Hindawi Publishing Corporation
Journal of Probability and Statistics
Volume 2010, Article ID 846320, 18 pages
doi:10.1155/2010/846320

*Research Article*

# An Explicit Representation of the Extended Skorokhod Map with Two Time-Dependent Boundaries

**Marek Slaby**

*Department of Mathematics, Computer Science and Physics, Fairleigh Dickinson University, 285 Madison Ave, M-AB2-02, Madison, NJ 07940, USA*

Correspondence should be addressed to Marek Slaby, mslaby@fdu.edu

Received 28 December 2009; Accepted 27 April 2010

Academic Editor: A. Thavaneswaran



We consider the extended Skorokhod problem for real-valued càdlàg functions with the constraining interval $[\alpha, \beta]$, where $\alpha$ and $\beta$ change in time as values of two càdlàg functions. We find an explicit form of the solution and discuss its continuity properties with respect to the uniform, $J_1$ and $M_1$, metrics on the space of càdlàg functions. We develop a useful technique of extending known results for the Skorokhod maps onto the larger class of extended Skorokhod maps.

## 1. Introduction

The Skorokhod problem (SP) was introduced originally in [1] as a tool for solving stochastic differential equations in a domain with one fixed reflective boundary. Given a function $\psi$, a solution of the Skorokhod problem on $[0, \infty)$ is a pair of functions $(\phi, \eta)$ such that $\phi \geq 0$, $\eta$ is nondecreasing, and $\int_0^\infty I_{\{\phi(s)>0\}} d\eta(s) = 0$. The mapping $\Gamma_0(\psi) = \phi$ is called a Skorokhod map (SM) on $[0, \infty)$ and is well defined for all càdlàg functions $\psi$.

The Skorokhod problem has been studied since [1] in more general settings. Chaleyat-Maural et al. studied in [2] a Skorokhod map constraining functions to $[0, a]$. A multidimensional version of the SP was introduced by Tanaka in [3]. Over the years, numerous applications were found for the SM particularly in queueing theory. In [4], Kruk et al. provided an explicit formula and studied the properties of the two-sided Skorokhod map constraining the function to remain in the interval $[0, a]$, where $a$ is a positive constant. More recently attempts were made towards relaxing the rigidity of the constrains. Burdzy et al. in [5] found an explicit representation for the so called extended Skorokhod map (ESM), which is a relaxed version of the SM. [The constraining interval in that paper varies with time.] Another explicit representation for the SM with two time-dependent boundaries, different from the representation in [5] and based on the approach in [4], was developed by the author in [6]. In addition, a number of properties of the SM were studied in [6] including its continuity and Lipschitz conditions.

In this paper we obtain an alternative form of the explicit formula for the ESM with two time-dependent boundaries developed in [5] that is simpler to understand and potentially more useful for applications and generalizations to higher dimensions. We develop methods of extending certain properties of the SM onto the ESM and use them to analyze continuity properties of the ESM.

Throughout the paper, $D[0,\infty)$ will denote real-valued càdlàg functions on $[0,\infty)$, $D^-[0,\infty)$, and $D^+[0,\infty)$ will denote càdlàg functions on $[0,\infty)$ taking values in $\mathbb{R}\cup\{-\infty\}$ and in $\mathbb{R}\cup\{\infty\}$, respectively. A function is càdlàg if it is right-continuous and has finite left limits at every $t\geq 0$. Similarly, we will use $I[0,\infty)$ and $\mathrm{BV}[0,\infty)$ to denote subspaces of $D[0,\infty)$ consisting of nondecreasing functions and functions with bounded variation on every finite interval, respectively. We will use $D_f$ to denote the effective domain of $f$, that is, $D_f = \{t\geq 0 \mid \infty < f(t) < \infty\}$.

On the space of the càdlàg functions we will consider the topology of the uniform convergence and the topology of the uniform convergence on compact sets. For every $T>0$, let $\|f\|_T = \sup_{0\leq t\leq T}|f(t)|$ and $\|f\| = \sup_{t\geq 0}|f(t)|$. Let $f_n$ be a sequence of functions in $D^+[0,\infty)$ or in $D^-[0,\infty)$. We say that $f_n$ converges to $f$ uniformly on compact sets if for every $T>0$, $\lim_{n\to\infty}\|f_n - f\|_T = 0$. Equivalently, we could say that $f_n$ converges to $f$ uniformly on compact sets if $f_n = f$ on $D_f^c$ for large enough $n$ and $f_n$ converges to $f$ uniformly on $[0,T]\cap D_f$ for every $T>0$.

*Definition 1.1* (extended Skorokhod problem). Let $\alpha\in D^-[0,\infty)$, $\beta\in D^+[0,\infty)$ be such that $\alpha\leq\beta$, and let $\psi\in D[0,\infty)$. A pair of real-valued càdlàg functions, $(\phi,\eta)$, is said to be a solution of the extended Skorokhod problem (ESP) on $[\alpha,\beta]$ for $\psi$ if the following three properties are satisfied:

(i) for every $t\geq 0$, $\phi(t) = \psi(t) + \eta(t) \in [\alpha(t),\beta(t)]$;

(ii) for every $0\leq s\leq t$,

$$\begin{aligned} &\eta(t)-\eta(s)\geq 0 \quad \text{if } \phi(r)<\beta(r)\ \forall r\in(s,t],\\ &\eta(t)-\eta(s)\leq 0 \quad \text{if } \phi(r)>\alpha(r)\ \forall r\in(s,t]; \end{aligned} \tag{1.1}$$

(iii) for every $t\geq 0$,

$$\begin{aligned} &\eta(t)-\overline{\eta}(t-)\geq 0 \quad \text{if } \phi(t)<\beta(t),\\ &\eta(t)-\eta(t-)\leq 0 \quad \text{if } \phi(t)>\alpha(t). \end{aligned} \tag{1.2}$$

Traditionally $\eta(0-)$ in (iii) is defined as zero so that $\eta$ has a jump at 0, whenever $\eta(0)>0$. The map $\overline{\Gamma}_{\alpha,\beta} : D[0,\infty)\to D[0,\infty)$ defined by $\overline{\Gamma}_{\alpha,\beta}(\psi)=\phi$ is called the extended Skorokhod map (ESM) on $[\alpha,\beta]$. In the traditional Skorokhod problem, conditions (ii) and (iii) are replaced by a stronger condition.

*Definition 1.2* (Skorokhod problem). Let $\alpha, \beta \in D[0,\infty)$ be such that $\alpha \leq \beta$. Given $\psi \in D[0,\infty)$, a pair of functions $(\phi, \eta) \in D[0,\infty) \times \mathrm{BV}[0,\infty)$ is said to be a solution of the Skorokhod problem on $[\alpha, \beta]$ for $\psi$ if the following two properties are satisfied:

(i) for every $t \in [0,\infty)$, $\phi(t) = \psi(t) + \eta(t) \in [\alpha(t), \beta(t)]$;

(ii) $\eta(0-) = 0$ and $\eta$ have the decomposition $\eta = \eta_l - \eta_u$, where $\eta_l, \eta_u \in I[0,\infty)$,

$$\int_0^\infty I_{\{\phi(s)>\alpha(t)\}} d\eta_l(s) = 0, \qquad \int_0^\infty I_{\{\phi(s)<\beta(s)\}} d\eta_u(s) = 0. \tag{1.3}$$

Burdzy et al. have shown in Theorem 2.6 of [5] that for any $\alpha \in D^-[0,\infty)$ and $\beta \in D^+[0,\infty)$ such that $\alpha \leq \beta$, there is a well-defined ESM $\overline{\Gamma}_{\alpha,\beta}$ and it is represented by

$$\overline{\Gamma}_{\alpha,\beta}(\psi) = \psi - \Xi_{\alpha,\beta}(\psi), \tag{1.4}$$

where $\Xi_{\alpha,\beta}(\psi) : D[0,\infty) \to D[0,\infty)$ is given by

$$\begin{aligned} \Xi_{\alpha,\beta}(\psi)(t) = \max\Bigg\{ &\left[ \left(\psi(0) - \beta(0)\right)^+ \wedge \inf_{0\leq r\leq t} \left(\psi(r) - \alpha(r)\right) \right], \\ &\sup_{0\leq s\leq t} \left[ \left(\psi(s) - \beta(s)\right) \wedge \inf_{s\leq r\leq t} \left(\psi(r) - \alpha(r)\right) \right] \Bigg\}. \end{aligned} \tag{1.5}$$

They obtained their result first for simple functions and then extended it by the limiting process. In Section 2 we will develop an alternative version of this formula.

It is easy to see that if $(\phi, \eta)$ is a solution of the SP on $[\alpha, \beta]$ for $\psi$, then it is also a solution of the ESP on $[\alpha, \beta]$ for $\psi$. Conversely, it is shown in Proposition 2.3 of [5] that a solution $(\phi, \eta)$ of the ESP solves also the corresponding SP whenever $\eta \in \mathrm{BV}[0,\infty)$. Furthermore, Corollary 2.4 of [5] shows that if $\inf_{t\geq 0}(\beta(t) - \alpha(t)) > 0$ and $(\phi, \eta)$ is a solution of the ESP on $[\alpha, \beta]$ for $\psi$, then $\eta \in \mathrm{BV}[0,\infty)$. Therefore we can identify the ESM with the SM in this special case.

