^{2(1-\alpha)}y, y).\end{aligned}$$

(2) $$\begin{aligned}|(SKx, y)|^2 &\leq \|S\|^{2\alpha}(|K|^{2\alpha}x, x)(|(SK)^*|^{2(1-\alpha)}y, y)\\ &\leq \|S\|^{2\alpha}\|K\|^{2(1-\alpha)}(|K|^{2\alpha}x, x)(|S^*|^{2(1-\alpha)}y, y).\end{aligned}$$

(3) *If SK is Hermitian, then*

$$\begin{aligned}|(SKx, y)|^2 &\leq \|K\|^{2\alpha}(|S^*|^{2\alpha}x, x)(|SK|^{2(1-\alpha)}y, y)\\ &\leq \|K\|^2(|S^*|^{2\alpha}x, x)(|S^*|^{2(1-\alpha)}y, y);\ \textit{and}\end{aligned}$$

$$\begin{aligned}|(SKx, y)|^2 &\leq \|K\|^{2(1-\alpha)}(|SK|^{2\alpha}x, x)(|S^*|^{2(1-\alpha)}y, y)\\ &\leq \|K\|^2(|S^*|^{2\alpha}x, x)(|S^*|^{2(1-\alpha)}y, y).\end{aligned}$$

(4) *If both S and SK are Hermitian, then*

$$\begin{aligned}|(SKx, y)|^2 &\leq \|K\|^{2\alpha}(|S|^{2\alpha}x, x)(|SK|^{2(1-\alpha)}y, y)\\ &\leq \|K\|^2(|S|^{2\alpha}x, x)(|S|^{2(1-\alpha)}y, y);\ \textit{and}\end{aligned}$$

$$\begin{aligned}|(SKx, y)|^2 &\leq \|K\|^{2(1-\alpha)}(|SK|^{2\alpha}x, x)(|S|^{2(1-\alpha)}y, y)\\ &\leq \|K\|^2(|S|^{2\alpha}x, x)(|S|^{2(1-\alpha)}y, y).\end{aligned}$$

(5) *If $S \geq O$ and SK is Hermitian, then*

$$\begin{aligned}|(SKx,y)|^2 &\leq \|K\|^{2\alpha}(S^{2\alpha}x,x)(|SK|^{2(1-\alpha)}y,y)\\ &\leq \|K\|^2(S^{2\alpha}x,x)(S^{2(1-\alpha)}y,y); \textit{ and}\\ |(SKx,y)|^2 &\leq \|K\|^{2(1-\alpha)}(|SK|^{2\alpha}x,x)(S^{2(1-\alpha)}y,y)\\ &\leq \|K\|^2(S^{2\alpha}x,x)(S^{2(1-\alpha)}y,y).\end{aligned}$$

Proof Firstly we notice that the inequality

$$|(SKx,y)|^2 \leq (|SK|^{2\alpha}x,x)(|(SK)^*|^{2(1-\alpha)}y,y)$$

holds by Lemma 1. It follows that inequalities (1) and (2) in Lemma 2 imply both (1) and (2) in Theorem 1. Each other inequality above follows from the corresponding inequality in Lemma 2 and we shall omit the details.

In particular let $y = x$ and $\alpha = 1/2$ in (5) of Theorem 1. Then we obtain Reid's inequality. We now consider sharpening of inequalities (3), (4) and (5) in Theorem 1, *i.e.*, replacing the norm of an operator by its spectral radius.

THEOREM 2 *Let $SK = V|SK|$ be the polar decomposition. Then the following inequalities hold for every $x, y \in$ H and $\alpha \in [0, 1]$.*

(1) *If $|S|^{2\alpha}K$ is Hermitian, then*

$$|(SKx,y)|^2 \leq [r(K)]^2(|S|^{2\alpha}x,x)(|S^*|^{2(1-\alpha)}y,y).$$

(2) *If both S and $|S|^{2\alpha}K$ are Hermitian, then*

$$|(SKx,y)|^2 \leq [r(K)]^2(|S|^{2\alpha}x,x)(|S|^{2(1-\alpha)}y,y).$$

(3) *If $S \geq O$ and $S^{2\alpha}K$ is Hermitian, then*

$$|(SKx,y)|^2 \leq [r(K)]^2(S^{2\alpha}x,x)(S^{2(1-\alpha)}y,y).$$

Proof *(1)* If $|S|^{2\alpha}K$ is Hermitian, *i.e.*, $K^*|S|^{2\alpha} = |S|^{2\alpha}K$, then clearly