*Remark 1.3.* If $\inf_{t\geq 0}(\beta(t) - \alpha(t)) > 0$, then $\overline{\Gamma}_{\alpha,\beta} = \Gamma_{\alpha,\beta}$.

## 2. Alternative Explicit Formula for the Two-Sided Extended Skorokhod Map with Time-Dependent Boundaries

We will make the explicit formula (1.4) more user friendly by developing a new expression for the constraining term $\Xi$ that is easier to understand and shows more promise for possible extensions to higher dimensions than (1.5). Given $\alpha \in D^-[0,\infty)$ and $\beta \in D^+[0,\infty)$ such that $\alpha \leq \beta$, we introduce two pairs of times

$$T_\alpha = \min\{t > 0 \mid \alpha(t) - \psi(t) \geq 0\}, \qquad T^\beta = \min\{t > 0 \mid \psi(t) - \beta(t) \geq 0\}, \tag{2.1}$$

$$\tau_\alpha = \inf\{t > 0 \mid \alpha(t) - \psi(t) > 0\}, \qquad \tau^\beta = \inf\{t > 0 \mid \psi(t) - \beta(t) > 0\}. \tag{2.2}$$

Note that

$$T_\alpha \le \tau_\alpha, \qquad T^\beta \le \tau^\beta. \tag{2.3}$$

Also note that the four times depend on $\psi$. When necessary we will indicate it by using full notation such as $\tau_\alpha(\psi)$ or $\tau^\beta(\psi)$.

*Remark 2.1.* Let $\psi \in D[0,\infty)$, $\alpha, \alpha_n, \in D^-[0,\infty)$, $n = 1,2,3,\ldots$, and $\beta, \beta_n \in D^+[0,\infty)$, $n = 1,2,3,\ldots$. If $\alpha_n \xrightarrow{n\to\infty} \alpha$ and $\beta_n \xrightarrow{n\to\infty} \beta$ pointwise, then

$$\limsup_{n\to\infty} \tau^{\beta_n} \le \tau^\beta, \qquad \limsup_{n\to\infty} \tau_{\alpha_n} \le \tau_\alpha. \tag{2.4}$$

*Proof.* Let $\epsilon > 0$. By (2.2) there is $\tau^\beta \le t_0 < \tau^\beta + \epsilon$ such that $\psi(t_0) - \beta(t_0) > 0$. Let $\psi(t_0) - \beta(t_0) = \delta$. There is $n_0$ such that $|\beta_n(t_0) - \beta(t_0)| < \delta/2$ for $n \ge n_0$. Hence

$$\psi(t_0) - \beta_n(t_0) = \psi(t_0) - \beta(t_0) + \beta(t_0) - \beta_n(t_0) \ge \delta_0 - \left|\beta(t_0) - \beta_n(t_0)\right| \ge \frac{\delta_0}{2} > 0. \tag{2.5}$$

Thus $\tau^{\beta_n} \le t_0 < \tau^\beta + \epsilon$. Since $\epsilon$ is arbitrary, we conclude that $\limsup_{n\to\infty} \tau^{\beta_n} \le \tau^\beta$. By similar argument we can show that $\limsup_{n\to\infty} \tau_{\alpha_n} \le \tau_\alpha$. □

The inequalities in Remark 2.1 distinguish $\tau_\alpha$ and $\tau^\beta$ from $T_\alpha$ and $T^\beta$ and will be essential for the proof of Theorem 2.11.

*Remark 2.2.* Let $\alpha \in D^-[0,\infty)$ and $\beta \in D^+[0,\infty)$ be such that $\inf_{t\ge 0}(\beta(t) - \alpha(t)) > 0$. For any $\psi \in D[0,\infty)$, there are three possibilities

$$\text{either} \quad \alpha(t) < \psi(t) < \beta(t) \quad \text{for every } t \ge 0, \quad T_\alpha < T^\beta \quad \text{or} \quad T^\beta < T_\alpha. \tag{2.6}$$

Similarly, in terms of $\tau_\alpha$ and $\tau^\beta$, the following three cases are possible:

$$\text{either} \quad \alpha(t) \le \psi(t) \le \beta(t) \quad \text{for every } t \ge 0, \quad \tau_\alpha < \tau^\beta \quad \text{or} \quad \tau^\beta < \tau_\alpha. \tag{2.7}$$

Clearly, $\alpha(t) < \psi(t) < \beta(t)$ for every $t \ge 0$ if and only if $T_\alpha = T^\beta = \infty$, and $\alpha(t) \le \psi(t) \le \beta(t)$ for every $t \ge 0$ if and only if $\tau_\alpha = \tau^\beta = \infty$.

*Remark 2.3.* It follows from the definition of the ESM that for every $0 \le t \le \tau_\alpha \wedge \tau^\beta$, $\alpha(t) \le \psi(t) \le \beta(t)$, $\overline{\Gamma}_{\alpha,\beta}(\psi)(t) = \psi(t)$ and $\Xi_{\alpha,\beta}(\psi)(t) = 0$. Similarly, we obtain that for every $0 \le t < T_\alpha \wedge T^\beta$, $\alpha(t) < \psi(t) < \beta(t)$ and $\overline{\Gamma}_{\alpha,\beta}(\psi)(t) = \psi(t)$.

Under the assumption that $\inf_{t\geq 0}(\beta(t)-\alpha(t))>0$, we define two increasing sequences of times $\{T_k \mid k=0,1,2,\ldots\}$ and $\{S_k \mid k=0,1,2,\ldots\}$ similar to the sequences $\tau_k$, $\sigma_k$ used in [6]. If $T^\beta < T_\alpha$, we set $T_0=0$ and $S_0=T^\beta$; for $k\geq 1$, we set

$$T_k = \min\left\{ t > S_{k-1} \mid \sup_{S_{k-1}\leq s\leq t}\left[\psi(s)-\beta(s)\right] \geq \psi(t)-\alpha(t)\right\}, \tag{2.8}$$

$$S_k = \min\left\{ t > T_k \mid \psi(t)-\beta(t) \geq \inf_{T_k\leq r\leq t}\left(\psi(r)-\alpha(r)\right)\right\}. \tag{2.9}$$

If $T_\alpha < T^\beta$, we set $T_0 = T_\alpha$, we define $S_k$ for all $k$ by (2.9), and we define $T_k$ for $k\geq 1$ by (2.8).

It is easy to see that $0\leq T_0\leq S_0 < T_1 < S_1 < T_2 < S_2 < \cdots$ unless one of the times equals $\infty$, at which point all the following times are also $\infty$. Also note that the time sequences depend on $\psi$, $\alpha$ and $\beta$. Finally, as in Proposition 2.1 of [6], if $\inf_{t\geq 0}(\beta(t)-\alpha(t))>0$, then

$$\lim_{k\to\infty} T_k = \infty, \qquad \lim_{k\to\infty} S_k = \infty. \tag{2.10}$$

The following observations follow immediately from the definition of $S_k$.

*Remark 2.4.* If $k>1$ or if $k=0$ and $T_\alpha < T^\beta$, then

$$\psi(S_k)-\beta(S_k) \geq \inf_{T_k\leq r\leq S_k}\left(\psi(r)-\alpha(r)\right) \tag{2.11}$$

and for every $s\in[T_k,S_k)$

$$\psi(s)-\beta(s) < \inf_{T_k\leq r\leq s}\left(\psi(r)-\alpha(r)\right). \tag{2.12}$$

If $T^\beta < T_\alpha$, then

$$\psi(S_0)-\beta(S_0)\geq 0, \tag{2.13}$$

and for every $s\in[T_0,S_0)$,

$$\psi(s)-\beta(s)<0. \tag{2.14}$$

Similarly, by definition of $T_k$, we make the following conclusions.

*Remark 2.5.* If $k>1$, then

$$\psi(T_k)-\alpha(T_k) \leq \sup_{S_{k-1}\leq s\leq T_k}\left[\psi(s)-\beta(s)\right], \tag{2.15}$$

and for every $t \in [S_{k-1}, T_k)$

$$\sup_{S_{k-1} \le s \le t} \left[\psi(s) - \beta(s)\right] < \psi(t) - \alpha(t). \tag{2.16}$$

If $T_\alpha < T^\beta$, then

$$\alpha(T_0) - \psi(T_0) \ge 0, \tag{2.17}$$

and for every $s \in [0, T_0)$,

$$\alpha(s) - \psi(s) < 0. \tag{2.18}$$

It follows from (2.16) that $[\psi(s) - \beta(s)] < [\psi(t) - \alpha(t)]$ whenever $S_{k-1} \le s \le t < T_k$, $k \ge 1$. Therefore

$$\left[\psi(s) - \beta(s)\right] \le \inf_{s \le r < T_k} \left[\psi(r) - \alpha(r)\right] \quad \text{whenever} \quad S_{k-1} \le s < T_k. \tag{2.19}$$

Also note that by (2.11),

$$\inf_{T_{k-1} \le r \le S_{k-1}} \left(\psi(r) - \alpha(r)\right) \le \sup_{S_{k-1} \le s \le t} \left[\psi(s) - \beta(s)\right]. \tag{2.20}$$

The following result establishes a straight-forward representation for the constraining term $\Xi_{\alpha,\beta}$ of the ESM similar to the representation of Theorem 2.2 of [6].

**Theorem 2.6.** *Let* $\alpha \in D^-[0,\infty)$, $\beta \in D^+[0,\infty)$ *be such that* $\inf_{t \ge 0}(\beta(t) - \alpha(t)) > 0$, *let* $\psi \in D[0,\infty)$ *and let* $\Xi$ *be defined by* (1.5).
*If* $T^\beta < T_\alpha$, *then*

$$\Xi_{\alpha,\beta}(\psi)(t) = \begin{cases} 0 & \text{if } 0 \le t < S_0, \\ \sup\limits_{S_{k-1} \le s \le t} \left[\psi(s) - \beta(s)\right] & \text{if } S_{k-1} \le t < T_k,\ k \ge 1, \\ \inf\limits_{T_k \le r \le t} \left[\psi(r) - \alpha(r)\right] & \text{if } T_k \le t < S_k,\ k \ge 1. \end{cases} \tag{2.21}$$

*If* $T_\alpha < T^\beta$, *then*

$$\Xi_{\alpha,\beta}(\psi)(t) = \begin{cases} 0 & \text{if } 0 \le t < T_0, \\ \inf\limits_{T_k \le r \le t} \left[\psi(r) - \alpha(r)\right] & \text{if } T_k \le t < S_k,\ k \ge 0, \\ \sup\limits_{S_{k-1} \le s \le t} \left[\psi(s) - \beta(s)\right] & \text{if } S_{k-1} \le t < T_k,\ k \ge 1. \end{cases} \tag{2.22}$$

We precede the proof with two technical lemmas. The first one examines $\Xi_{\alpha,\beta}(\psi)$ on $[S_{k-1}, T_k)$.

**Lemma 2.7.** *Under the assumptions of Theorem 2.6, for every $k \geq 1$ and for every $t \in [S_{k-1}, T_k)$,*

$$\Xi_{\alpha,\beta}(\psi)(t) = \sup_{S_{k-1}\leq s\leq t} [\psi(s) - \beta(s)]. \tag{2.23}$$

*Proof.* Let $t \in [S_{k-1}, T_k)$. Then $\Xi_{\alpha,\beta}(\psi)(t) = X_1(t) \vee X_2^{k-1}(t) \vee X_3^{k-1}(t) \vee X_4^{k-1}(t)$, where

$$X_1(t) = (\psi(0) - \beta(0))^+ \wedge \inf_{0\leq r\leq t} (\psi(r) - \alpha(r)), \tag{2.24}$$

$$X_2^{k-1}(t) = \sup_{0\leq s\leq T_{k-1}} \left[ (\psi(s) - \beta(s)) \wedge \inf_{s\leq r\leq t} (\psi(r) - \alpha(r)) \right], \tag{2.25}$$

$$X_3^{k-1}(t) = \sup_{T_{k-1}\leq s\leq S_{k-1}} \left[ (\psi(s) - \beta(s)) \wedge \inf_{s\leq r\leq t} (\psi(r) - \alpha(r)) \right], \tag{2.26}$$

$$X_4^{k-1}(t) = \sup_{S_{k-1}\leq s\leq t} \left[ (\psi(s) - \beta(s)) \wedge \inf_{s\leq r\leq t} (\psi(r) - \alpha(r)) \right]. \tag{2.27}$$

By (2.20),

$$X_1(t) \leq \inf_{0\leq r\leq t} (\psi(r) - \alpha(r)) \leq \inf_{T_{k-1}\leq r\leq S_{k-1}} (\psi(r) - \alpha(r)) \leq \sup_{S_{k-1}\leq s\leq t} [\psi(s) - \beta(s)], \tag{2.28}$$

$$X_2^{k-1}(t) \leq \sup_{0\leq s\leq T_{k-1}} \left[ \inf_{s\leq r\leq t} (\psi(r) - \alpha(r)) \right] \leq \inf_{T_{k-1}\leq r\leq S_{k-1}} (\psi(r) - \alpha(r)) \leq \sup_{S_{k-1}\leq s\leq t} [\psi(s) - \beta(s)]. \tag{2.29}$$

By (2.12) and (2.20),

$$\begin{aligned} X_3^{k-1}(t) &\leq \sup_{T_{k-1}\leq s\leq S_{k-1}} \left[ \inf_{T_{k-1}\leq r\leq s} (\psi(r) - \alpha(r)) \wedge \inf_{s\leq r\leq t} (\psi(r) - \alpha(r)) \right] \\ &\leq \inf_{T_{k-1}\leq r\leq S_{k-1}} (\psi(r) - \alpha(r)) \leq \sup_{S_{k-1}\leq s\leq t} [\psi(s) - \beta(s)]. \end{aligned} \tag{2.30}$$

Finally, by (2.19), $X_4^{k-1}(t) = \sup_{S_{k-1}\leq s\leq t} [\psi(s) - \beta(s)]$ which completes the proof. □

The next lamma examines $\Xi_{\alpha,\beta}(\psi)$ on $[T_k, S_k)$.

**Lemma 2.8.** *Under the assumptions of Theorem 2.6, if $k \geq 1$ or if $T_\alpha < T^\beta$ and $k = 0$, then for every $t \in [T_k, S_k)$,*

$$\Xi_{\alpha,\beta}(\psi)(t) = \inf_{T_k\leq s\leq t} [\psi(s) - \alpha(s)]. \tag{2.31}$$

*Proof.* Let $k$ be a nonnegative integer, and let $t \in [T_k, S_k)$. We can write $\Xi_{\alpha,\beta}(\psi)(t) = X_1(t) \vee X_2^k(t) \vee Y^k(t)$, where $X_1$ and $X_2^k$ are defined by (2.24) and (2.25), respectively, and

$$Y^k(t) = \sup_{T_k \le s \le t} \left[ \left(\psi(s) - \beta(s)\right) \wedge \inf_{s \le r \le t} \left(\psi(r) - \alpha(r)\right) \right]. \tag{2.32}$$

We first show the upper bound,

$$X_1(t) \le \inf_{0 \le r \le t} \left(\psi(r) - \alpha(r)\right) \le \inf_{T_k \le r \le t} \left(\psi(r) - \alpha(r)\right), \tag{2.33}$$

$$X_2^k(t) \le \sup_{0 \le s \le T_k} \inf_{s \le r \le t} \left(\psi(r) - \alpha(r)\right) \le \inf_{T_k \le r \le t} \left(\psi(r) - \alpha(r)\right). \tag{2.34}$$

By (2.12)

$$Y^k(t) \le \sup_{T_k \le s \le t} \left[ \inf_{T_k \le r \le s} \left(\psi(r) - \alpha(r)\right) \wedge \inf_{s \le r \le t} \left(\psi(r) - \alpha(r)\right) \right] = \inf_{T_k \le s \le t} \left(\psi(s) - \alpha(s)\right). \tag{2.35}$$

Thus we have shown that $\Xi_{\alpha,\beta}(\psi)(t) \le \inf_{T_k \le s \le t}(\psi(s) - \alpha(s))$. To show the opposite inequality for $k \ge 1$, it suffices to show that

$$X_2^k(t) \ge \inf_{T_k \le s \le t} \left(\psi(s) - \alpha(s)\right). \tag{2.36}$$

Let $\epsilon > 0$ be arbitrary and $k \ge 1$, and let $\rho \in [S_{k-1}, T_k]$ be such that

$$\sup_{S_{k-1} \le s \le T_k} \left[\psi(s) - \beta(s)\right] \le \psi(\rho) - \beta(\rho) + \epsilon. \tag{2.37}$$

Then, by (2.15),

$$\psi(\rho) - \beta(\rho) \ge \psi(T_k) - \alpha(T_k) - \epsilon. \tag{2.38}$$

Therefore, by (2.19) and (2.38),

$$\begin{aligned}
X_2^k(t) &\ge \left(\psi(\rho) - \beta(\rho)\right) \wedge \inf_{\rho \le r \le t} \left(\psi(r) - \alpha(r)\right) \\
&\ge \left(\psi(\rho) - \beta(\rho)\right) \wedge \inf_{\rho \le r < T_k} \left(\psi(r) - \alpha(r)\right) \wedge \inf_{T_k \le r \le t} \left(\psi(r) - \alpha(r)\right) \\
&\ge \left(\psi(\rho) - \beta(\rho)\right) \wedge \inf_{T_k \le r \le t} \left(\psi(r) - \alpha(r)\right) \\
&\ge \left(\psi(T_k) - \alpha(T_k) - \epsilon\right) \wedge \inf_{T_k \le r \le t} \left(\psi(r) - \alpha(r)\right) \ge \inf_{T_k \le r \le t} \left(\psi(r) - \alpha(r)\right) - \epsilon.
\end{aligned} \tag{2.39}$$

Since (2.39) holds for every $\epsilon > 0$, the proof of (2.36) is complete for $k \geq 1$. To complete the proof for $k = 0$ in the case of $T_\alpha < T^\beta$, it suffices to show that

$$X_1(t) = \inf_{T_0 \leq r \leq t} (\psi(r) - \alpha(r)), \quad \text{for every } t \in [T_0, S_0). \tag{2.40}$$

In this case $T^\beta > 0$, and so $\psi(0) - \beta(0) < 0$. Also $\inf_{0 \leq r < T_0}(\psi(r) - \alpha(r)) \geq 0$ since $T_0 = T_\alpha$ and $\inf_{T_0 \leq r \leq t}(\psi(r) - \alpha(r)) \leq \psi(T_0) - \alpha(T_0) \leq 0$. Therefore

$$X_1(t) = 0 \wedge \inf_{0 \leq r < T_0} (\psi(r) - \alpha(r)) \wedge \inf_{T_0 \leq r \leq t} (\psi(r) - \alpha(r)) = \inf_{T_0 \leq r \leq t} (\psi(r) - \alpha(r)), \tag{2.41}$$

which ends the proof. □

*Proof of Theorem 2.6.* Let $T^\beta < T_\alpha$. If $t \in [0, S_0)$, then $\Xi_{\alpha,\beta}(\psi)(t) = 0$ by Remark 2.3. If $t \in [S_{k-1}, T_k)$ for some $k \geq 1$, then (2.21) holds by Lemma 2.7, and if $t \in [T_k, S_k)$ for some $k \geq 1$, then (2.21) holds by Lemma 2.8.