$$(K^*)^n|S|^{2\alpha} = |S|^{2\alpha}K^n$$

for $n = 1, 2, \ldots$. Next we claim that

$$|(SKx, y)|^{2^n} \le (|S|^{2\alpha}K^{2^n}x, x)(|S|^{2\alpha}x, x)^{2^{n-1}-1}(|S^*|y, y)^{2^{n-1}},$$

and the proof will be done by induction. If $n = 1$, then

$$|(SKx, y)|^2 \le (|S|^{2\alpha}Kx, Kx)(|S^*|^{2(1-\alpha)}y, y)$$

by Lemma 1, which yields $|(SKx, y)|^2 \le (|S|^{2\alpha}K^2x, x)(|S^*|^{2(1-\alpha)}y, y)$. Now,

$$\begin{aligned}
|(SKx, y)|^{2^{n+1}} &= [|(SKx, y)|^{2^n}]^2 \\
&\le (|S|^{2\alpha}K^{2^n}x, x)^2(|S|x, x)^{2^n-2}(|S^*|^{2(1-\alpha)}y, y)^{2^n} \\
&\le (|S|^{2\alpha}K^{2^n}x, K^{2^n}x)(|S|^{2\alpha}x, x)(|S|^{2\alpha}x, x)^{2^n-2}(|S^*|^{2(1-\alpha)}y, y)^{2^n} \\
&= (|S|^{2\alpha}K^{2^{n+1}}x, x)(|S|^{2\alpha}x, x)^{2^n-1}(|S^*|^{2(1-\alpha)}y, y)^{2^n}.
\end{aligned}$$

Note that the second inequality above is due to Lemma 1, and the induction process is done. It follows that

$$|(SKx, y)|^{2^n} \le \||S|^{2\alpha}\| \|K^{2^n}\| \|x\|^2(|S|^{2\alpha}x, x)^{2^{n-1}-1}(|S^*|^{2(1-\alpha)}y, y)^{2^{n-1}},$$

which gives us

$$\begin{aligned}
|(SKx, y)| \le{}& \||S|^{2\alpha}\|^{1/2^n}\|K^{2^n}\|^{1/2^n}\|x\|^{2/2^n}(|S|^{2\alpha}x, x)^{1/2-1/2^n} \\
&\times (|S^*|^{2(1-\alpha)}y, y)^{1/2} \to r(K)(|S|^{2\alpha}x, x)^{1/2} \\
&\times (|S^*|^{2(1-\alpha)}y, y)^{1/2} \quad \text{as } n \to \infty,
\end{aligned}$$

and the inequality (1) follows.

Obviously inequalities (2) and (3) are special cases of (1) and the proof is finished.

In particular let $y = x$ and $\alpha = 1/2$ in (3) of Theorem 2. Then we obtain Halmos' inequality. It seems that there is no sharpening for (1) or (2) in Theorem 1 if no other conditions are attached to operators S and/or SK. Let us pose this as an open question, *i.e.*, in Theorem 1 can we replace the term $\|K\|^{2(1-\alpha)}$ in (1) by $r(K)^{2(1-\alpha)}$ and the term $\|S\|^{2\alpha}$ in (2) by $r(S)^{2\alpha}$? However, we know by the Cauchy–Schwarz inequality that $|(SKx, y)| \le \|SK\| \|x\| \|y\|$. Here $\|SK\|$ may be replaced by a weaker condition $r((SK)^*SK)^{1/2}$ as the following shows. For any operator E we claim by induction that

$$|(Ex, y)|^{2^n} \leq ((E^*E)^{2^{n-1}}x, x)\|x\|^{2^n-2}\|y\|^{2^n}$$

for every x, $y \in H$ and $n \geq 1$. It follows that $|(Ex, y)|^2 \leq \|(E^*E)^{2^{n-1}}\|^{1/2^{n-1}}\|x\|^2\ \|y\|^2$; and passing to the limit as $n \to \infty$ we obtain

$$|E(x, y)|^2 \leq r(E^*E)\|x\|^2\|y\|^2.$$

We leave the details to the readers.

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