Similarly, when $T_\alpha < T^\beta$, then $T_0 = T_\alpha$ and (2.22) holds for $t \in [0, T_0)$ by Remark 2.3, for $t \in [S_{k-1}, T_k)$, $k \geq 1$ by Lemma 2.7, and for $t \in [T_k, S_k)$, $k \geq 1$ by Lemma 2.8. □

We introduce two functions

$$H_{\alpha,\beta}(\psi)(t) = \sup_{0 \leq s \leq t} \left[ (\psi(s) - \beta(s)) \wedge \inf_{s \leq r \leq t} (\psi(r) - \alpha(r)) \right], \tag{2.42}$$

$$L_{\alpha,\beta}(\psi)(t) = \inf_{0 \leq s \leq t} \left[ (\psi(s) - \alpha(s)) \vee \sup_{s \leq r \leq t} (\psi(r) - \beta(r)) \right]. \tag{2.43}$$

It is easy to verify that the following relationship holds:

$$L_{\alpha,\beta}(\psi)(t) = -H_{-\beta,-\alpha}(-\psi)(t). \tag{2.44}$$

**Corollary 2.9.** *Let $\alpha \in D^-[0,\infty)$, $\beta \in D^+[0,\infty)$ be such that* $\inf_{t \geq 0}(\beta(t) - \alpha(t)) > 0$. *Then for every $\psi \in D[0,\infty)$,*

$$\Xi_{\alpha,\beta}(\psi)(t) = I_{\{T^\beta < T_\alpha\}} I_{[T^\beta,\infty)}(t) H_{\alpha,\beta}(\psi)(t) + I_{\{T_\alpha < T^\beta\}} I_{[T_\alpha,\infty)}(t) L_{\alpha,\beta}(\psi)(t). \tag{2.45}$$

*Proof.* If $T_\alpha = \infty$ and $T^\beta = \infty$, then both sides of (2.45) are zero.

Suppose that $T^\beta < T_\alpha$. Then $S_0 = T^\beta$. If $t \in [0, T^\beta)$, then $\Xi_{\alpha,\beta}(\psi)(t) = 0$ by Remark 2.3.

Let $t \in [S_{k-1}, T_k)$ for some $k \geq 1$. We have shown in the proof of Lemma 2.7 that

$$H_{\alpha,\beta}(\psi)(t) = X_2^{k-1}(t) \vee X_3^{k-1}(t) \vee X_4^{k-1}(t) = X_4^{k-1}(t). \tag{2.46}$$

Therefore (2.45) holds by (2.19) and (2.21).

Consider now $t \in [T_k, S_k)$ for some $k \geq 1$. We have shown in the proof of Lemma 2.8 that

$$H_{\alpha,\beta}(\psi)(t) = X_2^k(t) \vee Y^k(t) = \inf_{T_k \leq s \leq t}(\psi(s) - \alpha(s)). \tag{2.47}$$

Hence, again, we conclude (2.45) from (2.21).

Suppose now that $T_\alpha < T^\beta$ and set $\psi' = -\psi$, $\alpha' = -\beta$, and $\beta' = -\alpha$. Then $T^{\beta'}(\psi') = T_\alpha(\psi) < T^\beta(\psi) = T_{\alpha'}(\psi')$ and so we can apply the already proven case of (2.45) to $\psi'$, $\alpha'$ and $\beta'$. We obtain, by (2.44),

$$\Xi_{\alpha',\beta'}(\psi')(t) = I_{[T^{\beta'},\infty)}(t) H_{\alpha',\beta'}(\psi')(t) = -I_{[T_\alpha,\infty)}(t) L_{\alpha,\beta}(\psi)(t). \tag{2.48}$$

By Remark 1.3 and Remark 2.5 in [6], $\Xi_{\alpha,\beta}(\psi)(t) = -\Xi_{\alpha',\beta'}(\psi')(t)$, and so the proof of (2.45) is complete. □

We are going to show next that the times $T_\alpha$ and $T^\beta$ in (2.45) can be replaced by $\tau_\alpha$ and $\tau^\beta$. Their properties described in Remark 2.1 will be essential in expanding the representation to a general ESM.

**Corollary 2.10.** *Let $\alpha \in D^-[0,\infty)$, $\beta \in D^+[0,\infty)$ be such that $\inf_{t\geq 0}(\beta(t) - \alpha(t)) > 0$. Then for every $\psi \in D[0,\infty)$,*

$$\Xi_{\alpha,\beta}(\psi)(t) = I_{\{\tau^\beta < \tau_\alpha\}} I_{[\tau^\beta,\infty)}(t) H_{\alpha,\beta}(\psi)(t) + I_{\{\tau_\alpha < \tau^\beta\}} I_{[\tau_\alpha,\infty)}(t) L_{\alpha,\beta}(\psi)(t). \tag{2.49}$$

*Proof.* By Remark 2.2, there are three possible cases. If $\tau_\alpha = \tau^\beta = \infty$, then $\Xi_{\alpha,\beta}(\psi)(t) = 0$, and (2.49) holds trivially.

Consider the case when $\tau^\beta < \tau_\alpha$. If $t \leq \tau^\beta$, then, as in Remark 2.2, $\Xi_{\alpha,\beta}(\psi)(t) = 0$, and (2.49) holds.

If $t > \tau^\beta$, then by (2.3), $t > T^\beta$, and so, by (2.10), there is $k \geq 1$ such that $t \in [S_{k-1}, T_k)$ or $t \in [T_k, S_k)$. If $t \in [S_{k-1}, T_k)$, then, as in the proof of Lemma 2.7, $\Xi_{\alpha,\beta}(\psi)(t) = X_4^{k-1}(t)$, and therefore, by (2.46), $\Xi_{\alpha,\beta}(\psi)(t) = H_{\alpha,\beta}(\psi)(t)$.

If $t \in [T_k, S_k)$, then by (2.31) and (2.47), we have $\Xi_{\alpha,\beta}(\psi)(t) = H_{\alpha,\beta}(\psi)(t)$ again.

In the case of $\tau^\alpha < \tau_\beta$ we can apply the already proven case of (2.49) to $-\psi$, $-\beta$, and $-\alpha$, as in the proof of Corollary 2.9, to complete the proof. □

In the following final result, we extend the representation of (2.49) to a general case, thus producing an alternative representation formula for the ESM with two time-dependent reflective boundaries.

**Theorem 2.11.** *Let $\alpha \in D^-[0,\infty)$, $\beta \in D^+[0,\infty)$ be such that $\alpha \leq \beta$. Then for every $\psi \in D[0,\infty)$ and every $t \geq 0$,*

$$\Xi_{\alpha,\beta}(\psi)(t) = I_{\{\tau^\beta \leq \tau_\alpha\}} I_{[\tau^\beta,\infty)}(t) H_{\alpha,\beta}(\psi)(t) + I_{\{\tau_\alpha < \tau^\beta\}} I_{[\tau_\alpha,\infty)}(t) L_{\alpha,\beta}(\psi)(t). \tag{2.50}$$

*Proof.* Let $\alpha(t) \le \beta(t)$ for every $t \ge 0$, and define $\alpha_n(t) = \alpha(t) \wedge (\beta(t) - n^{-1})$ and $\beta_n(t) = (\alpha(t) + n^{-1}) \vee \beta(t)$. Then $\alpha_n \in D^-[0,\infty)$, $\beta_n \in D^+[0,\infty)$, $\alpha_n \uparrow \alpha$, $\beta_n \downarrow \beta$, $-n^{-1} \le \alpha_n - \alpha \le 0 \le \beta_n - \beta \le n^{-1}$, and $\inf_{t\ge 0}(\beta_n(t) - \alpha_m(t)) \ge n^{-1} \vee m^{-1}$. By Corollary 2.10, for every $n$, $m$ and every $t \ge 0$,

$$\Xi_{\alpha_m,\beta_n}(\psi)(t) = I_{\{\tau^{\beta_n}<\tau_{\alpha_m}\}} I_{[\tau^{\beta_n},\infty)}(t) H_{\alpha_m,\beta_n}(\psi)(t) + I_{\{\tau_{\alpha_m}<\tau^{\beta_n}\}} I_{[\tau_{\alpha_m},\infty)}(t) L_{\alpha_m,\beta_n}(\psi)(t). \tag{2.51}$$

To complete the proof of (2.50), it suffices to show that the right-hand side of (2.51) converges to the right-hand side of (2.50) uniformly on compact sets. By Proposition 2.5 in [5], we could then conclude that $\lim_{n\to\infty} \Xi_{\alpha_n,\beta_n}(\psi) = \Xi_{\alpha,\beta}(\psi)$. In fact we will show the uniform convergence of the right-hand sides.

It is easy to see that for every $n$

$$\tau_\alpha \le \tau_{\alpha_n}, \qquad \tau^\beta \le \tau^{\beta_n}. \tag{2.52}$$

Hence, by Remark 2.1,

$$\lim_{n\to\infty} \tau_{\alpha_n} = \tau_\alpha, \qquad \lim_{n\to\infty} \tau^{\beta_n} = \tau^\beta. \tag{2.53}$$

To show the convergence of $H_{\alpha_n,\beta_n}(\psi)$ we consider a mapping $R : D^+[0,\infty) \times D^-[0,\infty) \to D[0,\infty)$ defined by

$$R(f,g)(t) = \sup_{0\le s\le t} \left[ f(s) \wedge \inf_{s\le r\le t} g(r) \right]. \tag{2.54}$$

It is easy to see that $R$ is continuous in the uniform metric. In fact, it can be shown that

$$\| R(f_1,g_1) - R(f_2,g_2) \| \le \| f_1 - f_2 \| \vee \| g_1 - g_2 \|. \tag{2.55}$$

Since $H_{\alpha,\beta}(\psi) = R(\beta - \psi, \psi - \alpha)$, we get that $\lim_{n\to\infty} H_{\alpha_n,\beta_n}(\psi) = H_{\alpha,\beta}(\psi)$ uniformly. Similarly, $\lim_{n\to\infty} L_{\alpha_n,\beta_n}(\psi) = L_{\alpha,\beta}(\psi)$ uniformly.

We consider four possible cases: $\tau_\alpha = \tau^\beta = \infty$, $\tau_\alpha < \tau^\beta$, $\tau^\beta < \tau_\alpha$, and $\tau_\alpha = \tau^\beta < \infty$. If $\tau_\alpha = \tau^\beta = \infty$, then by (2.52), $\tau_{\alpha_n} = \tau^{\beta_n} = \infty$ for every $n$, and so the right-hand sides of both (2.51) and (2.50) are zero.

Suppose that $\tau^\beta < \tau_\alpha$, then also $\tau^{\beta_n} < \tau_{\alpha_n}$ for almost all $n$. We show first that $I_{[\tau^{\beta_n},\infty)} H_{\alpha_n,\beta_n}(\psi)$ converges to $I_{[\tau^\beta,\infty)} H_{\alpha,\beta}(\psi)$ uniformly. If $\beta - \psi$ has a jump at $\tau^\beta$, then $\tau^{\beta_n} = \tau^\beta$ and so also $I_{[\tau^{\beta_n},\infty)} = I_{[\tau^\beta,\infty)}$ for sufficiently large $n$. Thus for large enough $n$ we have that $I_{[\tau^{\beta_n},\infty)} H_{\alpha_n,\beta_n}(\psi) = I_{[\tau^\beta,\infty)} H_{\alpha_n,\beta_n}(\psi)$ converges uniformly to $I_{[\tau^\beta,\infty)} H_{\alpha,\beta}(\psi)$. If $\beta - \psi$ is continuous at $\tau^\beta$, then $\beta(\tau^\beta) = \psi(\tau^\beta)$ and so $H_{\alpha,\beta}(\psi)(\tau^\beta) = 0$. Because $H_{\alpha,\beta}(\psi)$ is right-continuous,

$\lim_{n\to\infty}\sup_{\tau^{\beta}\le t\le\tau^{\beta_n}} H_{\alpha,\beta}(\psi)(t) = 0$. Hence

$$\begin{aligned}
&\lim_{n\to\infty}\left\| I_{[\tau^{\beta_n},\infty)} H_{\alpha_n,\beta_n}(\psi) - I_{[\tau^{\beta},\infty)} H_{\alpha,\beta}(\psi)\right\| \\
&\quad\le \lim_{n\to\infty}\left\| I_{[\tau^{\beta_n},\infty)} H_{\alpha_n,\beta_n}(\psi) - I_{[\tau^{\beta_n},\infty)} H_{\alpha,\beta}(\psi)\right\| + \lim_{n\to\infty}\left\| I_{[\tau^{\beta_n},\infty)} H_{\alpha,\beta}(\psi) - I_{[\tau^{\beta},\infty)} H_{\alpha,\beta}(\psi)\right\| \\
&\quad\le \lim_{n\to\infty}\left\| H_{\alpha_n,\beta_n}(\psi) - H_{\alpha,\beta}(\psi)\right\| + \lim_{n\to\infty}\sup_{\tau^{\beta}\le t\le\tau^{\beta_n}}\left| H_{\alpha,\beta}(\psi)(t)\right| = 0.
\end{aligned} \tag{2.56}$$

Thus we have established that

$$\lim_{n\to\infty} I_{[\tau^{\beta_n},\infty)} H_{\alpha_n,\beta_n}(\psi) = I_{[\tau^{\beta},\infty)} H_{\alpha,\beta}(\psi), \tag{2.57}$$

and therefore

$$\begin{aligned}
&\lim_{n\to\infty}\left[ I_{\{\tau^{\beta_n}<\tau_{\alpha_n}\}} I_{[\tau^{\beta_n},\infty)} H_{\alpha_n,\beta_n}(\psi) - I_{\{\tau_{\alpha_n}<\tau^{\beta_n}\}} I_{[\tau_{\alpha_n},\infty)} L_{\alpha_n,\beta_n}(\psi)\right] \\
&\quad= \lim_{n\to\infty} I_{[\tau^{\beta_n},\infty)} H_{\alpha_n,\beta_n}(\psi) = I_{[\tau^{\beta},\infty)} H_{\alpha,\beta}(\psi) \\
&\quad= I_{\{\tau^{\beta}\le\tau_{\alpha}\}} I_{[\tau^{\beta},\infty)} H_{\alpha,\beta}(\psi) - I_{\{\tau_{\alpha}<\tau^{\beta}\}} I_{[\tau_{\alpha},\infty)} L_{\alpha,\beta}(\psi),
\end{aligned} \tag{2.58}$$

where the convergence is uniform.

By a similar argument we show convergence in the case of $\tau_\alpha < \tau^\beta$. We consider now the final case of $\tau_\alpha = \tau^\beta < \infty$. Because $\inf_{t\ge 0}[\beta_n(t) - \alpha(t)] \ge n^{-1}$, it follows by (2.7), that $\tau^{\beta_n} \ne \tau_\alpha$. Analogously we can show that $\tau_{\alpha_n} \ne \tau^\beta$. Hence, for every $n$,

$$\tau^{\beta} < \tau^{\beta_{n+1}} \le \tau^{\beta_n}, \qquad \tau_{\alpha} < \tau_{\alpha_{n+1}} \le \tau_{\alpha_n}. \tag{2.59}$$

By (2.53) and (2.59), we can find an increasing sequence of positive integers $\{n_k \mid k = 1, 2, \ldots\}$ such that

$$\tau^{\beta_{n_1}} > \tau_{\alpha_{n_2}} > \tau^{\beta_{n_3}} > \tau_{\alpha_{n_4}} > \tau^{\beta_{n_5}} > \cdots. \tag{2.60}$$

Let

$$\alpha'_{2k+1} = \alpha_{n_{2k+2}} \quad \text{for } k \ge 0, \qquad \alpha'_{2k} = \alpha_{n_{2k}} \quad \text{for } k \ge 1, \tag{2.61}$$

$$\beta'_{2k+1} = \beta_{n_{2k+1}} \quad \text{for } k \ge 1, \qquad \beta'_{2k} = \beta_{n_{2k+1}} \quad \text{for } k \ge 1. \tag{2.62}$$

Then it follows from (2.60) that

$$\tau_{\alpha'_{2k+1}} < \tau^{\beta'_{2k+1}} \quad \text{for every } k \geq 0, \qquad \tau_{\alpha'_{2k}} > \tau^{\beta'_{2k}} \quad \text{for every } k \geq 1. \tag{2.63}$$

Therefore, as in (2.58),

$$\begin{aligned}\lim_{k\to\infty}&\left[I_{\{\tau^{\beta'_{2k+1}}<\tau_{\alpha'_{2k+1}}\}} I_{[\tau^{\beta'_{2k+1}},\infty)} H_{\alpha'_{2k+1},\beta'_{2k+1}}(\psi) + I_{\{\tau_{\alpha'_{2k+1}}<\tau^{\beta'_{2k+1}}\}} I_{[\tau_{\alpha'_{2k+1}},\infty)} L_{\alpha'_{2k+1},\beta'_{2k+1}}(\psi)\right] \\ &= \lim_{k\to\infty} I_{[\tau_{\alpha'_{2k+1}},\infty)} L_{\alpha'_{2k+1},\beta'_{2k+1}}(\psi) = I_{[\tau_\alpha,\infty)} L_{\alpha,\beta}(\psi).\end{aligned} \tag{2.64}$$

Similarly,

$$\begin{aligned}\lim_{k\to\infty}&\left[I_{\{\tau^{\beta'_{2k}}<\tau_{\alpha'_{2k}}\}} I_{[\tau^{\beta'_{2k}},\infty)} H_{\alpha'_{2k},\beta_{2k}}(\psi) + I_{\{\tau_{\alpha'_{2k}}<\tau^{\beta'_{2k}}\}} I_{[\tau_{\alpha'_{2k}},\infty)} L_{\alpha'_{2k},\beta'_{2k}}(\psi)\right] \\ &= \lim_{k\to\infty} I_{[\tau^{\beta'_{2k}},\infty)}(t) H_{\alpha'_{2k},\beta'_{2k}}(\psi) = I_{[\tau^{\beta},\infty)}(t) H_{\alpha,\beta}(\psi).\end{aligned} \tag{2.65}$$

Both limits, in (2.64) and in (2.65), are in fact the uniform limits. Since, by Proposition 2.5 in [5], $\lim_{k\to\infty} \Xi_{\alpha'_{2k},\beta'_{2k}}(\psi) = \Xi_{\alpha,\beta}(\psi)$ and $\lim_{k\to\infty} \Xi_{\alpha'_{2k+1},\beta'_{2k+1}}(\psi) = \Xi_{\alpha,\beta}(\psi)$, the limits in (2.64) and (2.65) must be the same, that is,

$$\Xi_{\alpha,\beta}(\psi) = I_{[\tau_\alpha,\infty)}(t) L_{\alpha,\beta}(\psi)(t) = I_{[\tau^\beta,\infty)}(t) H_{\alpha,\beta}(\psi)(t). \tag{2.66}$$

Therefore $\lim_{n\to\infty} \Xi_{\alpha_n,\beta_n}(\psi) = \Xi_{\alpha,\beta}(\psi)$ uniformly, and so (2.50) holds again. □

## 3. Continuity Properties of the ESM in Metrics on $D[0,\infty)$

In [6] we have established a number of continuity properties of the Skorokhod map under the assumptions that $\alpha, \beta \in D[0,\infty)$ and $\inf_{t\geq 0}[\beta - \alpha] > 0$. We are going to extend some of these properties onto the ESM. This will be done in two steps. First we will allow $\alpha \in D^-[0,\infty)$, $\beta \in D^+[0,\infty)$, and secondly we will let $\inf_{t\geq 0}[\beta(t) - \alpha(t)] \geq 0$.

We begin by observing the following nesting property of the SP and the ESP constrains. It can be readily verified by checking the conditions of Definions 1.2 and 1.1.

*Remark 3.1.* Let $(\phi, \eta)$ be the solution of the SP [resp., ESP] for $\psi$ on $[\alpha_1, \beta_1]$ for some $\alpha_1 \in D^-[0,\infty)$ and $\beta_1 \in D^+[0,\infty)$. Consider another pair of constrains $\alpha_2 \in D^-[0,\infty)$ and $\beta_2 \in D^+[0,\infty)$. If $\alpha_1 \leq \alpha_2 \leq \phi \leq \beta_2 \leq \beta_1$, then $(\phi, \eta)$ is also the solution of the SP [resp., ESP] for $\psi$ on $[\alpha_2, \beta_2]$.

Instead of checking that the proofs of all the continuity properties in [6] are valid when the constraining functions $\alpha, \beta$ are allowed to take infinite values, we develop in the next lemma a convenient tool for expanding such properties to this more general case.

**Lemma 3.2.** *Let* $\alpha \in D^-[0,\infty)$ *and* $\beta \in D^+[0,\infty)$ *be such that* $\inf_{t\ge 0}[\beta(t)-\alpha(t)] > 0$. *For any* $\psi \in D[0,\infty)$ *there is a nonincreasing sequence* $\{\alpha_n \mid n \ge 1\} \subset D[0,\infty)$ *and a nondecreasing sequence* $\{\beta_n \mid n \ge 1\} \subset D[0,\infty)$ *such that the following conditions hold:*

(i) $\alpha \le \alpha_n \le \Gamma_{\alpha,\beta}(\psi) \le \beta_n \le \beta$ *for every* $n \ge 1$;

(ii) $\inf_{t\ge 0}[\beta_n(t)-\alpha_n(t)] = \inf_{t\ge 0}[\beta(t)-\alpha(t)]$ *for every* $n \ge 1$;

(iii) *for every* $T > 0$, *there is* $N_T$ *such that for all* $n \ge N_T$ *and all* $t \in [0,T]$,

$$\alpha_n(t) = \begin{cases} \alpha(t) & \text{if } t \in D_\alpha, \\ -n & \text{if } t \notin D_\alpha, \end{cases} \qquad \beta_n(t) = \begin{cases} \beta(t) & \text{if } t \in D_\beta, \\ n & \text{if } t \notin D_\beta; \end{cases} \tag{3.1}$$

(iv) $\Gamma_{\alpha_n,\beta_n}(\psi) = \Gamma_{\alpha,\beta}(\psi)$ *for every* $n \ge 1$.

*Proof.* Let $\alpha \in D^-[0,\infty)$, $\beta \in D^+[0,\infty)$ with $d = \inf_{t\ge 0}[\beta(t)-\alpha(t)] > 0$, let $\psi \in D[0,\infty)$, and let $\phi = \Gamma_{\alpha,\beta}(\psi)$. For every $n \ge 1$, we define

$$\begin{aligned} \alpha_n &= \alpha \vee \left[(-n) \wedge \phi \wedge 0.5(\alpha+\beta-d) I_{\{\alpha>-\infty,\ \beta<\infty\}}\right] \\ &= (\alpha \vee (-n)) \wedge \phi \wedge 0.5(\alpha+\beta-d) I_{\{\alpha>-\infty,\ \beta<\infty\}}, \\ \beta_n &= \beta \wedge \left[n \vee \phi \vee 0.5(\alpha+\beta+d) I_{\{\alpha>-\infty,\ \beta<\infty\}}\right] \\ &= (\beta \wedge n) \vee \phi \vee 0.5(\alpha+\beta+d) I_{\{\alpha>-\infty,\ \beta<\infty\}}. \end{aligned} \tag{3.2}$$

It is easy to verify that for every $n \ge 1$,

$$\alpha \le \alpha_{n+1} \le \alpha_n \le \phi \le \beta_n \le \beta_{n+1} \le \beta, \tag{3.3}$$

$$\alpha_n \le \frac{\alpha+\beta-d}{2} \le \frac{\alpha+\beta+d}{2} \le \beta_n \quad \text{on } D_\alpha \cap D_\beta. \tag{3.4}$$

Inequalities in (3.3) show the monotonic properties of $\alpha_n$ and $\beta_n$ and prove statement (i) as well.

To prove (ii) we note that, by (3.4), for $t \in D_\alpha \cap D_\beta$

$$\beta_n(t) - \alpha_n(t) \ge \frac{\alpha+\beta+d}{2} - \frac{\alpha+\beta-d}{2} = d \tag{3.5}$$

and so (ii) immediately follows.

For $T > 0$, define

$$N_T^\alpha = \sup_{t\in D_\alpha \cap [0,T]} |\alpha(t)|, \qquad N_T^\beta = \sup_{t\in D_\beta \cap [0,T]} \left|\beta(t)\right|, \qquad N_T^\phi = \sup_{t\in[0,T]} \left|\phi(t)\right|, \tag{3.6}$$

and set $N_T = N_T^\alpha \vee N_T^\beta \vee N_T^\phi$. Let $t \in [0,T]$ and let $n \ge N_T$. If $\beta(t) < \infty$, then $\beta_n(t) = (\beta(t) \wedge n) \vee \phi(t) \vee 0.5(\alpha(t)+\beta(t)+d) I_{\{\alpha(t)>-\infty\}} = \beta(t)$. If $\beta(t) = \infty$, then $\beta_n(t) = (\beta(t) \wedge n) \vee \phi(t) = n$. Similarly, we show that $\alpha_n(t) = \alpha(t)$ if $\alpha(t) > -\infty$ and $\alpha_n(t) = -n$ if $\alpha(t) = -\infty$, which completes the proof of (iii).

Finally, (iv) follows from (i) and Remark 3.1. □

The limiting process used in the proof of Theorem 2.11 provides a useful technique of extending properties of the Skorokhod map with separated constraining boundaries onto the general ESM. In the following proposition we set a formal method that will allow us to replace the assumption of $\inf_{t\geq 0}(\beta(t)-\alpha(t))>0$ by a weaker assumption of $\beta \geq \alpha$.

*Definition 3.3.* Consider $\mathcal{C}$, a family of mappings $T_{\alpha,\beta}$ from $D[0,\infty)$ to $D[0,\infty)$ indexed by a set of pairs $(\alpha,\beta)$, where $\alpha \in D^-[0,\infty)$, $\beta \in D^+[0,\infty)$. We will say that $\mathcal{C}$ is closed if the following condition is satisfied: for any sequence $T_{\alpha_n,\beta_n}$ in $\mathcal{C}$ if $\alpha_n \xrightarrow{n\to\infty} \alpha$, $\beta_n \xrightarrow{n\to\infty} \beta$, and $T_{\alpha_n,\beta_n}(\psi) \xrightarrow{n\to\infty} T_{\alpha,\beta}(\psi)$ for every $\psi \in D[0,\infty)$ uniformly, then $T_{\alpha,\beta} \in \mathcal{C}$.

**Proposition 3.4.** *Let $\mathcal{C}$ be a family of mappings $T_{\alpha,\beta}$ from $D[0,\infty)$ to $D[0,\infty)$ indexed by a set of pairs $(\alpha,\beta)$, where $\alpha \in D^-[0,\infty)$, $\beta \in D^+[0,\infty)$. If $\mathcal{C}$ is closed and contains all Skorokhod maps $\Gamma_{\alpha,\beta}$, such that* $\inf_{t\geq 0}(\beta(t)-\alpha(t))>0$*, then $\mathcal{C}$ contains all extended Skorokhod maps $\overline{\Gamma}_{\alpha,\beta}$, where $\alpha \leq \beta$.*

*Proof.* Suppose that $\mathcal{C}$ is closed and contains all Skorokhod maps $\Gamma_{\alpha,\beta}$ with $\inf_{t\geq 0}(\beta(t)-\alpha(t))>0$. Let $\overline{\Gamma}_{\alpha,\beta}$ be an ESM, where $\alpha \in D^-[0,\infty)$, $\beta \in D^+[0,\infty)$, and $\alpha \leq \beta$. As in the proof of Theorem 2.11 we can construct sequences $\{\alpha_n \mid n\geq 1\} \subset D^-[0,\infty)$ and $\{\beta_n \mid n \geq 1\} \subset D^+[0,\infty)$ such that $\alpha_n \uparrow \alpha$, $\beta_n \downarrow \beta$, and $\inf_{t\geq 0}(\beta_n(t)-\alpha_n(t)) \geq 1/n$, so that, by Remark 1.3, $\Gamma_{\alpha_n,\beta_n}$ is an SM for every $n\geq 1$, and $\Gamma_{\alpha_n,\beta_n}(\psi)$ converges to $\overline{\Gamma}_{\alpha,\beta}(\psi)$ uniformly for every $\psi \in D[0,\infty)$. Since $\Gamma_{\alpha_n,\beta_n} \in \mathcal{C}$, for every $n \geq 1$ we get $\overline{\Gamma}_{\alpha,\beta} \in \mathcal{C}$. □

**Theorem 3.5.** *For any $\psi_1,\psi_2 \in D[0,\infty)$, $\alpha_1,\alpha_2 \in D^-[0,\infty)$, $\beta_1,\beta_2 \in D^+[0,\infty)$ such that $\alpha_1 \leq \beta_1$ and $\alpha_2 \leq \beta_2$*

$$\left\|\overline{\Gamma}_{\alpha_1,\beta_1}(\psi_1)-\overline{\Gamma}_{\alpha_2,\beta_2}(\psi_2)\right\| \leq 4\left\|\psi_1-\psi_2\right\| + 3\left[\|\alpha_1-\alpha_2\| \vee \left\|\beta_1-\beta_2\right\|\right]. \tag{3.7}$$

*Proof.* Assume first that $\alpha_1,\alpha_2,\beta_1,\beta_2 \in D[0,\infty)$, and that the condition $\inf_{t\geq 0}(\beta_i(t)-\alpha_i(t))>0$ holds for $i=1,2$. By Proposition 4.1 and Remark 4.3 in [6], for any $\psi_1,\psi_2 \in D[0,\infty)$,

$$\left\|\Gamma_{\alpha_1,\beta_1}(\psi_1)-\Gamma_{\alpha_2,\beta_2}(\psi_2)\right\| \leq 4\left\|\psi_1-\psi_2\right\| + 2\|\alpha_1-\alpha_2\| + \left[\|\alpha_1-\alpha_2\| \vee \left\|\beta_1-\beta_2\right\|\right]. \tag{3.8}$$

By Remark 2.5 in [6], $\Gamma_{\alpha,\beta}(\psi) = -\Gamma_{-\beta,-\alpha}(-\psi)$; therefore applying (3.8) to $\Gamma_{-\beta,-\alpha}(-\psi)$, we get

$$\left\|\Gamma_{\alpha_1,\beta_1}(\psi_1)-\Gamma_{\alpha_2,\beta_2}(\psi_2)\right\| \leq 4\left\|\psi_1-\psi_2\right\| + 2\left\|\beta_1-\beta_2\right\| + \left[\|\alpha_1-\alpha_2\| \vee \left\|\beta_1-\beta_2\right\|\right]. \tag{3.9}$$

Combining (3.8) and (3.9), we get that (3.7) holds for Skorokhod maps when the constraining boundaries take finite values and are separated.

Suppose now that $\alpha_i \in D^-[0,\infty)$ and $\beta_i \in D^+[0,\infty)$ for $i = 1,2$. We can find sequences $\{\alpha_i^n \mid n \geq 1\}$, $\{\beta_i^n \mid n \geq 1\} \subset D[0,\infty)$, $i = 1,2$, satisfying (i)–(iv) of Lemma 3.2 and we already know that for every $n \geq 1$

$$\left\|\Gamma_{\alpha_1^n,\beta_1^n}(\psi_1) - \Gamma_{\alpha_2^n,\beta_2^n}(\psi_2)\right\| \leq 4\|\psi_1 - \psi_2\| + 3\left[\|\alpha_1^n - \alpha_2^n\| \vee \|\beta_1^n - \beta_2^n\|\right]. \tag{3.10}$$

If $D_{\alpha_1} \neq D_{\alpha_2}$ or $D_{\beta_1} \neq D_{\beta_2}$ then $\|\alpha_1 - \alpha_2\| = \infty$, or $\|\beta_1 - \beta_2\| = \infty$ and so (3.7) holds trivially. We can assume therefore that $D_{\alpha_1} = D_{\alpha_2}$ and $D_{\beta_1} = D_{\beta_2}$. By part (iii) of Lemma 3.2, $\|\alpha_1^n - \alpha_2^n\| = \|\alpha_1 - \alpha_2\|$ and $\|\beta_1^n - \beta_2^n\| = \|\beta_1 - \beta_2\|$ for large enough $n$. On the other hand, by part (iv) of Lemma 3.2, $\|\Gamma_{\alpha_1^n,\beta_1^n}(\psi_1) - \Gamma_{\alpha_2^n,\beta_2^n}(\psi_2)\| = \|\Gamma_{\alpha_1,\beta_1}(\psi_1) - \Gamma_{\alpha_2,\beta_2}(\psi_2)\|$. Thus, for large $n$, we can replace $\alpha_1^n$, $\beta_1^n$, $\alpha_2^n$, and $\beta_2^n$ in (3.10) by $\alpha_1$, $\beta_1$, $\alpha_2$, and $\beta_2$, and so (3.9) holds for Skorokhod maps with separated constraining boundaries.

Next, we are going to relax the assumption that $\alpha_i$, $\beta_i$ are separated for $i = 1,2$. For a fixed $\alpha_1 \in D^-[0,\infty)$ and $\beta_1 \in D^+[0,\infty)$ satisfying $\inf_{t\geq 0}[\beta_1(t) - \alpha_1(t)] > 0$, consider a family $\mathcal{C}_{\alpha_1,\beta_1}$ of mappings $T_{\alpha_2,\beta_2} : D[0,\infty) \to D[0,\infty)$ indexed by pairs $(\alpha_2,\beta_2)$, where $\alpha_2 \in D^-[0,\infty)$ and $\beta_2 \in D^+[0,\infty)$ and satisfying for every $\psi_1,\psi_2 \in D[0,\infty)$

$$\left\|\Gamma_{\alpha_1,\beta_1}(\psi_1) - T_{\alpha_2,\beta_2}(\psi_2)\right\| \leq 4\|\psi_1 - \psi_2\| + 3\left[\|\alpha_1 - \alpha_2\| \vee \|\beta_1 - \beta_2\|\right]. \tag{3.11}$$

We have already established that $\Gamma_{\alpha_2,\beta_2} \in \mathcal{C}_{\alpha_1,\beta_1}$ whenever $\inf_{t\geq 0}[\beta_2(t) - \alpha_2(t)] > 0$. It is also easy to verify that $\mathcal{C}_{\alpha_1,\beta_1}$ is closed and so, by Proposition 3.4, $\overline{\Gamma}_{\alpha_2,\beta_2} \in \mathcal{C}_{\alpha_1,\beta_1}$ for any $\alpha_2 \leq \beta_2$. Thus we have shown so far that (3.7) holds whenever $\inf_{t\geq 0}[\beta_1(t) - \alpha_1(t)] > 0$.

Finally, for a fixed $\alpha_2 \in D^-[0,\infty)$ and $\beta_2 \in D^+[0,\infty)$ such that $\alpha_2 \leq \beta_2$, let $\mathcal{C}_{\alpha_2,\beta_2}$ be a family of mappings $T_{\alpha_1,\beta_1} : D[0,\infty) \to D[0,\infty)$ indexed by pairs $(\alpha_1,\beta_1)$, where $\alpha_1 \in D^-[0,\infty)$ and $\beta_1 \in D^+[0,\infty)$ and satisfying for every $\psi_1,\psi_2 \in D[0,\infty)$

$$\left\|T_{\alpha_1,\beta_1}(\psi_1) - \overline{\Gamma}_{\alpha_2,\beta_2}(\psi_2)\right\| \leq 4\|\psi_1 - \psi_2\| + 3\left[\|\alpha_1 - \alpha_2\| \vee \|\beta_1 - \beta_2\|\right]. \tag{3.12}$$

Then $\mathcal{C}_{\alpha_2,\beta_2}$ contains all $\Gamma_{\alpha_1,\beta_1}$ with $\inf_{t\geq 0}[\beta_1(t) - \alpha_1(t)] > 0$, and it is closed. Applying again Proposition 3.4, we obtain that $\overline{\Gamma}_{\alpha_1,\beta_1} \in \mathcal{C}_{\alpha_2,\beta_2}$ for any $\alpha_1 \leq \beta_1$, and so the proof of (3.7) is complete. □

Applying Theorem 3.5 in the special case of $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, we can get the Lipschitz continuity of the ESM with the Lipschitz constant 4. However, reapplying our techniques based on Lemma 3.2 and Proposition 3.4, we will obtain the following stronger result, which is a generalization of Proposition 4.6 in [6].

**Theorem 3.6** (Lipschitz continuity). *Let $\alpha \in D^-[0,\infty)$, $\beta \in D^+[0,\infty)$ be such that $\alpha \leq \beta$. Then for any $\psi_1,\psi_2 \in D[0,\infty)$*

$$\left\|\overline{\Gamma}_{\alpha,\beta}(\psi_1) - \overline{\Gamma}_{\alpha,\beta}(\psi_2)\right\| \leq 2\|\psi_1 - \psi_2\|. \tag{3.13}$$

*Proof.* By Proposition 4.6 in [6], (3.13) holds for any $\alpha,\beta \in D[0,\infty)$ such that $\inf_{t\geq 0}(\beta(t) - \alpha(t)) > 0$. If $\alpha \in D^-[0,\infty)$ and $\beta \in D^+[0,\infty)$ then, as in the proof of Theorem 3.5, we find sequences $\{\alpha_n \mid n \geq 1\}$, $\{\beta_n \mid n \geq 1\} \subset D[0,\infty)$, $i = 1,2$, satisfying (i)–(iv) of Lemma 3.2. By part (iv) of Lemma 3.2, for any positive integer $n$,

$$\left\|\Gamma_{\alpha,\beta}(\psi_1) - \Gamma_{\alpha,\beta}(\psi_2)\right\| = \left\|\Gamma_{\alpha_n,\beta_n}(\psi_1) - \Gamma_{\alpha_n,\beta_n}(\psi_2)\right\| \leq 2\left\|\psi_1 - \psi_2\right\|. \tag{3.14}$$

Finally, to relax the assumption of $\inf_{t\geq 0}[\beta_1(t) - \alpha_1(t)] > 0$, we apply Proposition 3.4 to the family $\mathcal{C}$ of mappings $T_{\alpha,\beta} : D[0,\infty) \to D[0,\infty)$ indexed by pairs $(\alpha,\beta)$, where $\alpha \in D^-[0,\infty)$ and $\beta \in D^+[0,\infty)$ and satisfying

$$\left\|T_{\alpha,\beta}(\psi_1) - T_{\alpha,\beta}(\psi_2)\right\| \leq 2\left\|\psi_1 - \psi_2\right\|. \tag{3.15}$$

Since $\mathcal{C}$ contains all $\Gamma_{\alpha,\beta}$ with $\inf_{t\geq 0}[\beta(t) - \alpha(t)] > 0$ and it is closed, we conclude, by Proposition 3.4, that $\mathcal{C}$ must contain all $\overline{\Gamma}_{\alpha,\beta}$ with $\beta \geq \alpha$. Thus, (3.13) holds for every ESM. □

We next examine the continuity of the ESM under the Skorokhod $J_1$ metric $d_0$. The Skorokhod metric $d_0$ on $D[0,\infty)$ is defined by

$$d_0(f,g) = \inf_{\lambda}\left(\|\lambda - I\| \vee \left\|f - g\circ\lambda\right\|\right), \tag{3.16}$$

where the infimum is over all strictly increasing continuous bijections of $[0,\infty)$.

We are going to need the following scaling property of the ESM.

*Remark 3.7.* Let $\psi \in D[0,\infty)$, $\alpha \in D^-[0,\infty)$, and $\beta \in D^+[0,\infty)$ be such that $\alpha \leq \beta$. For any strictly increasing continuous bijection $\lambda$ of $[0,\infty)$

$$\overline{\Gamma}_{\alpha,\beta}(\psi)\circ\lambda = \overline{\Gamma}_{\alpha\circ\lambda,\beta\circ\lambda}(\psi\circ\lambda). \tag{3.17}$$

*Proof.* Using formula (2.42) we can verify that $H_{\alpha,\beta}(\psi)\circ\lambda = H_{\alpha\circ\lambda,\beta\circ\lambda}(\psi\circ\lambda)$. Then from (2.44) we obtain $L_{\alpha,\beta}(\psi)\circ\lambda = L_{\alpha\circ\lambda,\beta\circ\lambda}(\psi\circ\lambda)$. Using (2.2), we can verify that $\lambda^{-1}(\tau_\alpha(\psi)) = \tau_{\alpha\circ\lambda}(\psi\circ\lambda)$ and $\lambda^{-1}(\tau^\beta(\psi)) = \tau^{\beta\circ\lambda}(\psi\circ\lambda)$, which in turn implies that $I_{[\tau_\alpha(\psi),\infty)}\circ\lambda = I_{[\tau_{\alpha\circ\lambda(\psi\circ\lambda)},\infty)}$ and $I_{[\tau^\beta(\psi),\infty)}\circ\lambda = I_{[\tau^{\beta\circ\lambda(\psi\circ\lambda)},\infty)}$. Then by (2.50) we obtain $\Xi_{\alpha,\beta}(\psi)\circ\lambda = \Xi_{\alpha\circ\lambda,\beta\circ\lambda}(\psi\circ\lambda)$, and so, applying (1.4), we conclude (3.17). □

**Theorem 3.8.** *For any $\psi_1, \psi_2 \in D[0,\infty)$, $\alpha \in D^-[0,\infty)$, $\beta \in D^+[0,\infty)$ such that $\alpha \leq \beta$*

$$d_0\left(\overline{\Gamma}_{\alpha,\beta}(\psi_1), \overline{\Gamma}_{\alpha,\beta}(\psi_2)\right) \leq 4d_0(\psi_1,\psi_2) + 3\sup_{r,s>0}|\alpha(r) - \alpha(s)| \vee \sup_{r,s>0}\left|\beta(r) - \beta(s)\right|. \tag{3.18}$$

*Proof.* Let $\lambda$ be any strictly increasing continuous bijection of $[0,\infty)$. By (3.17) and (3.13),

$$
\begin{aligned}
&d_0\left(\overline{\Gamma}_{\alpha,\beta}(\psi_1),\overline{\Gamma}_{\alpha,\beta}(\psi_2)\right) \\
&\quad\le \|\lambda - I\| \vee \left\|\overline{\Gamma}_{\alpha,\beta}(\psi_1) - \overline{\Gamma}_{\alpha,\beta}(\psi_2)\circ\lambda\right\| = \|\lambda - I\| \vee \left\|\overline{\Gamma}_{\alpha,\beta}(\psi_1) - \overline{\Gamma}_{\alpha\circ\lambda,\beta\circ\lambda}(\psi_2\circ\lambda)\right\| \\
&\quad\le \|\lambda - I\| \vee \left[4\left\|\psi_1 - \psi_2\circ\lambda\right\| + 3\left(\|\alpha - \alpha\circ\lambda\| \vee \left\|\beta - \beta\circ\lambda\right\|\right)\right] \\
&\quad\le \|\lambda - I\| \vee 4\left\|\psi_1 - \psi_2\circ\lambda\right\| + 3\left(\|\alpha - \alpha\circ\lambda\| \vee \left\|\beta - \beta\circ\lambda\right\|\right) \\
&\quad\le 4\left(\|\lambda - I\| \vee \left\|\psi_1 - \psi_2\circ\lambda\right\|\right) + 3\left(\sup_{r,s>0}|\alpha(r) - \alpha(s)| \vee \sup_{r,s>0}\left|\beta(r) - \beta(s)\right|\right).
\end{aligned}
\tag{3.19}
$$

Taking $\inf_\lambda$, we obtain (3.18). □

Note that in cases when $D_\alpha$ or $D_\beta$ is a proper subset of $[0,\infty)$ the oscillation terms $\sup_{r,s>0}|\alpha(r)-\alpha(s)|$ or $\sup_{r,s>0}|\beta(r)-\beta(s)|$ become infinite thus rendering the upper bound of (3.18) useless.

*Remark 3.9.* In general $\overline{\Gamma}_{\alpha,\beta}$ is not continuous in $d_0$ metric on $D[0,\infty]$.

*Proof.* Example 4.1 in [6] shows how to construct $\alpha$, $\beta$, $\psi_1$, and $\psi_2$ so that $d_0(\psi_1,\psi_2)$ is arbitrarily small while $d_0(\Gamma_{\alpha,\beta}(\psi_1),\Gamma_{\alpha,\beta}(\psi_2))$ is arbitrarily large. Thus $\Gamma_{\alpha,\beta}$ is not continuous in $d_0$ metric and therefore neither is $\overline{\Gamma}_{\alpha,\beta}$. □

In fact the same example can be used to demonstrate that in general $\overline{\Gamma}_{\alpha,\beta}$ is not continuous in the Skorokhod $M_1$ metric as indicated in Example 4.2 of [6].

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